Maximal Order Reduction and Supremal (A, B)-Invariant and Controllability Subspaces

P. J. ANTSAKLIS, MEMBER, IEEE

Abstract—**Given the system $\{A, B, C, E\}$ the supremal (A, B)-invariant and controllability subspaces are studied and their dimensions are explicitly determined as functions of the number of zeros and the degree of the determinant of the interactor. This is done by solving the problem of the maximal order reduction via linear state feedback.**

I. Introduction

The geometric approach [9] has been used successfully in recent years in the analysis and synthesis of linear, multivariable, time-invariant systems. Among the key concepts of this approach are the concepts of the supremal (A, B)-invariant and controllability subspaces contained in a given subspace (often, the kernel of C), denoted by V^* and R^*, respectively. R^* is a subspace of V^* and it has been shown by a number of authors that $\dim V^* = q + \dim R^*$ where q is the number of zeros of the system considered. Note, however, that the actual dimension of R^* and consequently of V^* are unknown. These dimensions are important since they have a critical effect on the dynamic controller structure in some synthesis techniques, as for example in decoupling.

In this paper the dimensions of V^* and R^* of a completely controllable system are explicitly determined in terms of the degree of the determinant of the "interactor" introduced in [6]. Note that the relation between the interactor and the dimensions of V^* and R^* is not surprising in view of the importance of the interactor in the decoupling problem (Section III, Remark) and generally in the problem of system equivalence under dynamic compensation [6].

Given the controllable and observable system $\{A, B, C, E\}$, the dimension of V^* is determined by solving an equivalent problem, namely the problem of reducing the order[1] of the system via linear state feedback (LSF) by making unobservable the largest possible number of closed-loop poles (maximal order reduction problem). The equivalence of these two problems is established in Section II by using a well-known relation between V^* and an appropriate unobservable subspace (Lemma 1). In Section III the maximal order reduction problem is solved (Theorem 8) and the stability of the closed-loop system is examined (Lemma 9 and 10). These results are used in Section IV to study the subspaces V^* and R^* and determine their dimensions as functions of the number of zeros and the degree of the determinant of the interactor.

The advantage of studying the maximal order reduction problem instead of dealing directly with V^* and R^* is that one can avoid using abstract algebraic concepts in the proofs and talk simply about more familiar concepts as unobservable poles and their cancellation in the transfer matrix. Note that the maximal order reduction problem is of interest in its own right because, for example, of its relation to the decoupling problem (Section III, Remark); it can also be of interest in modeling where the dimension of the model can be reduced via LSF (or its equivalent dynamic compensation) by omitting the unobservable part which cancels out in the closed-loop transfer matrix.

It should be pointed out that the interactor is related to the transfer matrix or polynomial matrix representation of a system. Therefore, this paper establishes some of the key relations between the different approaches of studying linear, time-invariant systems, namely, between 1) the transfer matrix and polynomial matrix approach and 2) the geometric approach. Note that the connections between the polynomial matrix and geometric approach have recently attracted considerable interest and a variety of results, which differ from the present work in both nature and methodology, have been reported [13], [14].

Finally, note that the results of this paper extend similar results [12] developed under the assumptions: $p \leqslant m$, $E = 0$ and that the "decoupling condition" does hold.

Manuscript received March 5, 1979; revised July 30, 1979. Paper recommended by B. Francis, Chairman of the Linear Systems Committee.

The author is with the Department of Computing and Control, Imperial College of Science and Technology, London, England.

[1]This is taken to be equal to the McMillan degree of the transfer matrix.

II. Preliminaries and Problem Formulation

Assume that an nth order controllable and observable system $\dot{x} = Ax + Bu$, $y = Cx + Eu$ is given and let $T(s)$ be its $p \times m$ transfer matrix, which is assumed to be of full rank. Let $R(s)P^{-1}(s)$ be a right prime factorization of $T(s)$ where $R(s)$ and $P(s)$ are $p \times m$ and $m \times m$ polynomial matrices, respectively, with $P(s)$ column proper [1], i.e.,

$$T(s) = C(sI - A)^{-1}B + E = R(s)P^{-1}(s) \tag{1}$$

with (R, P) relatively right prime (rrp) polynomial matrices. The n poles of the given system (1) are the n zeros of $\det(sI - A) = \det P(s)$,[2] while the q zeros of (1) are those z_i (multiplicity included) for which [3]

$$\operatorname{rank}\begin{bmatrix} z_i - A & B \\ -C & E \end{bmatrix} < n + \min(p, m).$$

It can be shown using [2] that the above q zeros can be equivalently defined from $R(s)$ as follows.

The zeros of (1) are 1) $(p \geqslant m)$ the zeros of $\det G_R(s)$ where G_R is a greatest common right divisor (gcrd) of the p rows of $R(s)$ and 2) $(p < m)$ the zeros of $\det G_L(s)$, where G_L is a greatest common left divisor (gcld) of the m columns of $R(s)$. Note that when $p = m$, $G_R(s) = G_L(s) = R(s)$ and the zeros of (1) are simply the zeros of $\det R(s)$.

Assume that system (1) is compensated by a linear state feedback (LSF) control law of the form

$$u = Fx + v$$

where F is the state feedback matrix and v an external input. The state-space description of the closed-loop system is

$$\dot{x} = (A + BF)x + Bv; \quad y = (C + EF)x + Ev. \tag{2}$$

A subspace V of the state-space is an *output-nulling invariant subspace* [15] if and only if

$$\begin{bmatrix} A \\ C \end{bmatrix} V \subset \begin{bmatrix} I \\ 0 \end{bmatrix} V + \operatorname{Im}\begin{bmatrix} B \\ E \end{bmatrix}$$

or equivalently, if and only if for some F

$$(A + BF)V \subset V \subset \ker(C + EF) \tag{3}$$

where Im and ker are the image and the kernel of linear maps. Let V^* be the supremal output-nulling invariant subspace [15]. If $E = 0$, i.e., there is no direct feedthrough in (1), V^* is exactly the supremal (A, B)-invariant subspace in the $\ker C$ [9]. In the following, V^* will be referred to simply as the *supremal (A, B)-invariant subspace*.

Let O_F be the observability matrix of (2); then $\ker O_F = \cap_{i=0}^{n-1} \ker[(C + EF)(A + BF)^i]$ is the unobservable subspace of (2). Let M^* be the maximal unobservable subspace and assume that, given V^*, F_m is a matrix F which satisfies (3) with $V = V^*$. Then we have the following.

Lemma 1:

$$V^* = \ker O_{F_m} = M^*.$$

Proof: If $x \in V^*$, then $(C + EF_m)(A + BF_m)^i x = 0$, $i = 0, 1, \cdots$, i.e., $x \in \ker O_{F_m}$. Thus $V^* \subset \ker O_{F_m}$. Ker O_F is an (A, B)-invariant subspace in $\ker(C + EF)$ for all F. Since V^* is the supremal such element, $\ker O_F \subset V^*$ for all F. For $F = F_m$, $\ker O_{F_m} \subset V^*$ which implies that $V^* = \ker O_{F_m}$.

[2]Generally, $\det(sI - A) = a \cdot \det P(s)$ with a a nonzero scalar. Without loss of generality it is assumed that $a = 1$.

Note that $\ker O_F \subset V^* = \ker O_{F_m}$ for all F; that is, $\ker O_{F_m}$ is the maximal unobservable subspace M^*. Q.E.D.

Lemma 1, which is a simple extension ($E \neq 0$) of a well known result of the geometric approach [9], plays a central role in this paper.

Let $T_F(s)$ be the transfer matrix of (2); then

$$T_F(s) = (C+EF)(sI-A-BF)^{-1}B + E = R(s)P_F^{-1}(s) \tag{4}$$

where the nth degree polynomial $\det(sI-A-BF)$ equals $\det P_F(s)$ with $P_F(s) \triangleq P(s) - F(s)$; $F(s)$ is a column proper polynomial matrix, which depends on F, with column degrees strictly less than the column degrees of $P(s)$ [1].

The closed-loop system (2) is controllable for all F but it is not generally observable. The unobservable poles are canceled out in $T_F(s)$, thus causing a reduction in system order from n to n_F, where n_F is the McMillan degree of $T_F(s)$, i.e., the order of a minimal realization of $T_F(s)$. If $G_F(s)$ is a gcrd of $R(s)$ and $P_F(s)$, the zeros of $\det G_F(s)$ are the unobservable poles of (2). Furthermore, note that the number of unobservable poles equals the dimension of the unobservable subspace of (2) which is $\dim \ker O_F$ [3]. Therefore,

$$\dim \ker O_F = \text{degree}(\det G_F) = n - n_F. \tag{5}$$

In view of Lemma 1, (5) implies that

$$\dim V^* = \dim \ker O_{F_m} = n - n_{F_m}.$$

That is, $\dim V^*$ equals the largest possible number of unobservable poles of (2)[3] and it is also equal to the maximal order reduction possible using LSF; furthermore, the LSF matrices which cause maximal order reduction satisfy (3) for $V = V^*$.

The maximal order reduction problem will be solved in the following section to determine $\dim V^*$. The role of zeros of system (1) will emerge from the constructive proofs and the dimension of the largest controllability subspace R^* contained in V^* will be explicitly determined.

Note that although the maximal order reduction problem is introduced and solved here to explain concepts used in the analysis and synthesis of linear systems (geometric approach), the derived results can also be used to simplify system models by reducing their order via LSF (or the equivalent dynamic compensation). The LSF matrices F which achieve this are the matrices F_m which satisfy (3) for $V = V^*$.

III. Maximal Order Reduction via State Feedback

It will be shown in this section that if all the (q) zeros of the given system are canceled out (by making q of the poles of (1) unobservable), then maximal order reduction is achieved only when $p \geq m$. In the case when $p < m$, the order n can be reduced by $q+k$ where k is a nonnegative integer defined below. Thus, the two cases $p \geq m$ and $p < m$ will be studied separately. Note that in the following, F_m denotes an LSF matrix F which achieves maximal order reduction.

A. $p \geq m$

Lemma 2: If $p \geq m$, the maximal order reduction via LSF is q, the number of zeros.

Proof: It was shown that the reduction in order, caused by an LSF matrix F, is degree ($\det G_F$) [see (5)], where G_F is a gcrd of R and P_F. Note that G_F is a right divisor of G_R for all F where G_R is a gcrd of all the (p) rows of $R(s)$. This implies that $\det G_F$ is a factor of $\det G_R$ for all F and consequently, in view of the definition of zeros, it has as its zeros some of the zeros of (1). It is therefore clear that the order reduction is achieved by just canceling zeros of the system which implies that maximal order reduction can be achieved by choosing an F to cancel (if possible) all q zeros of (1).

It has now been shown that such an LSF matrix F exists (see among others [4] and [5]). Therefore, the maximal order reduction is

$$q = n - n_{F_m} \tag{6}$$

as claimed. Q.E.D.

B. $p < m$

This is a considerably more complicated case than $p \geq m$. Note that here the relation between G_F, the gcrd of R and P_F, and G_L, the gcld of the p columns of $R(s)$, is not clear; consequently, the above technique used in Lemma 2 which relates the largest possible number of unobservable poles to the number of zeros of (1) cannot be directly applied. The maximal order reduction will be found, in this case, using the "interactor" introduced in [6]. In particular, since $T(s)$ of (1) has full rank, there exists a unique $p \times p$ polynomial matrix $X_T(s)$,[4] called the interactor, such that

$$\lim_{s\to\infty} X_T(s)T(s) = K_T; \quad \text{rank}\, K_T = p \tag{7}$$

where $X_T(s) = H_T(s)\,\text{diag}(s^{f_1}, \cdots, s^{f_p})$ with $H_T(s)$ being a $p \times p$ polynomial matrix in lower left triangular form with 1s on the diagonal, and K_T being a real $p \times m$ matrix. Note that, as it can be easily seen from the construction of $X_T(s)$ in [6],

$$d_T \triangleq \text{degree}(\det X_T(s)) \leq n \tag{8}$$

where n is the order of the system (1), or equivalently, the McMillan degree of $T(s)$ of (1). Observe that d_T, which plays an important role in the following, can be easily found by inspection from $X_T(s)$. In particular, note that

$$d_T = \sum_{i=1}^{p} f_i.$$

The following lemma is now in order.

Lemma 3: $n_F \geq d_T$ for all F.

Proof: Let $X_{T_F}(s)$ be the interactor of $T_F(s)$ of (4). Then, in view of (8),

$$n_F \geq d_{T_F} \tag{9}$$

where n_F the McMillan degree of $T_F(s)$. From relation (4)

$$T_F(s) = R(s)P_F^{-1}(s) = (R(s)P^{-1}(s))(P(s)P_F^{-1}(s)) = T(s)T_{ec}(s) \tag{10}$$

where $T_{ec} \triangleq PP_F^{-1}$ is an equivalent to state feedback feedforward compensator [1], [7]. Note that $T_{ec}(s)$ is actually a solution to a model matching problem where $T_F(s)$ is considered to be the model. In view now of [6, Theorem 4.5] and the fact that $T_{ec}(s)$ is proper [1], $X_T(s)X_{T_F}^{-1}(s)$ is proper, which directly implies that

$$d_{T_F} \geq d_T. \tag{11}$$

Comparing (9) and (11), the desired relation

$$n_F \geq d_T \tag{12}$$

is derived. Q.E.D.

Lemma 3 shows that d_T ($= \sum_{i=1}^{p} f_i$) is a lower bound to the McMillan degree of the closed-loop transfer matrix. In the following (Lemma 5) it will be shown that there exists an LSF matrix F_m which achieves this lower bound. Before this can be done Lemma 4 must be shown.

Lemma 4: A realization of $\hat{T}(s) \triangleq X_T(s)T(s)$ is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \hat{y} &= \hat{C}x + K_T u. \end{aligned} \tag{13}$$

Proof: First note that $\lim_{s\to\infty} \hat{T}(s) = K_T$ [see (7)], i.e., $\hat{T}$ is a proper transfer matrix, and let $\hat{y} \triangleq \hat{T}u$. Assume now that $\{A^c, B^c, C^c, E\}$ is an equivalent to $\{A, B, C, E\}$ representation of (1) in controllable compan-

[3] $\dim V^*$ is also equal to the largest possible number of output decoupling zeros [3] of (2).

[4] The symbol $\xi_T(s)$ is used in [6] instead of $X_T(s)$.

ion form (Q the equivalence transformation matrix). Let also $T=R_cP_c^{-1}$ be an rrp factorization of T where R_c and P_c are polynomial matrices related to $\{A^c,B^c,C^c,E\}$ via the structure theorem of [1], i.e., $P_c(s)$ depends only on the controllability indices d_i, $i=1,\cdots,m$, of (1) and the nontrivial parts of A^c, B^c, while $R_c(s)=C^cS(s)+EP_c(s)$ where $S(s)=\operatorname{diag}(e_1,e_2,\cdots,e_m)$ with $e_i\triangleq[1,s,\cdots,s^{d_i-1}]$.[5]

Note that $\hat{T}=X_TT=X_TR_cP_c^{-1}=\hat{R}_cP_c^{-1}$ with $\hat{R}_c\triangleq X_TR_c$ which directly implies that a differential operator realization of $\hat{T}$ in controllable form [1] is $P_c(D)z(t)=u(t)$; $\hat{y}(t)=\hat{R}_c(D)z(t)$ where $D\triangleq d/dt$ and $z(t)$ is a "partial state." Since now $\hat{T}$ is a proper transfer matrix and P_c a column proper polynomial matrix, the column degrees of $\hat{R}_c$ are less than or equal to the column degrees (d_i) of P_c. In view of the structure theorem, $\hat{R}_c$ thus be written as $\hat{R}_c(s)=\hat{C}^cS(s)+K_TP_c(s)$ where $\hat{C}^c$ is an appropriate real matrix; this in turn implies that $\dot{\hat{x}}^c=A^c\hat{x}^c+B^cu$, $\hat{y}=\hat{C}^c\hat{x}^c+K_Tu$ is a realization of $\hat{T}$ equivalent to the above differential operator realization. If now the equivalence transformation matrix Q is used (which depends only on A and B) it is clear that (13) is a realization of $\hat{T}$ ($\hat{C}=\hat{C}^cQ$). Observe also that the state x of (13) is the state x of (1) since $\hat{x}^c(s)=S(s)z(s)=S(s)P_c^{-1}(s)u(s)=(sI-A^c)^{-1}B^cu(s)=Q(sI-A)^{-1}Bu(s)=Qx(s)$ [1]. Q.E.D.

Note that Lemma 4 can also be shown using the state-space approach of [8].

Lemma 5: There exists an LSF matrix F_m such that

$$n_{F_m}=d_T. \tag{14}$$

Proof: Let $\hat{T}(s)\triangleq X_T(s)T(s)$ and note that in view of Lemma 4, (13) is a realization of $\hat{T}$. If now the LSF law $u=Fx+v$ is applied to (13), then

$$\hat{y}=\hat{T}_Fv=\left[(\hat{C}+K_TF)(sI-A-BF)^{-1}B+K_T\right]v$$

which in view of the relation $\hat{y}=\hat{T}u=X_Ty=X_TT_Fv$ implies that

$$T_F=X_T^{-1}\hat{T}_F=X_T^{-1}\left[(\hat{C}+K_TF)(sI-A-BF)^{-1}B+K_T\right]. \tag{15}$$

If now the LSF matrix F_m is chosen to satisfy

$$\hat{C}+K_TF_m=0, \tag{16}$$

then $T_{F_m}(s)=X_T^{-1}(s)K_T$, a transfer matrix of McMillan degree equal to $d_T=\text{degree }(\det X_T(s))$ (this is because $X_T(s)$ and K_T are rlp matrices in view of Lemma 3). Finally, note that there always exists a (nonunique) solution F_m to (16) since $\operatorname{rank}K_T=p$. Q.E.D.

Lemmas 3 and 5 clearly show that the lowest McMillan degree of the closed-loop transfer matrix T_F of (4) is d_T. This implies that the maximal order reduction in this case is $n-d_T$, as the following shows.

Corollary 6: If $p<m$, the maximal order reduction via LSF is $n-d_T$.

It was shown in the proof of Lemma 2 that if $p\geq m$, the maximal order reduction is achieved by making q of the closed-loop poles equal to the values of the q zeros of the given system and it is now of interest to determine what values the unobservable poles of the closed-loop system must take when $p<m$. First note that there is a relation between q and d_T. In particular, it can be easily seen from the construction of $X_T(s)$ in [6] that

$$d_T+q\leq n. \tag{17}$$

Let k be a nonnegative integer such that

$$n-d_T=q+k. \tag{18}$$

In view of Corollary 6 it is now clear that the maximal order reduction is $n-n_{F_m}=q+k$ which is a number generally larger than the number of zeros q. It is shown in the following lemma that whenever the maximal order reduction is achieved via an LSF matrix F_m all the zeros of the system are canceled out in the closed-loop transfer matrix $T_{F_m}(s)$.

[5] It is assumed that $\operatorname{rank}B=m$. If $\operatorname{rank}B<m$ the above proof is still valid with slight modification.

Lemma 7: If the maximal order reduction is achieved via an LSF matrix F_m, the closed-loop transfer matrix T_{F_m} does not have any zeros.

Proof: It was shown in the proof of Lemma 2 that for $p\geq m$ the maximal order reduction (q) is achieved by canceling out all the q zeros of the system, i.e., T_F does not have any zeros. If $p<m$, then in view of (17)

$$d_{T_{F_m}}+q_{F_m}\leq n_{F_m}$$

where $d_{T_{F_m}}=\text{degree}(\det X_{T_{F_m}}(s))$ and $X_{T_{F_m}}$, q_{F_m}, and n_{F_m} are the interactor, the number of zeros, and the McMillan degree of T_{F_m}, respectively. Note however that Lemma 5 and relations (9) and (11) imply that $n_{F_m}=d_{T_{F_m}}$ ($=d_T$) which in view of the above inequality, implies that $q_{F_m}=0$, i.e., T_{F_m} does not have any zeros. Q.E.D.

Remark: The LSF matrix of Lemma 7 is any member of the class of the LSF matrices which achieve maximal order reduction. Note that if F_m is chosen to satisfy (16), then $T_{F_m}=X_T^{-1}(s)K_T$, which, clearly, has no zeros, i.e., Lemma 7 is satisfied.

It was shown in Lemma 7 that in the case when $p<m$, q out of the $q+k$ unobservable poles of the closed-loop system have values equal to the q zeros of the system. It will be shown in the following theorem, which is the main result of this section, that the remaining k unobservable poles can be arbitrarily assigned.

Theorem 8: Given system (1), the maximal order reduction via LSF $u=Fx+v$ is $q+k$ where q is the number of zeros of (1) and k is 1) zero when $p\geq m$ and 2) $n-q-d_T$ when $p<m$. Furthermore, this reduction is achieved by making $q+k$ closed-loop poles unobservable; q of these poles have always values equal to the zeros of (1) while the remaining k poles can be arbitrarily assigned.

Proof: Lemma 2 and Corollary 6, together with (18) prove the first part of the theorem. The fact that the maximal order reduction is equal to the largest possible number of poles which can be made unobservable via LSF is shown in Section II. Lemma 7 shows that q of the unobservable poles are equal to the zeros of the open-loop system. It remains to show that if $p<m$, k of the closed-loop unobservable poles can be arbitrarily assigned. This will be shown in a constructive way using the algorithm used in [7] to obtain stable proper inverses of a given transfer matrix. Assume that the realization (13), $\{A,B,\hat{C},K_T\}$, of $\hat{T}(s)=X_T(s)T(s)$ is given. There exists an $m\times m$ nonsingular matrix M such that $K_TM=[K_p\ \vdots\ 0]$ where K_p is a $p\times p$ nonsingular matrix ($\operatorname{rank}K_T=p<m$). Let $[\tilde{B}_1,\tilde{B}_2]\triangleq BM$ and assume that $\tilde{F}_2$ is an $(m-p)\times n$ LSF matrix which arbitrarily assigns the k controllable poles of $\{A-\tilde{B}_1K_p^{-1}\hat{C},\tilde{B}_2\}$. The existence of such an $\tilde{F}_2$ can be explained as follows: it was shown in [7, Lemma 4] that the uncontrollable poles of $\{A-\tilde{B}_1K_p^{-1}\hat{C},\tilde{B}_2\}$ are exactly equal to the zeros of $\{A,B,\hat{C},K_T\}$ which is a realization of $\hat{T}=X_TT$; in view of our definition of zeros it is clear that the zeros of $\{A,B,\hat{C},K_T\}$ are the zeros of system (1) together with the zeros of $\det X_T(s)$, i.e., they are $q+d_T$. This implies that the only eigenvalues of $A-\tilde{B}_1K_p^{-1}\hat{C}+\tilde{B}_2\tilde{F}_2$ which can be arbitrarily assigned using $\tilde{F}_2$ are the $n-q-d_T=k$ controllable poles of $\{A-\tilde{B}_1K_p^{-1}\hat{C},\tilde{B}_2\}$. It is now claimed that

$$F_m=M\begin{bmatrix}-K_p^{-1}\hat{C}\\ \tilde{F}_2\end{bmatrix}$$

is the desired feedback matrix. Note first that F_m satisfies (16) since

$$\hat{C}+K_TMM^{-1}F_m=\hat{C}+[K_p\ \vdots\ 0]\begin{bmatrix}-K_p^{-1}\hat{C}\\ \tilde{F}_2\end{bmatrix}=0$$

for any $\tilde{F}_2$. Furthermore, $A+BF_m=A+BMM^{-1}F_m=(A-\tilde{B}_1K_p^{-1}\hat{C})+\tilde{B}_2\tilde{F}_2$. That is, the nonuniqueness of the solution of (16) was used to arbitrarily assign k of the eigenvalues of $A+BF_m$; the remaining $n-k=q+d_T$ eigenvalues are equal to the q zeros of $T(s)$ and the d_T zeros of $\det X_T(s)$. In view now of (4) and the proof of Lemma 5

$$T_{F_m}(s)=(C+EF_m)(sI-A-BF_m)^{-1}B+E=X_T^{-1}(s)K_T \tag{19}$$

which in view of the above clearly implies that out of the $q+k$ unobservable poles of the closed-loop system (which cancel out) q are equal to the zeros of (1) and k are arbitrarily assignable, while the remaining $n-q-k$ eigenvalues of $A+BF_m$, (which appear in $T_{F_m}=X_T^{-1}(s)K_T$) are the zeros of $\det X_T(s)$. Q.E.D.

The following observation should be made at this point. The case $p=m$ was studied in the above together with $p>m$. Note, however, that it could have been studied together with $p<m$ since all the results developed for $p<m$ are valid for $p=m$ as well. In particular, for $p=m$ (17) becomes an equality [6], i.e., $d_T+q=n$ which in view of (18), implies that $k=0$. Furthermore, note that in the proof of Theorem 8 (16) has a unique solution F_m for $p=m$ ($\tilde{F}_2=0$) and the eigenvalues of $A+BF_m$ are in this case fixed and equal to the d_T+q zeros of $\hat{T}$, i.e., there are no arbitrarily assignable unobservable poles ($k=0$).

The following two lemmas deal with questions related to the stability of the closed-loop system. Lemma 9 deals with the arbitrary assignment of the poles of the closed-loop transfer matrix $T_{F_m}(s)$, while in Lemma 10, the problem of maximal order reduction, when only the stable zeros are allowed to cancel, is studied.

Lemma 9: When maximal order reduction is achieved via LSF, the poles of the closed-loop transfer matrix can be arbitrarily chosen.

Proof: Let the maximal order reduction be achieved by an LSF matrix F_m. Clearly, the $n-(q+k)$ observable poles of the closed-loop system $\dot{x}=(A+BF_m)x+Bv$, $y=(C+EF_m)x+Ev$ are the poles of $T_{F_m}(s)$. Let the first $q+k$ columns of the equivalence transformation matrix Q be a basis of the unobservable subspace of the closed loop system,[6] then

$$\left\{\begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [0,C_2], E\right\}$$

is the new equivalent representation where (A_2,B_2) is controllable and (A_2,C_2) is the observable part of the closed-loop system. The LSF matrix $[0,F_2]$ can now be used to arbitrarily assign the $n-(q+k)$ eigenvalues of $A_2+B_2F_2$ which implies that $F=F_m+[0,F_2]Q^{-1}$ is an LSF matrix which achieves maximal order reduction and, at the same time, arbitrary assignment of the poles of the closed-loop transfer matrix. Q.E.D.

Lemma 7 shows that whenever maximal order reduction is achieved all the q zeros of (1) cancel out in the closed-loop transfer matrix. Note, however, that in practice, only stable zeros are allowed to cancel in order to avoid unstable behavior of the system. The following lemma studies the problem of maximal order reduction when only q_s ($<q$) zeros of (1) are allowed to cancel.

Lemma 10: If only the stable (q_s) zeros of (1) are allowed to cancel, the maximal order reduction is q_s+k where k is 1) zero when $p>m$ and 2) $n-q-d_T$ when $p<m$. This reduction is achieved by making q_s+k closed-loop poles unobservable; q_s of these poles have values equal to the q_s stable zeros of (1) while the remaining k poles can be arbitrarily assigned.

Proof: First, the transfer matrix $T_s(s)$ is derived from $T(s)=R(s)P^{-1}(s)$ as follows. Let $\bar{S}(s)$ be the Smith form of $R(s)$ [3], i.e., $R(s)=U_1(s)\bar{S}(s)U_2(s)$ where U_1, U_2 are unimodular matrices and write $\bar{S}(s)=\bar{S}_1(s)\bar{S}_2(s)$ where $\bar{S}_1(s)$ is a $p\times p$ diagonal polynomial matrix with diagonal entries all the factors of the diagonal of $\bar{S}(s)$ which involve the $q-q_s$ (unstable) zeros[7] and $1s$ everywhere else. Define $T_s(s)\triangleq\bar{S}_2(s)U_2(s)P^{-1}(s)$ and observe that the zeros of $T_s(s)$ are exactly the q_s zeros of $T(s)$. If $X_{T_s}(s)$ is the interactor of $T_s(s)$ ($\lim_{s\to\infty}X_{T_s}(s)T_s(s)=K_{T_s}$) $\{A,B,\hat{C}_sK_{T_s}\}$ is a realization of $X_{T_s}(s)T_s(s)$ (see Lemma 4) where $\hat{C}_s$ an appropriate real matrix. If now the LSF law $u=Fx+v$ is applied to $\{A,B,\hat{C}_s,K_{T_s}\}$ and Theorem 8 is used, the maximal order reduction of the system $T_s(s)$ via an LSF matrix $\bar{F}_m$ will be q_s+k_s where $k_s=0$ for $p>m$ and $k_s=n-q_s-d_{T_s}$ for $p<m$. If $\bar{F}_m$ is applied to (1), the closed-loop transfer matrix for $p<m$ will be $T_{\bar{F}_m}=(U_1\bar{S}_1)X_{T_s}^{-1}K_{T_s}$. (Lemma 9 can be employed, if necessary, to avoid unstable cancellations between $U_1\bar{S}_1$ and X_{T_s}). Using the fact that $X_TT=(X_TU_1\bar{S}_1)T_s$ and the special structure of $\bar{S}_1$, it can be shown that $d_T=d_{T_s}+q-q_s$ which directly implies that $k_s=k$ of Theorem 8, i.e., there are again $k=n-q-d_T$ arbitrarily assignable unobservable poles. The reduction in order, therefore, is $n-d_{T_s}=q_s+k_s=q_s+k$. This is the maximal order reduction in this case as it can be intuitively seen; it can also be formally shown, if a technique similar to the one used to show that $n-d_T$ (of Theorem 8) is the maximal order reduction, is used. Q.E.D.

Remark: The computational difficulties associated with the evaluation of $X_T(s)$ and the calculation of the LSF matrix F_m which achieves maximal order reduction ($p<m$) are greatly reduced when $X_T(s)=\operatorname{diag}(s^{f_1},\cdots,s^{f_p})$, i.e.,

$$\lim_{s\to\infty}\operatorname{diag}(s^{f_1},\cdots,s^{f_p})T(s)=K_T;\qquad \operatorname{rank}K_T=p. \tag{20}$$

Note that if (20) is true, then system (1) can be diagonally decoupled via an LSF control law of the form $u=Fx+Gv$ [1]. In this case f_i $i=1,\cdots,p$ are the "decoupling indices" which can also be found from the state representation $\{A,B,C,E\}$ of (1) as follows [1], [8]: if

$$\begin{aligned} M[0]_i &\triangleq E_i \\ M[j]_i &\triangleq C_iA^{j-1}B \\ N[j]_i &\triangleq C_iA^{j-1} \end{aligned} \tag{21}$$

where $i=1,2,\cdots,p$, $j=1,2,\cdots$ and E_i, C_i denote the ith rows of E, C, respectively, then

$$f_i=\min\{j|M[j]_i\neq 0\}\qquad i=1,2,\cdots,p. \tag{22}$$

Furthermore, the matrices $\hat{C}$ and K_T of the realization $\{A,B,\hat{C},K_T\}$ of $\hat{T}(s)=X_T(s)T(s)$, which are used in the proof of Theorem 8 to determine an appropriate F_m via (16), can be directly written down as

$$K_T=\begin{bmatrix} M[f_1]_1 \\ M[f_2]_2 \\ \vdots \\ M[f_p]_p \end{bmatrix};\quad \hat{C}=\begin{bmatrix} N[f_1+1]_1 \\ N[f_2+1]_2 \\ \vdots \\ N[f_p+1]_p \end{bmatrix}. \tag{23}$$

It is clear that if (20) is satisfied ($p<m$), the maximal order reduction is $n-\sum_{i=1}^{p}f_i$ ($=q+k$) where f_i are the decoupling indices. Note, in addition, that if (20) is satisfied the poles of the closed-loop transfer matrix can be directly assigned without using the constructive proof of Lemma 9. Instead of $X_T(s)$, $\tilde{X}_T(s)\triangleq H(s)$. $\operatorname{Diag}(p_i(s))$ is used where $p_i(s)$ are arbitrary polynomials of the form $p_i(s)=s^{f_i}+\sum_{j=0}^{f_i-1}a_{ij}s^j$. It can be easily shown that

$$\lim_{s\to\infty}\tilde{X}_T(s)T(s)=K_T;\qquad \operatorname{rank}K_T=p \tag{24}$$

with K_T as in (7). If now the algorithm in the proof of Theorem 8 is applied to $\{A,B,\tilde{C},K_T\}$, a realization of $\tilde{X}_T(s)T(s)$ (see Lemma 4), an LSF matrix $\tilde{F}_m$ is derived which assigns the d_T observable poles of $\{A+B\tilde{F}_m,B,C+E\tilde{F}_m,E\}$ at arbitrary locations equal to the zeros of $\det\tilde{X}_T(s)=\prod_{i=1}^{p}p_i(s)$ (see Example).

Finally, note that if the LSF $u=\tilde{F}_mx+Gv$ is used with G such that $K_TG=I$, the closed-loop transfer matrix will be $T_{\tilde{F}_m}(s)=\tilde{X}_T^{-1}(s)$ which is a diagonal (stable) transfer matrix, i.e., the system (1) has been decoupled.

IV. Supremal (A,B)-Invariant and Controllability Subspaces

The results of the previous section will now be translated into properties of the supremal output-nulling invariant and controllability subspaces [15] V^* and R^*, respectively. The supremal output-nulling invariant subspace or simply the *supremal (A,B)-invariant subspace* V^*

[6] That is, a basis of V^* (Section II).

[7] The zeros of the diagonal entries of $\bar{S}(s)$ are the zeros (1) [2], [3].

was defined in Section II where a number of its properties were also discussed. The supremal output-nulling controllability subspace R^* is defined [15] by[8]

$$R^* = \langle A + BF_m | \hat{\mathfrak{B}} \cap V^* \rangle \tag{25}$$

where V^*, F_m satisfy (3) and $\hat{\mathfrak{B}}$ is the range of the map B restricted to $\ker E$, i.e., $\hat{\mathfrak{B}}$ is spanned by the columns of BG_1 where the columns of G_1 span $\ker E$. If $E=0$, then $\hat{\mathfrak{B}} = \operatorname{Im} B$ and R^* is the supremal controllability subspace in $\ker C$ [9]. In the following R^* will be referred to as *the supremal controllability subspace* R^*.

Observe that the previous section is based mainly on transfer matrix and differential operator [1] representations of a system and, therefore, by using those results to show properties of geometric approach concepts, certain relations between geometric and polynomial matrix approaches will emerge. In the following theorem, the dimensions of V^* and R^* are given as functions of q, the number of zeros, and d_T, the degree of the determinant of the interactor $X_T(s)$.

Theorem 11: $\operatorname{Dim} V^* = q + \dim R^*$ where $\dim R^*$ is 1) zero when $p > m$ and 2) $n - q - d_T$ when $p < m$.

Proof: It was shown in Section II that $\dim V^* = n - n_{F_m}$, the maximal order reduction. In view of Theorem 8, $\dim V^* = q + k$ where k is 1) zero when $p > m$ and 2) $n - q - d_T$ when $p < m$. Any output-nulling controllability subspace R is a subspace of V^* [15], which implies that R^* is the largest controllability subspace in V^*. Theorem 8 now shows that when maximal order reduction is achieved q of the unobservable poles have always (fixed) values equal to the q zeros of the given system, which implies that k is the maximum number of unobservable poles which can be arbitrarily chosen. Therefore, $\dim R^* = k$. Q.E.D.

Note that the relation $\dim V^* = q + \dim R^*$ was already known as it has been shown by a number of authors [10], [15], [16] using alternative methods.

The subspaces V^* and R^* can be determined from the corresponding $q+k$ eigenvectors of $A + BF_m$. A problem might arise if V^* is to be determined and the q eigenvectors corresponding to the fixed eigenvalues (equal to the q zeros) are not linearly independent. This problem might be avoided if the observability matrix O_{F_m} is used and V^* is found instead from $V^* = \ker O_{F_m}$; note, however, that all q eigenvectors are generically independent [11].

Remark: An interesting special case arises when $p \leq m$ and the interactor $X_T(s) = \operatorname{diag}(s^{f_i})$ (see Remark following Lemma 10). In this case $\dim V^*$ and $\dim R^*$ are expressed in terms of the "decoupling indices" f_i, $i = 1, \cdots, p$ since $d_T = \sum_{i=1}^{p} f_i$. If now $\operatorname{rank} E = p$, then $f_i = 0$ which implies that $\dim V^* = n$, i.e., the whole space, and $\dim R^* = n - q$; note that this is exactly the case when (1) has a proper right inverse [7].

The results of the previous section and especially Lemma 10 can be used to study another subspace of interest, V_g^*, which is used in the solution of the disturbance decoupling problem with stability (DDPS) [9]. V_g^* is the largest element of the subspaces V which satisfy (3) and for which $\sigma(A + BF/V)$ (the spectrum of the restriction of $A + BF$ on V) is stable. It can be easily shown, although not here, that $\dim V_g^*$ equals the maximum number of the stable unobservable poles of the closed-loop system. The following corollary of Theorem 11 is easily established using Lemma 10 and its proof is omitted.

Corollary 12: $\operatorname{Dim} V_g^* = q_s + \dim R^*$, where q_s is the number of stable zeros of (1).

Note that all the above properties are true for unobservable systems as well. This intuitively clear result can be shown as follows: consider the system $R(s)P^{-1}(s) = C(sI - A)^{-1}B + E$ with order[9] $n \triangleq \operatorname{degree}(\det P(s)) = \operatorname{degree}(\det(sI - A))$ where R, P are not necessarily rrp, i.e., the system $\{A, B, C, E\}$ is controllable but not necessarily observable. The interactor $X_T(s)$ can be found either from the corresponding controllable and observable system $T(s)$ or directly, as it can be shown, from $R(s)$ and the column degrees of $P(s)$. If now RP^{-1} is used instead of $T(s)$ all the above results refer to a system which is just controllable. It is of interest to note that if there are r unobservable poles in a system of order n with q zeros (defined in the Preliminaries) not all of these r poles correspond to zeros, when $p < m$, but only say $r_z \leq r$. In this case, $\dim V^* = q + k = n - d_T$ and $\dim R^* = k$ with $k = n - q - d_T = [(n - r) - (q - r_z) - d_T] + r - r_z$, i.e., the dimensions of R^* and V^* are larger than the dimensions of R^* and V^* of the corresponding controllable and observable system by $r - r_z$ and r, respectively. As an illustration consider the example.

Example: $P(D)z(t) = u(t)$; $y(t) = R(D)z(t)$ where

$$R(D) = [(D+1)(D+2), 0]$$

and

$$P(D) = \begin{bmatrix} D^2(D+2) & -D+1 \\ 0 & D(D-1) \end{bmatrix}.$$

The system is controllable but unobservable with unobservable poles at 1 and -2. The zeros of the system are at -1 and -2 ($q=2$) while the poles are at $1, -2, 0, 0, 0$ ($n=5$). The transfer matrix is $T(s) = R(s)P^{-1}(s) = [s + 1/s^2, s + 1/s^3]$. $X_T(s) = s$ which implies that $d_T = 1$. The maximal order reduction therefore is $n - d_T = q + k = 4 = \dim V^*$ and $\dim R^* = k = 2$. An appropriate LSF matrix F_m is found as follows. Let $\tilde{X}_T(s) = s + a$ (see Remark following Lemma 10). $\tilde{X}_T R = \tilde{C} S(s) + K_T P(s)$ where

$$S(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix} \quad K_T = [1 \quad 0] \quad \tilde{C} = [2a, 2 + 3a, a + 1, -1, 1].$$

An LSF matrix F_m which achieves maximal order reduction is

$$F_m = \begin{bmatrix} -\tilde{C} \\ \tilde{F}_2 \end{bmatrix}$$

with $\tilde{F}_2$ arbitrary, since $\tilde{C} + K_T F_m = 0$ for all $\tilde{F}_2$ (see proof of Theorem 8). For stable closed-loop eigenvalues, $a > 0$ and $\tilde{F}_2$ is chosen as follows. From the structure theorem [1] a realization of the given system is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad C = [2 \quad 3 \quad 1 \quad 0 \quad 0].$$

If $[\tilde{B}_1, \tilde{B}_2] = B$, $\{A - \tilde{B}_1 \tilde{C}, \tilde{B}_2\}$ has -1, -2, and $-a$ as uncontrollable poles. $\tilde{F}_2 = [0, 0, 0, -bc, -1 - (b + c)]$ assigns the controllable poles at $-b$, $-c$, i.e., the eigenvalues of $A + BF_m$ are at -1, -2 (equal to the $q=2$ zeros), at $-b$, $-c$ ($k=2$ arbitrarily assignable eigenvalues) and at $-a$ (which is the only observable pole of $\{A + BF_m, B, C\}$). $T_{F_m} = \tilde{X}_T^{-1} K_T = [1/(s+a), 0]$. The eigenvalues of $A + BF_m | V^*$ are -1, -2, $-b$, $-c$ and the eigenvalues of $A + BF_m | R^*$ are $-b$, $-c$. From the corresponding eigenvectors of $A + BF_m$

$$V^* = \operatorname{span} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad R^* = \operatorname{span} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The results developed above are precise results. It is of interest to determine what happens when generic properties [9] are important. This

[8] $\langle A | \mathfrak{B} \rangle \triangleq \mathfrak{B} + A\mathfrak{B} + \cdots + A^{n-1}\mathfrak{B}$.

[9] Note the different meaning of the term "order" for the given system.

is often the case in practice where the matrices A, B, C ($E=0$) are obtained from imprecise measured data [11]. It is clear that in those cases $T(s)=C(sI-A)^{-1}B=R(s)P(s)^{-1}$ will have (generically) full rank. If the zeros of the system are defined as the zeros of the greatest common divisor of the highest order minors of $R(s)$ [2], clearly $q=0$ for $p\neq m$. If $p=m$, then $q=n-p$ (rank $C=p$) as it can be also seen from the structure theorem of [1]. When $p\leqslant m$, $X_T(s)=\operatorname{diag}(s^{f_1},\cdots,s^{f_p})$, (the system can be diagonally decoupled) with $f_i=1$ and $d_T=p$ since the differences between denominator and numerator degrees in the rows of $T(s)$ (which define f_i) will be one. Therefore, (generically)

$$\text{for } p>m \quad V^*=R^*=0$$

$$\text{for } p=m \quad R^*=0 \quad V^*=\ker C \quad (\dim V^*=n-p).$$

This is because V^* is spanned by the $n-p$ (generically independent) eigenvectors corresponding to the $q=n-p$ (unobservable) eigenvalues of $A+BF_m$; these eigenvectors constitute a basis for $\ker C$ since if v_i is an eigenvector corresponding to an unobservable eigenvalue $Cv_i=0$ (also $\dim\ker C=n-p$). Similarly, it can be shown that

$$\text{for } p<m \quad R^*=V^*=\ker C \quad (\dim V^*=\dim R^*=n-p).$$

Note that all these results can be derived using purely geometric concepts [9], [11].

Concluding Remarks

The subspaces V^* and R^* were studied and their dimensions were explicitly determined as functions of the number of zeros of the system and the degree of the determinant of the interactor. This was achieved by solving the problem of maximal order reduction via linear state feedback. It should be noted at this point that the particular methods used in the constructive proofs of this paper are not, in general, computationally attractive. In a future paper it will be shown using eigenvectors [5] that the above subspaces can be found quite easily from the polynomial matrix description of the system; it will become then apparent that V^*, R^*, V_g^* depend only on the numerator $R(s)$ of the transfer matrix $T(s)$ and the column degrees of the denominator $P(s)$.

References

[1] W. A. Wolovich, *Linear Multivariable Systems*. New York: Springer-Verlag, 1974.
[2] ——, "On the numerators and zeros of rational transfer matrices," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 544–546, Oct. 1973.
[3] H. H. Rosenbrock, *State Space and Multivariable Theory*. London, England: Nelson, 1970.
[4] W. A. Wolovich, "On the cancellation of multivariable system zeros by state feedback," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 276–277, June 1974.
[5] P. J. Antsaklis, "Some new matrix methods applied to multivariable system analysis and design," Ph.D. dissertation, Brown Univ., June 1977.
[6] W. A. Wolovich and P. L. Falb, "Invariants and canonical forms under dynamic compensation," *SIAM J. Contr. Optimiz.*, vol. 14, pp. 996–1008, Nov. 1976.
[7] P. J. Antsaklis, "Stable proper nth order inverses," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 1104–1106, Dec. 1978.
[8] K. Furuta and S. Kamiyama, "State feedback and inverse system," *Int. J. Contr.*, vol. 25, no. 2, pp. 229–241, 1977.
[9] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*. New York: Springer-Verlag, 1974.
[10] J. P. Corfmat and A. S. Morse, "Control of linear systems through specified input channels," *SIAM J. Contr. Optimiz.*, vol. 14, no. 1, pp. 163–175, 1976.
[11] B. C. Moore and A. J. Laub, "Computation of supremal (A,B)-invariant and controllability subspaces," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 783–792, Oct. 1978.
[12] T. Mita, "On maximal unobservable subspaces, zeros and their applications," *Int. J. Contr.*, vol. 25, no. 6, pp. 885–899, 1977.
[13] P. N. R. Stoyle and A. I. G. Vardulakis, "On certain connections between the geometric and polynomial matrix approaches to linear system theory," Dep. Comput. Contr., Imperial College, London, Pub. 78/2, 1978.
[14] E. Emre and M. L. J. Hautus, "A polynomial characterization of (A,B)-invariant and reachability subspaces," Dep. Math., Eindhoven Univ. of Technol., The Netherlands, Memo. COSOR 78-19, Oct. 1978.
[15] B. D. O. Anderson, "Output-nulling invariant and controllability subspaces," in *Proc. IFAC 6th Triennial World Cong.*, 1975, preprints part IB, paper 43.6.
[16] B. C. Moore and L. M. Silverman, "A time domain characterization of the invariant factors of a system transfer function," in *Proc. 1974 Joint Automat. Contr. Conf.*, pp. 186–193.

An Improved Algorithm for Optimization Problems with Functional Inequality Constraints

C. GONZAGA, E. POLAK, FELLOW, IEEE, AND R. TRAHAN

***Abstract*—This paper presents an algorithm for optimization problems with distributed constraints. The algorithm is of the combined phase I–phase II feasible directions type, similar to one proposed by Polak and Mayne. It was developed as an improved version of the Polak–Mayne algorithm; by performing certain approximations in a different way it was possible to eliminate an expensive test required by Polak and Mayne.**

I. Introduction

Recently, Polak and Mayne [1] presented an algorithm for solving problems of the form

$$\min\{f^0(z)\,|\,g^j(z)\leqslant 0, j=1,2,\cdots,p;\ f^j(z)\leqslant 0, j=1,2,\cdots,m\} \tag{1a}$$

where f^0: $\mathbb{R}^n\to\mathbb{R}$ and g^j: $\mathbb{R}^n\to\mathbb{R}, j=1,2,\cdots,p$, are continuously differentiable functions and f^j: $\mathbb{R}^n\to\mathbb{R}, j=1,2,\cdots,m$, are functional constraints of the form

$$f^j(z)=\max_{\omega\in\Omega}\phi^j(z,\omega) \tag{1b}$$

where ϕ^j $\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$. It is assumed that $\phi^j(\cdot,\cdot)$ is continuous and that $\nabla_z\phi^j(\cdot,\cdot)$ is continuous for each $j=1,2,\cdots,m$. The set Ω is a compact interval of the real line. As noted in [1]–[4], an important class of engineering design problems can be formulated in the form of (1a).

The method in [1] is a phase I–phase II type method of feasible directions. Algorithms of this kind, for the case when all $f^j(\cdot)\equiv 0$ (i.e., when the f^j are not present) in (1a), were described in [5]. Since the algorithm in [1], as well as the one in this paper, generalizes the ideas in [5], it is worth while to summarize the principal concepts found in [5]. Thus, consider the simpler problem

$$\min\{f^0(z)\,|\,g^j(z)\leqslant 0, j=1,2,\cdots,p\} \tag{2a}$$

with all functions as in (1a). Let

$$\psi(z)=\max_{j\in\mathbf{p}} g^j(z) \tag{2b}$$

with $\mathbf{p}\triangleq\{1,2,\cdots,p\}$. Then (2a) can be rewritten as

$$\min\{f^0(z)\,|\,\psi(z)\leqslant 0\} \tag{2c}$$

where $\psi(\cdot)$ is only directionally differentiable, with directional derivative at z, in the direction h, given by

$$D\psi(z;h)=\max_{j\in I_0(z)}\langle\nabla f^j(z),h\rangle \tag{2d}$$

where $I_0(z)=\{j\in\mathbf{p}\,|\,f^j(z)=\psi(z)\}$.

Now consider three cases: $z\in\mathbb{R}^n$ is such that 1) $\psi(z)<0$, 2) $\psi(z)=0$, and 3) $\psi(z)>0$. For 1), any $h\in\mathbb{R}^n$ such that

$$\langle\nabla f^0(z),h\rangle<0 \tag{2e}$$

defines a cost decrease direction which can be followed for some distance without violating constraints. For 2) any $h\in\mathbb{R}^n$ such that

$$\max\{\langle\nabla f^0(z),h\rangle, D\psi(z;h)\}<0 \tag{2f}$$

is again a feasible direction of cost decrease. Finally, for 3), any $h\in\mathbb{R}^n$

Manuscript received October 30, 1978; revised September 19, 1979. Paper recommended by A. Z. Manitius, Chairman of the Optimal Systems Committee. This work was supported by the National Science Foundation under Grants ENV76-04264 and ENG73-08214-A01.

The authors are with the Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720.

0018-9286/80/0200-0049$00.75 ©1980 IEEE