Universal Inputs
for Identifiability and Observability

Symposium to Honor Bill Wolovich

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(with Y. Wang, A. Megretski)

Outline

The Setting

what classes of input signals are sufficient
to completely identify the i/o behavior of a system?

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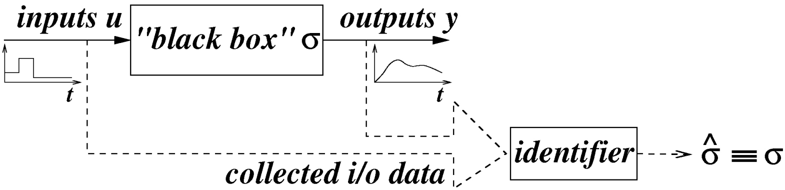

Two types of problems

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if so, does this input need to be very special? (genericity)

Problem(s) 2: what about *"simple"* inputs?

Why these problems?

restricted class of experiments

esp. in systems biology, no experiments w/arbitrary input profiles
sometimes only steps, pulses

or, at the other extreme: a “random” input
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note: for *linear* (0 initial state), any single $\neq 0$ input OK (steps, pulses)

e.g. for $m = p = 1$ just do $W(s) = \dfrac{\hat{y}(s)}{\hat{u}(s)}$

[no noise; also, not talking about *steady-state* ID]

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- no regard to computational effort
- deterministic
- finite-time (no stability assumed)

Outline

Grasselli and Isidori's 1977 paper

universal inputs for observability of bilinear systems:

system observable $\Rightarrow$

$\exists$ inputs that distinguish any pair

(or any state from zero)

idea: construct descending sequence of spaces

$K_1 \supset K_2 \supset \ldots \supset K_n = \{0\}$

of states indistinguishable from zero

Many follow-up universal input theorems

EDS'78: polynomial d.t., analytic c.t. on compacts

Sussmann'79: general theorem for c.t. analytic; and *genericity*

can be interpreted as *parameter identifiability*

(params as constant states)

here:

- universal (and generic) over *all possible (analytic) systems*
- back to bilinear: very concrete classes of inputs
  (motivated by biological applications)

(with Yuan Wang, and with YW & Sasha Megretski)

Outline

Informal statement of universal results (m=p=1)

$$\dot{x} \;=\; f(x) + g(x)u, \quad x(0) = x_o, \quad y = h(x)$$

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- no possible $\mathcal{C}^\omega$ input can work

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to be precise: under non-degeneracy conditions $\sim$
controllability/observability

Systems & distinguishability

single-input single-output initialized σ and $\widehat{\sigma}$:

$$\sigma : \dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_o, \quad y = h(x(t))$$

and

$$\widehat{\sigma} : \dot{x}(t) = \widehat{f}(x(t)) + \widehat{g}(x(t))u(t), \quad x(0) = \widehat{x}_o, \quad y = \widehat{h}(x(t))$$

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$\Omega :=$ all (m.e.b.) inputs $u : [0, T_u] \to \mathbb{R}$
given two systems σ, $\widehat{\sigma}$, input u, solutions defined for $t \in [0, T_u]$,

$$\varphi(t, u) = x(t), \quad y(t) = h(\varphi(t, u))$$

σ, $\widehat{\sigma}$ *indistinguishable under* u if

$$h(\varphi(t, u)) = \widehat{h}(\widehat{\varphi}(t, u)) \quad \forall\, t \in [0, T_u]$$

I/O Equivalence

$\sigma, \widehat{\sigma}$ *i/o equivalent* ($\sigma \underset{\mathcal{U}}{\equiv} \widehat{\sigma}$) *w.r.t. all inputs* $\mathcal{U} \subseteq \Omega$

if no input in $\mathcal{U}$ distinguishes:

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subset $\mathcal{U} \subseteq \Omega$ of inputs *sufficient for identifying system class* Σ if:

for each pair $\sigma, \widehat{\sigma}$ in Σ,

$$\sigma \underset{\mathcal{U}}{\equiv} \widehat{\sigma} \quad \Rightarrow \quad \sigma \equiv \widehat{\sigma}$$

i.e., not i/o equivalent $\Rightarrow \exists$ input in set $\mathcal{U}$ which distinguishes

E.g., linear

linear systems (finite-dimensional, continuous-time)

$$\dot{x} = Ax + bu\,, \quad x(0) = 0\,, \quad y = cx$$

($A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $c \in \mathbb{R}^{1 \times n}$)

identifiable by any single nonzero input on a nontrivial interval
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what about interesting classes of nonlinear systems?

Outline

Bilinear systems

we consider two different classes of bilinear systems
$\mathcal{S}_n^I :=$ n-dimensional *bilinear systems of type I*: f_0 linear, f_1 affine, $x_0 = 0$, h linear:

$$\begin{aligned} \dot{x} &= (A + uN)x + bu, \quad x(0) = 0 \\ y &= cx \end{aligned}$$

$$(A, N, b, c) \;\; \text{where} \;\; A, N \in \mathbb{R}^{n\times n}, \; b \in \mathbb{R}^{n\times 1}, \; c \in \mathbb{R}^{1\times n}$$

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$\mathcal{S}_n^{II} :=$ n-dimensional *bilinear systems of type II*: f_0, f_2, h all linear, $x(0)$ allowed nonzero:

$$\begin{aligned} \dot{x} &= (A + uN)x, \quad x(0) = b \\ y &= cx \end{aligned}$$

Algebraic characterization of equivalence

recall (Isidori, Fliess, 1970s):

no need to test all possible inputs

σ and $\widehat{\sigma}$ (both type I or type II) i/o equivalent iff

$$cA_{i_1} \dots A_{i_k} b \;=\; \widehat{c}\widehat{A}_{i_1} \dots \widehat{A}_{i_k} \widehat{b}$$

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(enough check sequences of length $n + \widehat{n}$)

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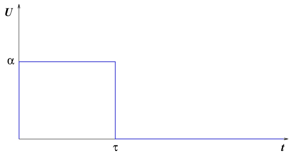
Theorem: same for class-II.

(*"generic"*:= set of 4-tuples $S \subseteq \mathbb{R}^{2n^2+2n}$ w/full measure & open dense)

Negative results: single pulses



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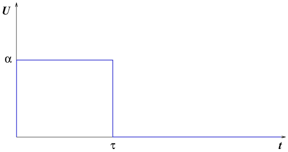

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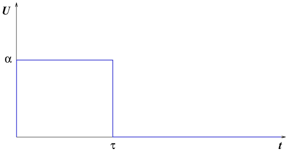

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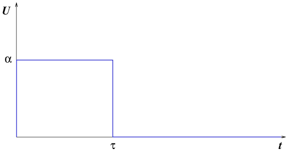

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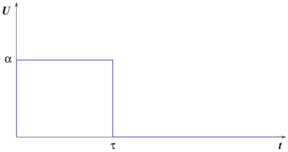

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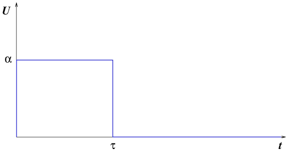

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Positive results: pulses of fixed amplitude

for any fixed $\alpha \in \mathbb{R}$, $\mathcal{V}_\alpha :=$ set of pulses of magnitude α:

$$\mathcal{V}_\alpha := \{\mathbf{u}_{\tau,\alpha} | \ \tau \geq 0\}$$

Theorem: for each $\alpha \neq 0$, $\exists$ generic $\mathcal{M} \subseteq \mathcal{S}_n^I$ s.t.,

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i.e.: set of pulses of amplitude α (and varying length) sufficient

Outline

Sketch of proof of negative results

let $\mathcal{C}$:= set consisting of all those 4-tuples

$$(Q, N, b_0, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n}$$

which satisfy the following conditions:

(a) (Q, b_0, c) is canonical

(b) $(Q - N, b_0, c)$ is canonical

(c) $e^Q - I$ is invertible

(d) $N \notin \mathcal{B}(T(Q, b_0, c))$

where: for each canonical $\sigma = (A, b, c)$, pick (unique, self-adjoint) $T = T(\sigma)$ such that

$$AT = TA', \ b = Tc', \ cT = b'$$

and for each nonzero $n \times n$ matrix S, ("commutator") proper linear subspace of $\mathbb{R}^{n\times n}$:

$$\mathcal{B}(S) := \{N \in \mathbb{R}^{n\times n} \mid NS = SN'\}$$

(continued sketch of neg)

let $\mathcal{X} = \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$, and
consider the analytic map $\psi : \mathcal{X} \to \mathcal{X}$ defined by

$$\psi : (Q, N, b_0, c) \mapsto (Q - N,\ N,\ (\rho(Q))^* b_0, c)\,,$$

where $\rho(Q) = \int_0^1 e^{sQ} ds$, and $\rho(Q)^*$ denotes adjoint matrix of $\rho(Q)$

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for general τ and α, rescale inputs and time scale

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let $\mathbf{u} = \mathbf{u}_{\tau,\alpha}$ with $\tau = 1, \alpha = 1$.

Lemma: consider systems of type I
$\forall\ \sigma^o \in \mathcal{D}$, $\exists\ \widehat{\sigma^o} \in \mathcal{D}$ s.t.:

1. σ^o and $\widehat{\sigma^o}$ i/o equivalent under the pulse function $\mathbf{u}$, but
2. σ^o and $\widehat{\sigma^o}$ not i/o equivalent

for general τ and α, rescale inputs and time scale
similarly for type II

Construction of $\widehat{\sigma}^o$

given $\sigma^o = (A, N, b, c) \in \mathcal{D}$, there exist

$$(Q, N, b_0, c) \in \mathcal{C} \quad \text{s/t}:$$

$$A = Q - N, \;\; b = [\det(\rho(Q)](\rho(Q))^{-1} b_0.$$

let $b_1 = \rho(Q) b$
then $b_1 = \det(\rho(Q)) b_0$, so:

$$\begin{aligned} \mathcal{R}(Q, b_1) &= \det(\rho(Q)) \mathcal{R}(Q, b_0) \\ \mathcal{R}(Q - N, b_1) &= \det(\rho(Q)) \mathcal{R}(Q - N, b_0) \end{aligned}$$

since $\det(\rho(Q)) \neq 0$, both (Q, b_1) and $(Q - N, b_1)$ reachable

construction (ctd)

moreover, it can be shown that:

$$\begin{aligned} T(Q, b_1, c) &= \det(\rho(Q))\, T(Q, b_0, c), \\ &\Downarrow \\ \mathcal{B}(T(Q, b_1, c)) &= \mathcal{B}(T(Q, b_0, c)). \end{aligned}$$

so $(Q, N, b_1, c) \in \mathcal{C}$
with $M = TN'T^{-1}$, one has:

- $M \neq N$, and
- $ce^{t(Q+\gamma N)} b_1 = ce^{t(Q+\gamma M)} b_1$ for all $\gamma \in \mathbb{R}$ and $t \geq 0$

in particular, for $\gamma = -1$:

$$ce^{t(Q-N)} b_1 = ce^{t(Q-M)} b_1 \qquad \forall\, t \geq 0$$

let $\widehat{\sigma^o} = ((A + N - M), M, b, c) = (Q - M, M, b, c)$
(compare w/ $\sigma^o = (Q - N, N, b, c)$)
then: $\widehat{\sigma^o} \in \mathcal{D}$, and $\sigma^o \underset{\mathbf{u}}{\equiv} \widehat{\sigma^o}$, but $\sigma^o \not\equiv \widehat{\sigma^o}$

Outline

The generic set for positive results

let $\alpha \neq 0$ be given

$\mathcal{M} :=$ be the set of 4-tuples satisfying the following two properties:

1. (A, b, c) is canonical
2. $(A + \alpha N, b)$ is controllable

this is a proper algebraic set, so generic
and shown to work

Tools: Similarity

bilinear systems $\sigma \sim \widehat{\sigma}$ ("similar" or "internally equivalent")
if $\exists$ change of variables $x = Tz$
s.t. equations of σ get transformed into those of $\widehat{\sigma}$

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s.t. equations of σ get transformed into those of $\widehat{\sigma}$

i.e. 4-tuples (A, N, b, c) and $(\widehat{A}, \widehat{N}, \widehat{b}, \widehat{c})$ in same $GL(n)$-orbit under similarity action:
same dimension n, and $\exists\ T \in \mathbb{R}^{n \times n}$ invertible s.t.

$$A = T\widehat{A}T^{-1}, \quad N = T\widehat{N}T^{-1}, \quad b = T\widehat{b}, \quad c = \widehat{c}T^{-1}$$

Canonical Systems and Uniqueness

suppose that both σ and $\widehat{\sigma}$ canonical

Lemma. $\sigma \equiv \widehat{\sigma} \iff \sigma \sim \widehat{\sigma}$

(and similarity is unique)

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recall: canonical σ means span-reachable and observable 4-tuple (A, N, b, c):

- no proper subspace of $\mathbb{R}^n$ contains b and is invariant under $x \mapsto Ax$ and $x \mapsto Nx$
- no proper subspace of $\mathbb{R}^n$ is contained in the nullspace of $x \mapsto cx$ and is invariant under $x \mapsto Ax$ and $x \mapsto Nx$

Sketch of positive result (type I)

pick $\sigma_1^o = (A_1, N_1, b_1, c_1)$ and $\sigma_2^o = (A_2, N_2, b_2, c_2)$ in $\mathcal{M}$, s.t. same outputs for each $u \in \mathcal{V}_\alpha$

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must show that $\sigma_1^o \equiv \sigma_2^o$

fix any $\tau > 0$
applying $\mathbf{u}_{\tau,\alpha} \in \mathcal{V}_\alpha$ to the two systems:

$$\begin{aligned} \dot{x} &= (A_1 + uN_1)x + b_1 u, \quad x(0) = 0, \quad y = c_1 x \\ \dot{z} &= (A_2 + uN_2)z + b_2 u, \quad z(0) = 0, \quad y = c_2 z, \end{aligned}$$

one has:

$$c_1 e^{A_1(t-\tau)} x(\tau) = c_2 e^{A_2(t-\tau)} z(\tau) \qquad \forall\, t \geq \tau,$$

(easy:) for generic $\tau_0 > 0$, $(A_1, x(\tau_0), c_1)$ & $(A_2, z(\tau_0), c_2)$ canonical
$\therefore\ \exists\ T \in GL(n)$ s.t.

$$A_2 = T^{-1} A_1 T,\ z(\tau_0) = T^{-1} x(\tau_0),\ c_2 = c_1 T$$

(ctd.)

so using $c_2 e^{A_2 s} = c_1 e^{A_1 s} T$ for all s, above becomes:

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from observability of (A_1, c_1), it follows that

$$x(\tau) = T z(\tau) \qquad \forall\, \tau > 0$$

or equivalently:

$$\int_0^\tau e^{(A_1+\alpha N_1)s} ds\, b_1 = T \int_0^\tau e^{(A_2+\alpha N_2)s} ds\, b_2 \qquad \forall\, \tau > 0$$

taking $d/d\tau$:

$$e^{(A_1+\alpha N_1)\tau} b_1 = T e^{(A_2+\alpha N_2)\tau} b_2$$

this is true for all $\tau \geq 0$
so in particular:

$$b_1 = T b_2$$

(ctd.)

on the other hand, taking repeated derivatives in τ
and then setting $\tau = 0$, one obtains:

$$(A_1 + \alpha N_1)^k b_1 = T(A_2 + \alpha N_2)^k b_2 \qquad \forall\, k \geq 0$$

which implies $(0 \leq k \leq n-1)$:

$$\mathcal{R}(A_1 + \alpha N_1, b_1) = T[\mathcal{R}(A_2 + \alpha N_2, b_2)]$$

and $(1 \leq k \leq n)$:

$$(A_1 + \alpha N_1)[\mathcal{R}(A_1 + \alpha N_1, b_1)] \;=\; T(A_2 + \alpha N_2)[\mathcal{R}(A_2 + \alpha N_2, b_2)]$$

so

$$(A_1 + \alpha N_1)[\mathcal{R}(A_1 + \alpha N_1, b_1)] = T(A_2 + \alpha N_2)T^{-1}[\mathcal{R}(A_1 + \alpha N_1, b_1)]$$

(ctd.)

as $\mathcal{R}(A_1 + \alpha N_1, b_1)$ invertible
(because $(A_1 + \alpha N_1, b_1)$ is controllable), $\Rightarrow$

$$T(A_2 + \alpha N_2)T^{-1} = (A_1 + \alpha N_1)$$

so again follows from above, and the fact that $\alpha \neq 0$,

that $N_2 = T^{-1}N_1 T$

so, the 4-tuples are similar, and the systems are i/o equiv

Outline

A construction

given any analytic function $\kappa(r)$, consider:

$$\begin{aligned} \dot{x} &= 1 \\ \dot{z} &= z \\ \dot{w} &= (\kappa(x) - u)z \\ y &= w \end{aligned}$$

analytic system

σ and $\widehat{\sigma}$ same system, but just different initial states:

$x_0 := (0, 1, 0)$ and $\widehat{x}_0 := (0, 0, 0)$

"observables": $(\{h,\ L_f h,\ L_g h,\ L_f^2 h,\ L_f L_g h,\ \ldots\})$ are:

$\{w,\ z,\ \kappa(x)z,\ \kappa'(x)z,\ \kappa''(x)z,\ \ldots\}$

$\Rightarrow y(t) \neq \widehat{h}(t)$ for some u, i.e., $\sigma \not\equiv \widehat{\sigma}$

but, $\sigma \underset{\kappa}{\equiv} \widehat{\sigma} \quad (y(t) = \widehat{y}(t) \equiv 0)$

Outline

Main result

Theorem: there exists a generic set $\mathcal{U}$ of smooth inputs s/t

$$(\forall\, u \in \mathcal{U}) \qquad \sigma \underset{u}{\equiv} \widehat{\sigma} \;\; \Rightarrow \; \sigma \equiv \widehat{\sigma}$$

i.e.: every u in $\mathcal{U}$ distinguishes any two σ and $\widehat{\sigma}$

generic:= contains a countable intersections of open dense sets seeing for each $T > 0$, $C^\infty[0,\, T]$ w/Whitney topology

Identification by jets

$\sigma \not\equiv \widehat{\sigma}$ if there is some u such that

$$\left.\frac{d^k}{dt^k}\right|_{t=0} y(t,u) \neq \left.\frac{d^k}{dt^k}\right|_{t=0} \hat{y}(t,u)$$

for some k.

Theorem: there is a generic set $\mathcal{W} \subseteq \mathbb{R}^\infty$ such that

for each $\mu = (\mu_0,\, \mu_1, \mu_2, \ldots)$ in $\mathcal{W}$, $\exists\, u$ with $u^{(i)}(0) = \mu_i$ so that for any σ and $\widehat{\sigma}$,

$$\left.\frac{d^k}{dt^k}\right|_{t=0} y(t,u) \quad = \quad \left.\frac{d^k}{dt^k}\right|_{t=0} \hat{y}(t,u) \quad \forall\, k$$

$$\Downarrow$$

$$\sigma \quad \equiv \widehat{\sigma}$$

(generic: containing a countable intersection of open dense subsets, with the product topology on $\mathbb{R}^\infty$)

Outline

Review: Chen/Fliess encoding of inputs (m=1)

consider:

$$\dot{C}(t) = (X_0 + \sum_{i=1}^{m} u_i X_i) C(t), \quad C(0) = 1$$

solution exists by Peano-Baker formula:

$$C[u](t) = \sum_w V_w[u](t) w$$

w: word in $\{X_0, X_1, \ldots, X_m\}$
$V_w[u]$: iterated integrals of u defined recursively:

$$V_\phi[u](t) = 1$$

$$V_{X_i w}[u](t) = \int_0^t u_i(s) V_w[u](s)\, ds$$

where $u_0(t) \equiv 1$

e.g.: $\quad V_{X_i}[u](t) = \int_0^t u_i(s)\, ds, \quad V_{X_i X_j}[u](t) = \int_0^t u_i(s) \int_0^s u_j(\tau) d\tau ds$

Review: Fliess Series

$$
\begin{aligned}
c &= \sum_w c(w)w \qquad\qquad (c(w) \in \mathbb{R}) \\
&= c(\phi) + c(X_0)X_0 + c(X_1)X_1 + \ldots c(X_m)X_m \\
&\qquad + c(X_0X_0)X_0X_0 + c(X_0X_1)X_0X_1 + \ldots
\end{aligned}
$$

c is convergent if: $|c(w)| \leq CM^l l! \quad C, M = \text{const}, \ \ l = |w|$

Fliess operator: $u \mapsto y = F_c[u]$ defined by:

$$
y(t) = < c,\ C[u](t) > = \sum_w c(w) V_w[u](t)
$$

$$
\begin{aligned}
\text{e.g.}(m=1): \quad y(t) &= c(\phi) + c(X_0)\int_0^t ds + c(1)\int_0^t u(s)\,ds \\
&+ c(X_0X_1)\int_0^t\int_0^{s_1} u(s_2)\,ds_2\,ds_1 \\
&+ c(X_1X_1)\int_0^t u(s_1)\int_0^{s_1} u(s_2)\,ds_2\,ds_1 + \ldots
\end{aligned}
$$

$F_c[u]$ well defined on $[0, T)$, some $T > 0$, if c convergent

Look at operators

Fact: *i/o behaviors of initialized state space system are defined by appropriate* $\sigma = F_c$, i.e.,

for each u in Ω, $y(t) = F_c[u](t)$

in fact,

$$c(X_{i_1} X_{i_2} \ldots X_{i_r}) = L_{g_{i_r}} \ldots L_{g_{i_1}} h(x_0)$$

$(g_0 := f)$

so enough to show $\exists$ generic set $\mathcal{W}$ in $\mathbb{R}^\infty$ s/t for each $\mu = (\mu_0, \mu_1 \ldots)$ in $\mathcal{W}$, there is u with $u^{(i)}(0) = \mu_i$ such that:

for any $c, \widehat{c}$,

$$\left.\frac{d^k}{dt^k}\right|_{t=0} F_c[u](t) = \left.\frac{d^k}{dt^k}\right|_{t=0} F_{\widehat{c}}[u](t) \quad \forall\, k$$

$$\Downarrow$$

$$c = \widehat{c}$$

Sketch of proof

to prove result on series,

enough to deal with c and a specific series $\widehat{c}$ – the zero series

infinite jet $\mu = (\mu_0, \mu_1, \ldots)$ said to be *universal* for a set S of series

if $\exists$ u with $u^{(i)}(0) = \mu_i$ s.t.

$$F_c[u] \not\equiv 0 \quad \text{for each } c \in S$$

Definition: family S of Fliess series is:

1. *equiconvergent* if $\exists$ $r, M > 0$ s.t.

$$|c(w)| \;\leq\; M\, r^{|w|}\,(|w|)! \qquad \forall\, c \in S$$

2. *compact* if compact in weak topology

(seen as a family of sequences indexed by words in $X_i's$)

Main Lemma

Lemma: S compact, equiconvergent, $0 \notin S$

$\Rightarrow$ {infinite jets universal for S} open dense.

Corollary:

the set of uniformly universal jets is generic

because:

set of nonzero conv Fliess series $= \bigcup_{w,k} S_{w,k}$

where $S_{w,k} =$ all series such that

$$|c(w)| \geq 1/k$$

and

$$|c(w)| \leq k^l l! \qquad \text{where} \;\; l = |w|$$

A technical Lemma

Lemma: Let S compact, equiconv,

and assume we know $\exists$ at least one u which is universal for S, i.e.:

$$F_c[u](t) \not\equiv 0 \quad \forall c \in S$$

now let μ be **any** (arbitrary) finite jet

then, $\exists\ \nu$, finite extension of μ, universal for S

main idea: for any given finite μ,

$\exists$ analytic inputs v_j with $v_j^{(i)}(0) = \mu_i$ s/t $v_j \to u$ (in L_1 topology)

together with compactness and equiconv of S,

$\exists\ j_0$ s.t. v_{j_0} universal for S

Proof of Main Lemma

given S: compact, equiconvergent, $0 \notin S$

want: univ jets for S is open, dense

openness follows from compactness of S

Density:

let μ be a jet, and let $\mathcal{W}$ be any nbhd of μ

$$\mathcal{W} = \mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2 \times \ldots \times \mathcal{W}_r \times \mathbb{R}^m \times \mathbb{R}^m \times \ldots$$

let μ^r:= restriction of μ to first r terms

$0 \notin S \Rightarrow \forall\ c \in S$, $\exists$ a jet ν "good" for c

compactness of $S \Rightarrow \exists\ \mathcal{V}_1,\ \mathcal{V}_2,\ \ldots \mathcal{V}_s,\ \ni$:

- $\{\mathcal{V}_i\}$ covers S
- each $\mathcal{V}_i$ has a (finite) univ jet

tech lemma $\Rightarrow$:

$\exists\ \nu_1$, extension of μ^r, univ for $\mathcal{V}_1$

$\exists\ \nu_2$, extension of ν_1, univ for $\mathcal{V}_1 \bigcup \mathcal{V}_2$

Repeat^s

Outline

Many open problems...

- special classes: pulses with varying amplitudes?
- cascades of bilinear?
- other classes of systems?
- unif univ theorems for $\mathcal{C}^\infty$ classes with appropriate transversality assumptions?

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