Universal Inputs
for Identifiability and Observability

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(with Y. Wang, A. Megretski)
Outline

Introduction

Background

New results

Special classes of inputs for special classes of systems

Proofs of negative results for bilinear

Proofs of positive result for bilinear

No single $C^\omega$ input works for $C^\omega$ systems

Uniformly universal inputs

Proof of uniformly universal result

Closing Remarks
The Setting

*what classes of input signals are sufficient to completely identify the i/o behavior of a system?*
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We look for classes $\mathcal{U}$ of inputs and classes of systems s.t.:

If system $\sigma$ stimulated with inputs from the set $\mathcal{U}$ and corresponding time record of outputs is recorded,

possible theoretically to uniquely (i/o) identify system in class
The Setting

**what classes of input signals are sufficient to completely identify the i/o behavior of a system?**

we look for classes $\mathcal{U}$ of inputs and classes of systems s.t.:

if system $\sigma$ stimulated with inputs from the set $\mathcal{U}$ and corresponding time record of outputs is recorded,

possible theoretically to uniquely (i/o) identify system in class $\hat{\sigma} \equiv \sigma$
Two types of problems

Problem(s) 1: is a single input sufficient?
Two types of problems

**Problem(s) 1:** is a single input sufficient?
if so, does this input need to be very special? (genericity)

**Problem(s) 2:** what about “simple” inputs?
Why these problems?

*restricted class of experiments*

esp. in systems biology, no experiments w/arbitrary input profiles
sometimes only steps, pulses

or, at the other extreme: a “random” input
(observed, but originating from a “black box” subsystem)
Why these problems?

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(observed, but originating from a “black box” subsystem)

note: for \textit{linear} (0 initial state), any single \( \neq 0 \) input OK (steps, pulses)
e.g. for \( m = p = 1 \) just do \( W(s) = \frac{\hat{y}(s)}{\hat{u}(s)} \)

[no noise; also, not talking about \textit{steady-state ID}]
(1) generic $C^\infty$ inputs enough for class of all analytic systems
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(2) results for bilinear systems, steps and pulses
   • a class of nonlinear systems
   • theoretically, approximate fading-memory . . .
   • enzymatic signaling cascades far from saturation
(1) generic $C^\infty$ inputs enough for class of all analytic systems

(2) results for bilinear systems, steps and pulses
• a class of nonlinear systems
• theoretically, approximate fading-memory . . .
• enzymatic signaling cascades far from saturation

• no regard to computational effort
• deterministic
• finite-time (no stability assumed)
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universal inputs for observability of bilinear systems:

system observable $\Rightarrow$

$\exists$ inputs that distinguish any pair

(or any state from zero)

idea: construct descending sequence of spaces

$K_1 \supset K_2 \supset \ldots \supset K_n = \{0\}$

of states indistinguishable from zero
Many follow-up universal input theorems

EDS’78: polynomial d.t., analytic c.t. on compacts

Sussmann’79: general theorem for c.t. analytic; and genericity

can be interpreted as parameter identifiability

(params as constant states)

here:

▶ universal (and generic) over all possible (analytic) systems
▶ back to bilinear: very concrete classes of inputs
  (motivated by biological applications)

(with Yuan Wang, and with YW & Sasha Megretski)
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Informal statement of universal results \((m=p=1)\)

\[
\dot{x} = f(x) + g(x)u, \quad x(0) = x_0, \quad y = h(x)
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- there is a \(C^\infty\) input that serves to distinguish any two \(C^\omega\) systems (independently of the pair, truly universal)

- in fact, a generic \(C^\infty\) input works

- no possible \(C^\omega\) input can work
Informal statement of bilinear results \((m=p=1)\)

\[
\dot{x} = Ax + u Nx, \quad x(0) = x_0, \quad y(t) = cx(t),
\]
Informal statement of bilinear results \((m=p=1)\)

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- step inputs not enough for identifying bilinear systems
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- nor single pulses

{pulses of a fixed amplitude (but varying widths)}

OK to be precise: under non-degeneracy conditions $\sim$ controllability/observability
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- \{pulses of a fixed amplitude (but varying widths)\} OK

to be precise: under non-degeneracy conditions \(\sim\) controllability/observability
single-input single-output initialized $\sigma$ and $\hat{\sigma}$:

$$\sigma : \dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_o, \quad y = h(x(t))$$

and

$$\hat{\sigma} : \dot{x}(t) = \hat{f}(x(t)) + \hat{g}(x(t))u(t), \quad x(0) = \hat{x}_o, \quad y = \hat{h}(x(t))$$

(all analytic)
single-input single-output initialized $\sigma$ and $\hat{\sigma}$:

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(all analytic)

$\Omega := \text{all (m.e.b.) inputs } u: [0, T_u] \to \mathbb{R}$

given two systems $\sigma, \hat{\sigma}$, input $u$, solutions defined for $t \in [0, T_u]$,

$$\varphi(t, u) = x(t), \quad y(t) = h(\varphi(t, u))$$

$\sigma, \hat{\sigma}$ indistinguishable under $u$ if

$$h(\varphi(t, u)) = \hat{h}(\hat{\varphi}(t, u)) \quad \forall \ t \in [0, T_u]$$
\(\sigma, \hat{\sigma}\) i/o equivalent \((\sigma \equiv \hat{\sigma})\) w.r.t. all inputs \(\mathcal{U} \subseteq \Omega\)

if no input in \(\mathcal{U}\) distinguishes:

\[
h(\varphi(t, u)) = \hat{h}(\hat{\varphi}(t, u)) \quad \forall u \in \mathcal{U}, \ t \in [0, T_u]
\]

when \(\mathcal{U} = \Omega\), just write \(\sigma \equiv \hat{\sigma}\), or “systems i/o equivalent”:

cannot be distinguished at all based on “black box” i/o behavior
$\sigma, \hat{\sigma}$ i/o equivalent ($\sigma \equiv_{\mathcal{U}} \hat{\sigma}$) w.r.t. all inputs $\mathcal{U} \subseteq \Omega$

if no input in $\mathcal{U}$ distinguishes:

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when $\mathcal{U} = \Omega$, just write $\sigma \equiv \hat{\sigma}$, or “systems i/o equivalent”: cannot be distinguished at all based on “black box” i/o behavior

subset $\mathcal{U} \subseteq \Omega$ of inputs sufficient for identifying system class $\Sigma$ if:

for each pair $\sigma, \hat{\sigma}$ in $\Sigma$,

$$\sigma \equiv_{\mathcal{U}} \hat{\sigma} \quad \Rightarrow \quad \sigma \equiv \hat{\sigma}$$

i.e., not i/o equivalent $\Rightarrow \exists$ input in set $\mathcal{U}$ which distinguishes
linear systems (finite-dimensional, continuous-time)

\[ \dot{x} = Ax + bu, \quad x(0) = 0, \quad y = cx \]

\((A \in \mathbb{R}^{n \times n}, \, b \in \mathbb{R}^{n \times 1}, \, c \in \mathbb{R}^{1 \times n})\)

identifiable by any single nonzero input on a nontrivial interval
e.g. constant function (step) or pulse
linear systems (finite-dimensional, continuous-time)

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what about interesting classes of nonlinear systems?
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we consider two different classes of bilinear systems

\( S^I_n := \text{n-dimensional bilinear systems of type I: } f_0 \text{ linear, } f_1 \text{ affine, } x_0 = 0, \ h \text{ linear:} \)

\[
\begin{align*}
\dot{x} &= (A + uN)x + bu, \quad x(0) = 0 \\
y &= cx \\
(A, N, b, c) \quad \text{where } A, N \in \mathbb{R}^{n \times n}, \ b \in \mathbb{R}^{n \times 1}, \ c \in \mathbb{R}^{1 \times n}
\end{align*}
\]

write "\( \sigma^o = (A, N, b, c) \)"
we consider two different classes of bilinear systems
$S^I_n := n$-dimensional bilinear systems of type $I$: $f_0$ linear, $f_1$ affine, $x_0 = 0$, $h$ linear:

$$
\dot{x} = (A + uN)x + bu, \quad x(0) = 0 \\
y = cx
$$

$(A, N, b, c)$ where $A, N \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $c \in \mathbb{R}^{1 \times n}$

write “$\sigma^o = (A, N, b, c)$”

(linear systems: just $N = 0$)
Bilinear systems

we consider two different classes of bilinear systems

$S_n^I := n$-dimensional bilinear systems of type I: $f_0$ linear, $f_1$ affine, $x_0 = 0$, $h$ linear:

\[
\dot{x} = (A + uN)x + bu, \quad x(0) = 0
\]
\[
y = cx
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\[(A, N, b, c) \quad \text{where} \quad A, N \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^{n \times 1}, \quad c \in \mathbb{R}^{1 \times n}
\]

write \( \sigma = (A, N, b, c) \) (linear systems: just $N = 0$)

$S_n^{II} := n$-dimensional bilinear systems of type II: $f_0, f_2, h$ all linear, $x(0)$ allowed nonzero:

\[
\dot{x} = (A + uN)x, \quad x(0) = b
\]
\[
y = cx
\]
Algebraic characterization of equivalence

recall (Isidori, Fliess, 1970s):
no need to test all possible inputs

\( \sigma \) and \( \hat{\sigma} \) (both type I or type II) i/o equivalent iff

\[
cA_{i_1} \ldots A_{i_k} b = \hat{c} \hat{A}_{i_1} \ldots \hat{A}_{i_k} \hat{b}
\]

for all sequences of matrices \( A_j \) picked out of \( A \) and \( N \)
recall (Isidori, Fliess, 1970s):
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\[ \sigma \text{ and } \hat{\sigma} \text{ (both type I or type II) i/o equivalent iff} \]
\[ cA_{i_1} \ldots A_{i_k} b = \hat{c}\hat{A}_{i_1} \ldots \hat{A}_{i_k} \hat{b} \]
for all sequences of matrices \( A_j \) picked out of \( A \) and \( N \)
(Enough check sequences of length \( n + \hat{n} \))
Theorem: \( \exists \) generic subset \( S \subseteq S'_n \) s.t.:

- \( \sigma_o \) and \( \hat\sigma_o \) are i/o equivalent under all constant inputs (steps)
- \( \sigma_o \) and \( \hat\sigma_o \) are not i/o equivalent

Theorem: same for class-II.

("generic" := set of 4-tuples \( S \subseteq \mathbb{R}^{2n+2w} \) full measure & open dense)
Negative results: all steps

**Theorem:** \( \exists \) generic subset \( S \subseteq S_n^l \) s.t.:

\( \forall \sigma^o \in S \ \exists \sigma^o \in S \) so that:

- \( \sigma^o \) and \( \sigma^o \) are i/o equivalent under all constant inputs (steps)
- But \( \sigma^o \) and \( \sigma^o \) are not i/o equivalent

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Theorem: \( \exists \) generic subset \( S \subseteq \mathcal{S}_n^I \) s.t.:

\[ \forall \sigma^o \in S \; \exists \sigma^\hat{o} \in S \text{ so that:} \]

- \( \sigma^o \) and \( \sigma^\hat{o} \) are i/o equivalent under all constant inputs (steps)
- but \( \sigma^o \) and \( \sigma^\hat{o} \) are not i/o equivalent
Negative results: all steps

**Theorem:** $\exists$ generic subset $S \subseteq S^n_1$ s.t.:

$\forall \sigma^o \in S \ \exists \ \hat{\sigma}^o \in S$ so that:

- $\sigma^o$ and $\hat{\sigma}^o$ are i/o equivalent under all constant inputs (steps)
- but $\sigma^o$ and $\hat{\sigma}^o$ are not i/o equivalent

**Theorem:** same for class-II.

("generic" := set of 4-tuples $S \subseteq \mathbb{R}^{2n^2+2n}$ w/full measure & open dense)
Negative results: single pulses

Theorem: \( \forall \tau \geq 0, \alpha \in \mathbb{R}, \exists \text{ generic subset } S \subseteq S_{\text{I}} \text{s.t.}: \forall \sigma_o \in S \exists \hat{\sigma}_o \in S \text{ so that:} \)

- \( \sigma_o \) and \( \hat{\sigma}_o \) are i/o equivalent under the pulse function \( u_{\tau,\alpha}(t) \)
- but \( \sigma_o \) and \( \hat{\sigma}_o \) are not i/o equivalent

Theorem: same for class-II systems.
Negative results: single pulses

Theorem: \( \forall \tau \geq 0, \alpha \in \mathbb{R}, \exists \) generic subset \( S \subseteq S_n \) s.t.:

\[
u_{\tau, \alpha}(t)
\]

**Theorem:** \( \forall \tau \geq 0, \alpha \in \mathbb{R}, \exists \) generic subset \( S \subseteq S_n \) s.t.:
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Theorem: \( \forall \tau \geq 0, \alpha \in \mathbb{R}, \exists \) generic subset \( S \subseteq S_n^I \) s.t.:

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\[ u_{\tau, \alpha}(t) \]

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Negative results: single pulses

**Theorem:** $\forall \tau \geq 0, \alpha \in \mathbb{R}, \exists$ generic subset $S \subseteq S_n^I$ s.t.:

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**Theorem:** \( \forall \tau \geq 0, \alpha \in \mathbb{R}, \ \exists \text{ generic subset } S \subseteq S_n^I \text{ s.t.:} \)

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- but \( \sigma^o \) and \( \widehat{\sigma}^o \) are not i/o equivalent

**Theorem:** same for class-II systems.
for any fixed $\alpha \in \mathbb{R}$, $V_\alpha :=$ set of pulses of magnitude $\alpha$:

$$V_\alpha := \{u_{\tau,\alpha} | \tau \geq 0\}$$

**Theorem:** for each $\alpha \neq 0$, $\exists$ generic $M \subseteq S^l$ s.t.,

for every pair of systems $\sigma^o_1, \sigma^o_2 \in M$,

$$\forall \alpha \in \mathbb{R}, V_\alpha, \quad \sigma^o \equiv \sigma^o_1 \iff \sigma^o \equiv \sigma^o_2$$
Positive results: pulses of fixed amplitude

for any fixed $\alpha \in \mathbb{R}$, $\mathcal{V}_\alpha := \text{set of pulses of magnitude } \alpha$:

$$\mathcal{V}_\alpha := \{u_{\tau,\alpha} | \tau \geq 0\}$$

**Theorem:** for each $\alpha \neq 0$, $\exists$ generic $\mathcal{M} \subseteq S_n^I$ s.t.,

for every pair of systems $\sigma_1^\circ, \sigma_2^\circ \in \mathcal{M}$,

$$\sigma^\circ \equiv \sigma^\circ \iff \mathcal{V}_\alpha \sigma^\circ$$

**Theorem:** same for type-II
Positive results: pulses of fixed amplitude

for any fixed $\alpha \in \mathbb{R}$, $\mathcal{V}_\alpha :=$ set of pulses of magnitude $\alpha$:

$$\mathcal{V}_\alpha := \{u_{\tau,\alpha} \mid \tau \geq 0\}$$

**Theorem:** for each $\alpha \neq 0$, $\exists$ generic $M \subseteq S^I_n$ s.t.,

for every pair of systems $\sigma_1^o, \sigma_2^o \in M$,

$$\sigma^o \equiv \overset{\sim}{\sigma}^o \iff \sigma^o \equiv \overset{\sim}{\sigma}^o$$

**Theorem:** same for type-II

i.e.: set of pulses of amplitude $\alpha$ (and varying length) sufficient
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Sketch of proof of negative results

let $\mathcal{C} :=$ set consisting of all those 4-tuples

$$(Q, N, b_0, c) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$$

which satisfy the following conditions:

(a) $(Q, b_0, c)$ is canonical
(b) $(Q - N, b_0, c)$ is canonical
(c) $e^Q - I$ is invertible
(d) $N \notin \mathcal{B}(T(Q, b_0, c))$

where: for each canonical $\sigma = (A, b, c)$, pick (unique, self-adjoint) $T = T(\sigma)$ such that

$$AT = TA', \ b = Tc', \ cT = b'$$

and for each nonzero $n \times n$ matrix $S$, (“commutator”) proper linear subspace of $\mathbb{R}^{n \times n}$:

$$\mathcal{B}(S) := \{ N \in \mathbb{R}^{n \times n} \mid NS = SN' \}$$
let $\mathcal{X} = \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$, and consider the analytic map $\psi : \mathcal{X} \to \mathcal{X}$ defined by

$$
\psi : (Q, N, b_0, c) \mapsto (Q - N, N, (\rho(Q))^* b_0, c),
$$

where $\rho(Q) = \int_0^1 e^{sQ} ds$, and $\rho(Q)^*$ denotes adjoint matrix of $\rho(Q)$.
let $\mathcal{X} = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n}$, and consider the analytic map $\psi : \mathcal{X} \to \mathcal{X}$ defined by

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where $\rho(Q) = \int_0^1 e^{sQ} ds$, and $\rho(Q)^*$ denotes adjoint matrix of $\rho(Q)$.

**Lemma:** the set $\mathcal{D} = \psi(\mathcal{C})$ is generic.
let $\mathcal{X} = \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$, and consider the analytic map $\psi : \mathcal{X} \to \mathcal{X}$ defined by

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**Lemma:** the set $\mathcal{D} = \psi(\mathcal{C})$ is generic

let $u = u_{\tau, \alpha}$ with $\tau = 1, \alpha = 1$. 
let $X = \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$, and consider the analytic map $\psi : X \rightarrow X$ defined by

$$\psi : (Q, N, b_0, c) \mapsto (Q - N, N, (\rho(Q))^* b_0, c),$$

where $\rho(Q) = \int_0^1 e^{sQ} ds$, and $\rho(Q)^*$ denotes adjoint matrix of $\rho(Q)$.

**Lemma:** the set $D = \psi(C)$ is generic

let $u = u_{\tau, \alpha}$ with $\tau = 1, \alpha = 1$.

**Lemma:** consider systems of type I

$\forall \sigma^o \in D$, $\exists \hat{\sigma}^o \in D$ s.t.:
let $\mathcal{X} = \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$, and consider the analytic map $\psi : \mathcal{X} \rightarrow \mathcal{X}$ defined by

$$\psi : (Q, N, b_0, c) \mapsto (Q - N, N, (\rho(Q))^* b_0, c),$$

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**Lemma:** the set $D = \psi(C)$ is generic

let $u = u_{\tau, \alpha}$ with $\tau = 1, \alpha = 1$.

**Lemma:** consider systems of type I

$\forall \sigma^o \in D, \exists \hat{\sigma}^o \in D$ s.t.:

1. $\sigma^o$ and $\hat{\sigma}^o$ i/o equivalent under the pulse function $u$, but
2. $\sigma^o$ and $\hat{\sigma}^o$ not i/o equivalent
let $\mathcal{X} = \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$, and consider the analytic map $\psi : \mathcal{X} \to \mathcal{X}$ defined by

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$\forall \sigma^o \in \mathcal{D}$, $\exists \widehat{\sigma}^o \in \mathcal{D}$ s.t.:

1. $\sigma^o$ and $\widehat{\sigma}^o$ i/o equivalent under the pulse function $u$, but
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for general $\tau$ and $\alpha$, rescale inputs and time scale.
let $\mathcal{X} = \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n}$, and consider the analytic map $\psi : \mathcal{X} \to \mathcal{X}$ defined by

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\psi : (Q, N, b_0, c) \mapsto (Q - N, N, (\rho(Q))^* b_0, c),
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where $\rho(Q) = \int_0^1 e^{sQ} ds$, and $\rho(Q)^*$ denotes adjoint matrix of $\rho(Q)$

**Lemma:** the set $\mathcal{D} = \psi(\mathcal{C})$ is generic

let $u = u_{\tau, \alpha}$ with $\tau = 1, \alpha = 1$.

**Lemma:** consider systems of type I

$\forall \sigma^0 \in \mathcal{D}, \exists \hat{\sigma}^0 \in \mathcal{D}$ s.t.:

1. $\sigma^0$ and $\hat{\sigma}^0$ i/o equivalent under the pulse function $u$, but
2. $\sigma^0$ and $\hat{\sigma}^0$ not i/o equivalent

for general $\tau$ and $\alpha$, rescale inputs and time scale similarly for type II
Construction of $\hat{\sigma}^o$

given $\sigma^o = (A, N, b, c) \in D$, there exist

$$(Q, N, b_0, c) \in C \quad \text{s/t:} \quad A = Q - N, \; b = [\det(\rho(Q))(\rho(Q))^{-1}]b_0.$$

let $b_1 = \rho(Q)b$

then $b_1 = \det(\rho(Q))b_0$, so:

$$\mathcal{R}(Q, b_1) = \det(\rho(Q))\mathcal{R}(Q, b_0)$$

$$\mathcal{R}(Q - N, b_1) = \det(\rho(Q))\mathcal{R}(Q - N, b_0)$$

since $\det(\rho(Q)) \neq 0$, both $(Q, b_1)$ and $(Q - N, b_1)$ reachable
moreover, it can be shown that:

\[
T(Q, b_1, c) = \det(\rho(Q)) \cdot T(Q, b_0, c),
\]

\[
\downarrow
\]

\[
\mathcal{B}(T(Q, b_1, c)) = \mathcal{B}(T(Q, b_0, c)).
\]

so \((Q, N, b_1, c) \in \mathcal{C}\)

with \(M = TN' T^{-1}\), one has:

- \(M \neq N\), and
- \(ce^{t(Q+\gamma N)} b_1 = ce^{t(Q+\gamma M)} b_1\) for all \(\gamma \in \mathbb{R}\) and \(t \geq 0\)

in particular, for \(\gamma = -1\):

\[
ce^{t(Q-N)} b_1 = ce^{t(Q-M)} b_1 \quad \forall t \geq 0
\]

let \(\hat{\sigma}^o = ((A + N - M), M, b, c) = (Q - M, M, b, c)\)

(compare w/ \(\sigma^o = (Q - N, N, b, c)\))

then: \(\hat{\sigma}^o \in \mathcal{D}\), and \(\sigma^o \equiv \hat{\sigma}^o\), but \(\sigma^o \not\equiv \hat{\sigma}^o\)
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let $\alpha \neq 0$ be given

$\mathcal{M} :=$ be the set of 4-tuples satisfying the following two properties:

1. $(A, b, c)$ is canonical
2. $(A + \alpha N, b)$ is controllable

this is a proper algebraic set, so generic and shown to work
bilinear systems $\sigma \sim \hat{\sigma}$ ("similar" or "internally equivalent")
if $\exists$ change of variables $x = Tz$

s.t. equations of $\sigma$ get transformed into those of $\hat{\sigma}$
bilinear systems $\sigma \sim \hat{\sigma}$ ("similar" or "internally equivalent") if $\exists$ change of variables $x = Tz$

s.t. equations of $\sigma$ get transformed into those of $\hat{\sigma}$

i.e. 4-tuples $(A, N, b, c)$ and $(\hat{A}, \hat{N}, \hat{b}, \hat{c})$ in same $GL(n)$-orbit under similarity action:

same dimension $n$, and $\exists T \in \mathbb{R}^{n \times n}$ invertible s.t.

$$A = T\hat{A}T^{-1}, \quad N = T\hat{N}T^{-1}, \quad b = T\hat{b}, \quad c = \hat{c}T^{-1}$$
suppose that both $\sigma$ and $\hat{\sigma}$ canonical

Lemma. $\sigma \equiv \hat{\sigma} \iff \sigma \sim \hat{\sigma}$

(and similarity is unique)
suppose that both $\sigma$ and $\hat{\sigma}$ canonical

Lemma. $\sigma \equiv \hat{\sigma} \iff \sigma \sim \hat{\sigma}$

(and similarity is unique)

recall: canonical $\sigma$ means span-reachable and observable 4-tuple $(A, N, b, c)$:

- no proper subspace of $\mathbb{R}^n$ contains $b$ and is invariant under $x \mapsto Ax$ and $x \mapsto Nx$
- no proper subspace of $\mathbb{R}^n$ is contained in the nullspace of $x \mapsto cx$ and is invariant under $x \mapsto Ax$ and $x \mapsto Nx$
Sketch of positive result (type I)

pick $\sigma^o_1 = (A_1, N_1, b_1, c_1)$ and $\sigma^o_2 = (A_2, N_2, b_2, c_2)$ in $\mathcal{M}$, s.t. same outputs for each $u \in \mathcal{V}_\alpha$
Sketch of positive result (type I)

pick $\sigma_1^o = (A_1, N_1, b_1, c_1)$ and $\sigma_2^o = (A_2, N_2, b_2, c_2)$ in $M$, s.t. same outputs for each $u \in V_\alpha$

must show that $\sigma_1^o \equiv \sigma_2^o$
Sketch of positive result (type I)

pick \( \sigma^o_1 = (A_1, N_1, b_1, c_1) \) and \( \sigma^o_2 = (A_2, N_2, b_2, c_2) \) in \( \mathcal{M} \), s.t. same outputs for each \( u \in \mathcal{V}_\alpha \)

must show that \( \sigma^o_1 \equiv \sigma^o_2 \)

fix any \( \tau > 0 \)

applying \( u_{\tau, \alpha} \in \mathcal{V}_\alpha \) to the two systems:

\[
\begin{align*}
\dot{x} &= (A_1 + u N_1) x + b_1 u, \quad x(0) = 0, \quad y = c_1 x \\
\dot{z} &= (A_2 + u N_2) z + b_2 u, \quad z(0) = 0, \quad y = c_2 z,
\end{align*}
\]

one has:

\[
c_1 e^{A_1(t-\tau)} x(\tau) = c_2 e^{A_2(t-\tau)} z(\tau) \quad \forall \ t \geq \tau,
\]

(easy:) for generic \( \tau_0 > 0 \), \((A_1, x(\tau_0), c_1)\) & \((A_2, z(\tau_0), c_2)\) canonical

\[
\therefore \exists \ T \in GL(n) \ s.t.
\]

\[
A_2 = T^{-1} A_1 T, \quad z(\tau_0) = T^{-1} x(\tau_0), \quad c_2 = c_1 T
\]
so using $c_2 e^{A_2 s} = c_1 e^{A_1 s} T$ for all $s$, above becomes:

$$c_1 e^{A_1 (t-\tau)} x(\tau) = c_1 e^{A_1 (t-\tau)} Tz(\tau) \quad \forall t \geq \tau$$
so using \( c_2 e^{A_2 s} = c_1 e^{A_1 s} T \) for all \( s \), above becomes:

\[
c_1 e^{A_1(t-\tau)} x(\tau) = c_1 e^{A_1(t-\tau)} T z(\tau) \quad \forall \ t \geq \tau
\]

from observability of \((A_1, c_1)\), it follows that

\[
x(\tau) = T z(\tau) \quad \forall \ \tau > 0
\]

or equivalently:

\[
\int_0^\tau e^{(A_1 + \alpha N_1) s} \, ds \ b_1 = T \int_0^\tau e^{(A_2 + \alpha N_2) s} \, ds \ b_2 \quad \forall \ \tau > 0
\]

taking \( d/d\tau \):

\[
e^{(A_1 + \alpha N_1)_\tau} b_1 = T e^{(A_2 + \alpha N_2)_\tau} b_2
\]

*this is true for all \( \tau \geq 0 \)*

so in particular:

\[
b_1 = Tb_2
\]
on the other hand, taking repeated derivatives in $\tau$ and then setting $\tau = 0$, one obtains:

\[(A_1 + \alpha N_1)^k b_1 = T(A_2 + \alpha N_2)^k b_2 \quad \forall \, k \geq 0\]

which implies $(0 \leq k \leq n - 1)$:

\[\mathcal{R}(A_1 + \alpha N_1, b_1) = T[\mathcal{R}(A_2 + \alpha N_2, b_2)]\]

and $(1 \leq k \leq n)$:

\[(A_1 + \alpha N_1)[\mathcal{R}(A_1 + \alpha N_1, b_1)] = T(A_2 + \alpha N_2)[\mathcal{R}(A_2 + \alpha N_2, b_2)]\]

so

\[(A_1 + \alpha N_1)[\mathcal{R}(A_1 + \alpha N_1, b_1)] = T(A_2 + \alpha N_2) T^{-1}[\mathcal{R}(A_1 + \alpha N_1, b_1)]\]
as $\mathcal{R}(A_1 + \alpha N_1, b_1)$ invertible
(because $(A_1 + \alpha N_1, b_1)$ is controllable), $\Rightarrow$

$$T(A_2 + \alpha N_2) T^{-1} = (A_1 + \alpha N_1)$$

so again follows from above, and the fact that $\alpha \neq 0$,
that $N_2 = T^{-1} N_1 T$

so, the 4-tuples are similar, and the systems are i/o equiv
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A construction

given any analytic function $\kappa(r)$, consider:

\[
\begin{align*}
\dot{x} &= 1 \\
\dot{z} &= z \\
\dot{w} &= (\kappa(x) - u)z \\
y &= w
\end{align*}
\]

analytic system

$\sigma$ and $\hat{\sigma}$ same system, but just different initial states:

$x_0 := (0, 1, 0)$ and $\hat{x}_0 := (0, 0, 0)$

“observables”: $(\{h, L_fh, L_g h, L_f^2 h, L_f L_g h, \ldots\})$ are:

$\{w, z, \kappa(x)z, \kappa'(x)z, \kappa''(x)z, \ldots\}$

$\Rightarrow y(t) \neq \hat{h}(t)$ for some $u$, i.e., $\sigma \neq \hat{\sigma}$

but, $\sigma \equiv \kappa \hat{\sigma}$ \quad (y(t) = \hat{y}(t) \equiv 0)$
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Closing Remarks
**Theorem:** there exists a generic set $\mathcal{U}$ of smooth inputs s/t

$$(\forall u \in \mathcal{U}) \quad \sigma \equiv_u \hat{\sigma} \implies \sigma \equiv \hat{\sigma}$$

i.e.: every $u$ in $\mathcal{U}$ distinguishes any two $\sigma$ and $\hat{\sigma}$

*generic:* contains a countable intersections of open dense sets

seeing for each $T > 0$, $C^\infty[0, \ T]$ w/Whitney topology
Identification by jets

\[ \sigma \not\equiv \hat{\sigma} \text{ if there is some } u \text{ such that } \]
\[ \left. \frac{d^k}{dt^k} \right|_{t=0} y(t, u) \neq \left. \frac{d^k}{dt^k} \right|_{t=0} \hat{y}(t, u) \]

for some \( k \).

**Theorem:** there is a generic set \( \mathcal{W} \subseteq \mathbb{R}^\infty \) such that

for each \( \mu = (\mu_0, \mu_1, \mu_2, \ldots) \) in \( \mathcal{W} \), \( \exists u \) with \( u^{(i)}(0) = \mu_i \) so that for any \( \sigma \) and \( \hat{\sigma} \),

\[ \left. \frac{d^k}{dt^k} \right|_{t=0} y(t, u) = \left. \frac{d^k}{dt^k} \right|_{t=0} \hat{y}(t, u) \quad \forall k \]

\[ \Downarrow \]

\[ \sigma \equiv \hat{\sigma} \]

(generic: containing a countable intersection of open dense subsets, with the product topology on \( \mathbb{R}^\infty \))
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Review: Chen/Fliess encoding of inputs ($m=1$) consider:

$$\dot{C}(t) = (X_0 + \sum_{i=1}^{m} u_i X_i) C(t), \quad C(0) = 1$$

solution exists by Peano-Baker formula:

$$C[u](t) = \sum_{w} V_w[u](t) w$$

$w$: word in \{ $X_0, X_1, \ldots, X_m$ \}

$V_w[u]$: iterated integrals of $u$ defined recursively:

$$V_{\phi}[u](t) = 1$$

$$V_{X_i w}[u](t) = \int_{0}^{t} u_i(s) V_w[u](s) \, ds$$

where $u_0(t) \equiv 1$

e.g.: $V_{X_i}[u](t) = \int_{0}^{t} u_i(s) \, ds$, $V_{X_i X_j}[u](t) = \int_{0}^{t} u_i(s) \int_{0}^{s} u_j(\tau) \, d\tau \, ds$
\[ c = \sum_{w} c(w)w \quad (c(w) \in \mathbb{R}) \]
\[ = c(\phi) + c(X_0)X_0 + c(X_1)X_1 + \ldots c(X_m)X_m \]
\[ + c(X_0X_0)X_0X_0 + c(X_0X_1)X_0X_1 + \ldots \]

\( c \) is convergent if: \( |c(w)| \leq CM^l/l! \quad C, M = \text{const}, \quad l = |w| \)

Fliess operator: \( u \mapsto y = F_c[u] \) defined by:

\[ y(t) = \langle c, C[u](t) \rangle = \sum_{w} c(w) V_w[u](t) \]

\( e.g. (m = 1) : \quad y(t) = c(\phi) + c(X_0) \int_{0}^{t} ds + c(1) \int_{0}^{t} u(s) \, ds \]
\[ + c(X_0X_1) \int_{0}^{t} \int_{0}^{s_1} u(s_2) \, ds_2 \, ds_1 \]
\[ + c(X_1X_1) \int_{0}^{t} u(s_1) \int_{0}^{s_1} u(s_2) \, ds_2 \, ds_1 + \ldots \]

\( F_c[u] \) well defined on [0, \( T \)), some \( T > 0 \), if \( c \) convergent
Fact: i/o behaviors of initialized state space system are defined by appropriate $\sigma = F_c$, i.e., 

\[ \text{for each } u \in \Omega, \ y(t) = F_c[u](t) \]

in fact,

\[ c(X_{i_1}X_{i_2} \ldots X_{i_r}) = L_{g_{i_r}} \ldots L_{g_{i_1}} h(x_0) \]

($g_0 := f$)

so enough to show $\exists$ generic set $\mathcal{W}$ in $\mathbb{R}^\infty$ s/t for each $\mu = (\mu_0, \mu_1 \ldots)$ in $\mathcal{W}$, there is $u$ with $u^{(i)}(0) = \mu_i$ such that:

\[ \text{for any } c, \hat{c}, \]

\[ \frac{d^k}{dt^k} \bigg|_{t=0} F_c[u](t) = \frac{d^k}{dt^k} \bigg|_{t=0} F_{\hat{c}}[u](t) \quad \forall k \]

\[ \Downarrow \]

\[ c = \hat{c} \]
to prove result on series, enough to deal with $c$ and a specific series $\widehat{c}$ – the zero series
infinite jet $\mu = (\mu_0, \mu_1, \ldots)$ said to be *universal* for a set $S$ of series if $\exists u$ with $u^{(i)}(0) = \mu_i$ s.t.

$$F_c[u] \neq 0 \text{ for each } c \in S$$

**Definition:** family $S$ of Fliess series is:

1. *equiconvergent* if $\exists r, M > 0$ s.t.

   $$|c(w)| \leq M r^{|w|} (|w|)! \quad \forall c \in S$$

2. *compact* if compact in weak topology

(see as a family of sequences indexed by words in $X_i$'s)
Lemma: $S$ compact, equiconvergent, $0 \notin S$

$\Rightarrow \{\text{infinite jets universal for } S\}$ open dense.

Corollary:

*the set of uniformly universal jets is generic*

because:

set of nonzero conv Fliess series $= \bigcup_{w,k} S_{w,k}$

where $S_{w,k} = \text{all series such that}$

$$|c(w)| \geq 1/k$$

and

$$|c(w)| \leq k^l/l! \quad \text{where } l = |w|$$
Lemma: Let $S$ compact, equiconv, and assume we know $\exists$ at least one $u$ which is universal for $S$, i.e.:

$$F_c[u](t) \not\equiv 0 \quad \forall c \in S$$

now let $\mu$ be any (arbitrary) finite jet
then, $\exists \nu$, finite extension of $\mu$, universal for $S$

*main idea*: for any given finite $\mu$,

$\exists$ analytic inputs $v_j$ with $v_j^{(i)}(0) = \mu_i$ s/t $v_j \to u$ (in $L_1$ topology)

together with compactness and equiconv of $S$,

$\exists j_0$ s.t. $v_{j_0}$ universal for $S$
given $S$: compact, equiconvergent, $0 \not\in S$

*want*: univ jets for $S$ is open, dense

openness follows from compactness of $S$
let $\mu$ be a jet, and let $\mathcal{W}$ be any nbhd of $\mu$

$$\mathcal{W} = \mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2 \times \ldots \times \mathcal{W}_r \times \mathbb{R}^m \times \mathbb{R}^m \times \ldots$$

let $\mu^r :=$ restriction of $\mu$ to first $r$ terms

$0 \notin S \Rightarrow \forall c \in S, \exists$ a jet $\nu$ “good” for $c$

compactness of $S \Rightarrow \exists \mathcal{V}_1, \mathcal{V}_2, \ldots \mathcal{V}_s, \exists$:

- $\{\mathcal{V}_i\}$ covers $S$
- each $\mathcal{V}_i$ has a (finite) univ jet

tech lemma $\Rightarrow$:

$\exists \nu_1$, extension of $\mu^r$, univ for $\mathcal{V}_1$

$\exists \nu_2$, extension of $\nu_1$, univ for $\mathcal{V}_1 \cup \mathcal{V}_2$

Repeat $s$
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Many open problems...

- special classes: pulses with varying amplitudes?
- cascades of bilinear?
- other classes of systems?
- unif univ theorems for $C^\infty$ classes with appropriate transversality assumptions?

References: