P.J. Antsaklis and X.D. Koutsoukos, "On Hybrid
Control of Complex Systems: A Survey," Proc. of the 3rd
International Conference on Automation o f M ixed Processes:
Dynamic Hybrid Systems (ADPM'98), pp. 1-8, Reims, France, March 19-20 1998.

POLYNOMIAL AND RATIONAL MATRIX INTERPOLATION: THEORY AND CONTROL APPLICATIONS

Control Systems Technical Report # 71
Department of Electrical Engineering
University of Notre Dame
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(See 123 - IJC93)

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Abstract

In this report, a generalization of polynomial interpolation to the matrix case is introduced and applied to problems in systems and control. It is shown that this generalization is most general and it includes all other such interpolation schemes that have appeared in the literature. The polynomial matrix interpolation theory is developed and then applied to solving matrix equations; solutions to the Diophantine equation are also derived. The relation between a polynomial matrix and its characteristic values and vectors is established and it is used in pole assignment and other control problems. Rational matrix interpolation is discussed; it can be seen as a special case of polynomial matrix interpolation. It is then used to solve rational matrix equations including the model matching problem.

I. INTRODUCTION

A theory of polynomial and rational matrix interpolation is introduced in this paper and its application to certain Systems and Control problems is discussed at length. Note that many system and control problems can be formulated in terms of matrix equations where polynomial or rational solutions with specific properties are of interest. It is known that equations involving just polynomials can be solved by either equating coefficients of equal power of the indeterminate s or equivalently by using the values obtained when appropriate values for s are substituted in the given polynomials; in the latter case one uses results from the theory of polynomial interpolation. Similarly one may solve polynomial matrix equations using the theory of polynomial matrix interpolation presented here; this approach has significant advantages and these are discussed below. In addition to equation solving, there are many instances where interpolation type constraints are being used in systems and control without adequate justification; the theory presented here provides such justification and thus it clarifies and builds confidence into these methods.

Polynomial interpolation has fascinated researchers and practitioners alike. This is probably due to the mathematical simplicity and elegance of the theory complemented by the wide applicability of its results to areas such as numerical analysis among others. Note that although for the scalar polynomial case, interpolation is an old and very well studied problem, only recently polynomial matrix interpolation appears to have been addressed in any systematic way [1-5]. Rational, mostly scalar interpolation has been of interest to systems and control researchers recenly. Note that the rational interpolation results presented here are distinct from other literature results as they refer to matrix case and concentrate on fundamental representation questions. Other results in the literature attempt to characterize rational functions that satisfy certain interpolation constraints and are optimal in some sense and so they rather complement our results than compete with them.

In this report polynomial matrix interpolation of the type $Q(s_j)\, a_j = b_j$, where $Q(s)$ is a matrix and a_j, b_j vectors, is introduced as a generalization of the scalar polynomial interpolation of the form $q(s_j) = b_j$. This generalization appears to be well suited to study and solve a variety of multivariable system and control problems. The original motivation for the development of the matrix interpolation theory was to be able to solve polynomial matrix equations, which appear in the theory of Systems and Control and in particular the Diophantine equation; the results presented here however go well beyond solving that equation. It should be pointed out that the driving force while developing the theory and the properties of matrix interpolation has always been system and control needs. This explains why no attempt has been made to generalize more of the classical polynomial interpolation theory results to the matrix case. This was certainly not because it was felt that it would be impossible, quite the contrary. The emphasis on system and control properties in this paper simply reflects the main research interests of the authors.

Characteristic values and vectors of polynomial matrices are also discussed in this paper. Note that contrary to the polynomial case, the zeros of the determinant of a square polynomial matrix $Q(s)$ do not adequately characterize $Q(s)$; additional information is needed that is contained in the characteristic vectors of $Q(s)$, which must also be given together with the characteristic values, to characterize $Q(s)$.

The use of interpolation type constraints in system and control theory is first discussed and a number of examples are presented.

Motivation: Interpolation type constraints in Systems and Control theory

Many control system constraints and properties that are expressed in terms of conditions on a polynomial or rational matrix $R(s)$, can be written in an easier to handle form in terms of $R(s_j)$, where $R(s_j)$ is $R(s)$ evaluated at certain (complex) values $s = s_j$ $j = 1, \ell$. We shall call such conditions in terms of $R(s_j)$, interpolation (type) conditions on $R(s)$. This is because in order to understand the exact implications of these constraints on the structure and properties of $R(s)$, one needs to use results from polynomial interpolation theory. Next, a number of examples from Systems and Control theory where polynomial and polynomial matrix interpolation constraints are used, are outlined. This list is not complete, by far.

Eigenvalue / eigenvector controllability tests: It is known that all the uncontrollable eigenvalues of $\dot{x} = Ax + Bu$ are given by the roots of the determinant of a greatest left divisor of the polynomial matrices $sI - A$ and B. An alternative, and perhaps easier to handle, form of this result is that s_j is an uncontrollable eigenvalue if and only if $\text{rank}[s_jI-A, B] < n$ where A is $n\text{x}n$ (PBH controllability test [11]). This is a more restrictive version of the previous result which involves left divisors, since it is not clear how to handle multiple eigenvalues when it is desirable to determine all uncontrollable eigenvalues. The results presented here can readily provide the solution to this problem.

Selecting T(s): In the Model Matching Problem, the plant H(s) and the desired transfer function matrix T(s) are given and a proper and stable M(s) is to be found so that T(s) = H(s)M(s), The selection of T(s) for such M(s) to exist can be handeled with matrix interpolation.

The *state feedback pole assignment problem* has a rather natural formulation in terms of interpolation type constraints; similarly the *output feedback pole assignment problem.*

More recently, stability constraints in the *H^∞ formulation of the optimal control problem* have been expressed in terms of interpolation type constraints[18-20]. It is rather interesting that [18, 19] discuss a "directional" approach which is in the same spirit of the approach taken here (and in [1-7]).

The above are just a few of the many examples of the strong presence of interpolation type conditions in the systems and control literature; this is because they represent a convenient way to handle certain types of constraints. However, a closer look reveals that the relationships between conditions on $R(s_j)$ and properties of the matrix R(s) are not clear at all and this needs to be explained. Only in this way one can take full advantage of the method and develop new approaches to handle control problems. Our research on matrix interpolation and its applications addresses this need.

The main ideas of the polynomial matrix interpolation results can be found in earlier publications [1-5], with state and static output feedback applications appearing in [6, 7]; some of the material on rational matrix interpolation has appeared before in [5]. Here a rather complete theory of polynomial and rational matrix interpolation with applications is presented. Note that all the algorithms in this paper have been successfully implemented in Matlab. In summary, the contents of the paper are as follows:

Summary

Section II presents the main results of polynomial matrix interpolation. In particular, Theorem 2.1 shows that a pxm polynomial matrix Q(s) of column degrees d_i i =

1, m can be uniquely represented, under certain conditions, by $\ell = \Sigma d_i + m$ triplets (s_j, a_j, b_j) j = 1, ℓ where s_j is a complex scalar and a_j, b_j are vectors such that $Q(s_j)\, a_j = b_j$ j = 1, ℓ. It is shown that this formulation is most general and it includes as special cases other interpolation constraints which have been used in the literature.

In Section III, equations involving polynomial matrices are studied using interpolation. All solutions of degree r are characterized and it is shown how to impose additional constraints on the solutions. The Diophantine equation is an important special case and it is examined at length. The conditions under which a solution to the Diophantine equation of degree r does exist are established and a method based on the interpolation results to find all such solutions is also given.

In Section IV the characteristic values and vectors of a polynomial matrix Q(s) are discussed and all matrices with given characteristic values and vectors are characterized. Based on these results it is possible to impose restrictions on Q(s) of the form $Q(s_j)\, a_j = 0$ that imply certain characteristic value locations with certain algebraic and geometric multiplicity. This problem is completely solved here. The cases when the desired multiplicities require the use of conditions involving derivatives of Q(s) are handled in Appendix A.

In Section V, the results developed in the previous section on the characteristic values and vectors of a polynomial matrix Q(s) are used to study several Systems and Control problems. The pole or eigenvalue assignment is a problem studied extensively in the literature. It is shown how this problem can be addressed using interpolation, in a way which is perhaps more natural and effective; dynamic (and static) output feedback as well as state feedback is used and assignment of both characteristic values and vectors is studied. Tests for controllability and observability and control design problem with interpolation type of constraints are also discussed.

Section VI introduces rational matrix interpolation. It is first shown that rational matrix interpolation can be seen as a special case of polynomial matrix interpolation and the conditions under which a rational matrix H(s) is uniquely represented by interpolation triplets are derived in Theorem 6.1. It is also shown how additional constraints on H(s) can be incorporated. These results are then applied to rational matrix equations and results analogous to the results on polynomial matrix equations derived in the previous sections are obtained.

Appendix A contains the general versions of the results in Section IV, that are valid for repeated values of s_j, with multiplicities beyond those handled in that section. Smith forms are defined and the relation between Smith and Jordan canonical forms is shown.

Appendix B contains one of the key references [1] for completeness. Appendix C contains Matlab code that implements the polynomial and rational matrix interpolation results presented here.

II. POLYNOMIAL MATRIX INTERPOLATION

In this section the theory of polynomial matrix interpolation is introduced. The main result is given by Theorem 2.1 where it is shown that a pxm polynomial matrix Q(s) of column degrees d_i $i = 1, m$ can be uniquely represented, under certain conditions, by $\ell = \Sigma\, d_i + m$ triplets (s_j, a_j, b_j) $j = 1, \ell$ where s_j a complex scalar and a_j, b_j are vectors such that $Q(s_j)\, a_j = b_j$ $j = 1, \ell$. One may have fewer than $\Sigma\, d_i + m$ interpolation points ℓ in which case the matrix (with column degrees d_i) can satisfy additional constraints. This is very useful in applications and it is shown in (2.6); in Corollary 2.2 the leading coefficient is assigned. Connections to the eigenvalues and eigenvectors are established in Corollary 2.3. In Lemma 2.4 the choice of the interpolation points is discussed. In Theorem 2.1 the vector a_j postmultiplies Q(s); in Corollary 2.5 premultiplication of Q(s) by a vector is considered and similar (dual) results are derived. The theory of polynomial matrix interpolation presented here is a generalization of the interpolation theory of polynomials and there are of course alternative approaches which are discussed; they are shown to be special cases of the formulation in Theorem 2.1. In particular, Q(s) is seen as a matrix polynomial and alternative expressions are derived in Corollary 2.6; in Corollary 2.7 interpolation constraints of the form $Q(z_k) = R_k$ $k = 1, q$ are considered, which may be seen as a direct generalization of polynomial constraints. Finally in Theorem 2.8 derivatives of Q(s) are used to generalize the main interpolation results.

The basic theorem of polynomial interpolation can be stated as follows:
Given ℓ distinct complex scalars s_j $j = 1, \ell$ and ℓ corresponding complex values b_j, there exists a unique polynomial q(s) of degree $n = \ell - 1$ for which

$$q(s_j) = b_j \qquad j = 1, \ell \tag{2.1}$$

That is, an nth degree polynomial q(s) can be uniquely represented by the $\ell = n+1$ interpolation (points or doublets or) pairs (s_j, b_j) $j = 1, \ell$. To see this, write the n-th degree polynomial q(s) as $q(s) = q\,[1, s, \ldots, s^n]'$ where q is the $(1x(n+1))$ row vector of the coefficients and [ ]' denotes the transpose. The $\ell = n+1$ equations in (2.1) can then be written as

$$qV = q\begin{bmatrix} 1 & \cdots & 1 \\ s_1 & & s_\ell \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ s_1^{\ell-1} & \cdots & s_\ell^{\ell-1} \end{bmatrix} = [b_1, \ldots, b_\ell] = B_\ell$$

Note that the matrix V (ℓxℓ) is the well known Vandermonde matrix which is nonsingular if and only if the ℓ scalars s_j $j = 1, \ell$ are distinct. Here s_i are distinct and therefore V is nonsingular. This implies that the above equation has a unique solution q, that is there exists a unique polynomial q(s) of degree n which satisfies (2.1). This proves the above stated basic theorem of polynomial interpolation .

There are several approaches to generalize this result to the polynomial matrix case and a number of these are discussed later in this section. It is shown that they are special cases of the basic polynomial matrix interpolation theorem that follows:

Let $S(s) := \text{blk diag}\{[1, s, ..., s^{d_i}]'\}$ where d_i $i = 1, m$ are non-negative integers; let $a_j \neq 0$ and b_j denote (mx1) and (px1) complex vectors respectively and s_j complex scalars.

<u>Theorem 2.1</u>: Given interpolation (points) triplets (s_j, a_j, b_j) $j = 1, \ell$ and nonnegative integers d_i with $\ell = \sum d_i + m$ such that the $(\sum d_i + m)$xℓ matrix

$$S_\ell := [S(s_1)a_1, ..., S(s_\ell)a_\ell] \tag{2.2}$$

has full rank, there exists a unique (pxm) polynomial matrix Q(s), with ith column degree equal to d_i, $i = 1, m$ for which

$$Q(s_j)\, a_j = b_j \qquad j = 1, \ell \tag{2.3}$$

<u>Proof:</u> Since the column degrees of Q(s) are d_i, Q(s) can be written as

$$Q(s) = QS(s) \tag{2.4}$$

where Q $(px(\sum d_i + m))$ contains the coefficients of the polynomial entries. Substituting in (2.3), Q must satisfy

$$QS_\ell = B_\ell \tag{2.5}$$

where $B_\ell := [b_1, ..., b_\ell]$. Since S_ℓ is nonsingular, Q and therefore Q(s) are uniquely determined. □

It should be noted that when $p = m = 1$ and $d_1 = \ell - 1 = n$ this theorem reduces to the polynomial interpolation theorem. To see this, note that in this case the nonzero scalars a_j $j = 1, \ell$, can be taken to be equal to 1, in which case $S_\ell = V$ the well known Vandermonde matrix; V is nonsingular if and only if s_j $j = 1, \ell$ are distinct.

<u>Example 2.1</u>: Let Q(s) be a 1x2 (=pxm) polynomial matrix and let $\ell = 3$ interpolation points or triplets be specified:

$\{(s_j, a_j, b_j)\ j = 1, 2, 3\} = \{ (-1,[1, 0]', 0), (0,[-1, 1]', 0), (1, [0, 1]', 1)\}$.

In view of Theorem 2.1, Q(s) is uniquely specified when d_1 and d_2 are chosen so that $\ell(=3) = \Sigma d_i + m = (d_1 + d_2) + 2$ or $d_1 + d_2 = 1$ assuming that S_3 has full rank. Clearly there are more than one choices for d_1 and d_2; the resulting Q(s) depends on the particular choice for the column degrees d_i and different combinations of d_i will result to different matrices Q(s). In particular:

(i) Let $d_1 = 1$, and $d_2 = 0$. Then $S(s) = \text{blk diag}\{[1,s]',1\}$ and (2.5) becomes:

$$Q\,S_3 = Q\,[S(s_1)a_1, S(s_2)a_2, S(s_3)a_3] = Q\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [0, 0, 1] = B_3$$

from which $Q = [1, 1, 1]$ and $Q(s) = QS(s) = [s+1, 1]$.

(ii) Let $d_1 = 0$, $d_2 = 1$. Then $S(s) = \text{blk diag}\{1, [1, s]'\}$ and (2.5) gives $Q = [0, 0, 1]$ from which $Q(s) = [0, s]$, clearly different from (i) above. □

Discussion of the Interpolation Theorem

Representations of Q(s)

Theorem 2.1 provides an alternative way to represent a polynomial matrix, or a polynomial, other than by its coefficients and degree of each entry. More specifically:

A polynomial q(s) is specified uniquely by its degree, say, n and its n+1 ordered coefficients. Alternatively, in view of (2.1) the ℓ pairs (s_j, b_j) $j = 1, \ell$ uniquely specify the nth degree polynomial q(s) provided that $\ell = n+1$ and the scalars s_j are distinct.

Similarly, a polynomial matrix Q(s) is specified uniquely by its dimensions pxm, its column degrees d_i $i = 1, m$ and the d_i+1 coefficients in each entry of column i. In view of Theorem 2.1, given the dimensions pxm, the polynomial matrix Q(s) is uniquely specified by its column degrees d_i $i = 1, m$ and the ℓ triplets (s_j, a_j, b_j) $j = 1, \ell$ provided that $\ell = \Sigma d_i + m$ and (s_j, a_j) are so that S_ℓ in (2.2) has full rank. Notice that when $p = m = 1$ these conditions reduce to the well known polynomial interpolation conditions described above, namely that s_j must be distinct.

Number of Interpolation Points

It is of interest to examine what happens when the number of interpolation points ℓ, in Theorem 2.1, is different from the required number determined by the number of columns m and the desired column degrees of Q(s), d_i $i = 1, m$. That is what happens when $\ell \neq \Sigma\, d_i + m$:

The equation of interest is $QS_\ell = B_\ell$ in (2.5). A solution Q $(px(\Sigma d_i + m))$ of this equation exists if and only if

$$\text{rank}\begin{bmatrix} S_\ell \\ B_\ell \end{bmatrix} = \text{rank } S_\ell$$

This implies that there exists a solution Q for any B_ℓ if and only if rank $(S_\ell) = \ell$, that is if and only if S_ℓ, a $(\Sigma d_i + m)x\ell$ matrix has full column rank.

(i) When $\ell > \Sigma d_i + m$, the system of equations in (2.5) is over specified; there are more equations than unknowns as S_ℓ is a $(\Sigma d_i + m)x\ell$ matrix. If now the additional $(\ell - (\Sigma d_i + m))$ equations are linearly dependent upon the previous $(\Sigma d_i + m)$ ones, then a Q(s) with column degrees d_i $i = 1, m$ is uniquely determined provided that $(\Sigma d_i + m)$ interpolation triplets (s_j, a_j, b_j) satisfy the conditions of Theorem 2.1. Otherwise there is no matrix of column degrees d_i $i = 1, m$ which satisfies these interpolation constraints. In this case these interpolation points represent a matrix of column degrees greater than d_i.

(ii) When $\ell < \Sigma d_i + m$, then Q(s) with column degrees d_i $i = 1, m$ is not uniquely specified, since there are more unknowns than equations in (2.5). That is, in this case there are many (pxm) matrices Q(s) with the same column degrees d_i which satisfy the ℓ interpolation constraints (2.3) and therefore can be represented by these ℓ interpolation triplets (s_j, a_j, b_j).

Additional Constraints

This additional freedom (in (ii) above) can be exploited so that Q(s) satisfies additional constraints. In particular, $k := (\Sigma d_i + m) - \ell$ additional linear constraints, expressed in terms of the coefficients of Q(s) (in Q), can be satisfied in general. The equations describing the constraints can be used to augment the equations in (2.5). This is a very useful characteristic and it is used extensively in later sections. In this case the equations to be solved become

$$Q\,[S_\ell\,, C] = [B_\ell, D] \tag{2.6}$$

where $QC = D$ represent $k := (\Sigma d_i + m) - \ell$ linear constraints imposed on the coefficients Q; C and D are matrices (real or complex) with k columns each. The following examples illustrate the above.

<u>Example 2.2</u> (i) Consider a 1x2 polynomial matrix Q(s) and $\ell = 3$ interpolation points: $\{(s_j, a_j, b_j)\ j = 1, 2, 3\} = \{\ (-1,[1, 0]', 0), (0,[-1, 1]', 0), (1, [0, 1]', 1)\}$.
as in Example 2.1. Let $d_1 = 1, d_2 = 0$. It was shown in Example 2.1 (i) that the above uniquely represent $Q(s) = [s+1, 1]$. Suppose now that an additional interpolation point $(s_4, a_4, b_4) = (1, [1, 0]', 2)$ is specified. Here $\ell = 4 > \Sigma d_i + m = 1+2 = 3$ and

$$Q\,S_4 = Q\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = [0, 0, 1, 2] = B_4$$

Notice however that the last equation $Q[1\ 1\ 0]' = 2$ can be obtained from $QS_3 = B_3$, by a postmultiplication of $[-1\ -2\ 2]'$. Clearly the additional interpolation point does not impose any additional constraints on $Q(s)$ as it does not contain any new information about $Q(s)$. If now the new interpolation point is taken to be $(s_4, a_4, b_4) = [1, [1, 0]', 3)$ then, as it can be easily verified, there is no $Q(s)$ with $d_1 + d_2 = 1$ which satisfies all 4 interpolation constraints. In this case one should consider $Q(s)$ with higher column degrees, namely $d_1 + d_2 = 2$.

(ii) Consider again a 1x2 polynomial matrix $Q(s)$ but with $\ell = 2$ interpolation points:
$\{(s_j, a_j, b_j)\ j = 1, 2\} = \{\ (-1,[1, 0]', 0), (0,[-1, 1]', 0)\}$
from Example 2.1. Let $d_1 = 1$, $d_2 = 0$. Here $\ell = 2 < \Sigma d_i + m = 1+2 = 3$. In this case it is possible, in general, to satisfy $(\Sigma d_i + m) - \ell = 1$ additional (linear) constraint. In particular

$$Q[S_2, C] = Q\begin{bmatrix} 1 & -1 & c_1 \\ -1 & 0 & c_2 \\ 0 & 1 & c_3 \end{bmatrix} = [0, 0, d] = [B_2, D]$$

where $Q[c_1\ c_2\ c_3]' = d$ is the additional constraint on the coefficients Q of

$$Q(s) = QS(s) = [q_1\, q_2\, q_3]\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}$$

For example, if it is desired that the coefficient $q_1 = 2$, this can be enforced by taking $c_1 = c_3 = 0$ and $c_2 = 1$, $d = 2$. Then $Q = [2\ 2\ 2]$ and $Q(s) = [2s+2\ \ 2]$ satisfies all requirements.

□

The additional constraints on $Q(s)$ (or Q) do not have of course to be linear. They can be described for example by nonlinear algebraic equations or inequalities. However, in contrast to the linear constrains, it is difficult in this case to show general results.

Determination of the Leading Coefficients

It is well known that if the leading coefficient of an nth degree polynomial is given, then n, not n+1, distinct points suffice to uniquely determine this polynomial. A corresponding result is true in the polynomial matrix case:

Let C_c denote the matrix with ith column entries the coefficients of s^{d_i}, in the ith column of $Q(s)$; that is the leading matrix coefficient (with respect to columns) of $Q(s)$. Let also $S_1 :=$ blk diag$\{[1, s, \ldots, s^{d_i-1}]'\}$ $i = 1, m$ where the assumption that d_i is greater than zero is made for S_1 to be well defined. Note that this assumption is relaxed in the alternative expression of these results discussed immediately following the Corollary.

<u>Corollary 2.2</u>: Given (s_j, a_j, b_j) $j = 1, \ell$ and nonnegative integers d_i with $\ell = \Sigma\, d_i$ such that the $(\Sigma\, d_i) x \ell$ matrix $S_{1\ell} := [S_1(s_1)\, a_1, \ldots, S_1(s_\ell) a_\ell]$ has full rank, there exists a unique (pxm) polynomial matrix Q(s), with ith column degree d_i, and a given leading coefficient matrix C_c which satisfies (2.3).

<u>Proof:</u> $Q(s) = C_c D(s) + Q_l S_l(s)$ with $D(s) := \mathrm{diag}[s^{d_i}]$ for some coefficient $px(\Sigma d_i)$ matrix Q_1. (2.3) implies

$$Q_l\, S_{1\ell} = B_\ell - C_c\, [D(s_1)a_1\,, \ldots, D(s_\ell)a_\ell] \qquad (2.7)$$

which has a unique solution Q_1 since $S_{1\ell}$ is nonsingular. Q(s) is therefore uniquely determined. □

Note that here the given C_c provides the additional m constraints (for a total of Σd_i+m) needed to uniquely determine Q(s) in view of Theorem 2.1. It is also easy to see that when p = m = 1, the corollary reduces to the polynomial interpolation result mentioned above.

The results in Corollary 2.2 can be seen in view of our previous discussion for the case when only $\ell < \Sigma d_i + m$ interpolation points are given. In that case it was possible to satisfy, in general $k := (\Sigma\, d_i + m) - \ell$ additional constraints. Here, the requirement that the leading coefficients should be C_c can be written as

$$Q\, [S_\ell\,, C] = [B_\ell, C_c\,] \qquad (2.8)$$

where C is chosen to extract the leading coefficients from Q. Since C_c has k = m columns, $\ell = \Sigma\, d_i$ interpolation points will suffice to generate $\Sigma\, d_i + m$ equations with $\Sigma\, d_i + m$ unknowns, to uniquely determine Q(s).

<u>Example 2.3</u> Consider a 1x2 polynomial matrix Q(s) with column degrees $d_1 = 1, d_2 = 0$. Assume that the interpolation point $(\ell = \Sigma d_i = 1)$ is $(s_1, a_1, b_1) = (-1, [1, 0]', 0)$ and the desired leading coefficient is $C_c = [c_1, c_2]$. Then

$$Q[S_1, C] = Q\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0\ c_1\ c_2] = [B_1, C_c]$$

from which $Q = [c_1, c_1, c_2]$ and $Q(s) = [c_1 + c_1 s, c_2]$. □

Interpolation Constraints with $B_\ell = 0$

Often the interpolation constraints (2.3) are of the form

$$Q(s_j)\, a_j = 0 \qquad j = 1, \ell \tag{2.9}$$

leading to a system of equations

$$QS_\ell = 0 \tag{2.10}$$

where S_ℓ is a $(\Sigma\, d_i + m)x\ell$ matrix; see Theorem 2.1. In this case, if the conditions of Theorem 2.1 are satisfied then the unique Q(s) which is described by the $\ell = (\Sigma\, d_i + m)$ interpolation points is Q(s) = 0. It is perhaps instructive to point out what this result means in the polynomial case. In the polynomial case this result simply states that the only nth degree polynomial with n+1 distinct roots s_j is the zero polynomial, a rather well known fact. It is useful to determine nonzero solutions of Q of (2.10). One way to achieve this is to use:

$$Q\,[S_\ell\,, C] = [0, D] \tag{2.11}$$

where again S_ℓ is a $(\Sigma\, d_i + m)x\ell$ matrix but the number of interpolation points ℓ is taken to be $\ell < \Sigma d_i + m$. In this way Q(s) is not necessarily equal to a zero matrix. The matrices C and D each have $k := (\Sigma\, d_i + m) - \ell$ columns, so that Q(s) can satisfy in general k additional constraints; see Example 2.3.

Eigenvalues and Eigenvectors

An interesting result is derived when Corollary 2.2 is applied to an (nxn) matrix Q(s) = sI - A. In this case $d_i = 1$, $i = 1, n$. $C_c = I$, $S_1(s) = I$ and $Q_1 = A$; also $\ell = n$, $S_{1n} = [a_1, ..., a_n]$ and (2.7) can be written as:

$$A\,[a_1, ..., a_n] = B_n - [a_1, ..., a_n]\,\mathrm{diag}[s_j] \tag{2.12}$$

When $[b_1, ..., b_n] = B_n = 0$ then in view of (2.12) and Corollary 2.2 the following is true:

<u>Corollary 2.3</u>: Given (s_j, a_j) j = 1, n such that the (nxn) matrix $S_{1n} = [a_1, ..., a_n]$ has full rank, there exists a unique nxn polynomial matrix Q(s) with column degrees all equal to 1 and a leading coefficient matrix equal to I which satisfies (2.3) with all $b_j = 0$; that is $Q(s_j)a_j = (s_jI - A)\, a_j = 0$.

The above corollary simply says that A is uniquely determined by its n eigenvalues s_j and the n corresponding linearly independent eigenvectors a_j, a well-known result from matrix algebra. Here this result was derived from our polynomial matrix interpolation theorem, thus pointing to a *strong connection between the polynomial matrix interpolation theory developed here and the classical eigenvalue eigenvector matrix algebra results.*

Choice of Interpolation Points

The main condition of Theorem 2.1 is that S_ℓ , a $(\Sigma\, d_i + m)x\ell$ matrix, has full (column) rank ℓ. This guarantees that the solution Q in (2.5) exists for any B_ℓ and it is unique. In the polynomial case S_ℓ can be taken to be a Vandermonde matrix which has full rank if and only if $s_j \;\; j = 1, \ell$ are distinct, and this has already been pointed out. In general however, in the matrix case, $s_j \;\; j = 1, \ell$ *do not have to be distinct*; repeated values for s_j , coupled with appropriate choices for a_j will still produce full rank in S_ℓ in many instances, as it can be easily verified by example. This is a property unique to matrix case.

Example 2.4 Consider a 1x2 polynomial matrix Q(s) with $d_1 = 1, d_2 = 0$ (as in Example 2.1). Suppose that $\ell = 3$ interpolation points are given:
$\{(s_j, a_j, b_j)\; j = 1, 2, 3\} = \{ (0,[1, 0]', 1), (0,[0, 1]', 1), (1, [1, 0]', 2)\}$.
Here $S(s) = \text{blk diag}\{[1, s]', 1\}$ and

$$Q\,S_3 = Q\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = [1, 1, 2] = B_3$$

from which $Q(s) = QS(s) = [1\;\;1\;\;1]S(s) = [s+1,\; 1]$. Note that the first two columns of S_3 are $S(0)[1\;\;0]'$ and $S(0)[0\;\;1]'$. They correspond to the same $s_j = 0 \;\; j = 1, 2$ and they are linearly independent. □

If $s_j \;\; j = 1, \ell$ are taken to be distinct, then there always exist $a_j \neq 0$ such that S_ℓ has full rank. An obvious choice is $a_j = e_1$ for $j = 1, d_1 + 1$, $a_j = e_2$ for $j = d_1 + 2, ..., d_1 + d_2 + 2$ etc, where the entries of column vector e_i are zero except the ith entry which is 1; in this way, S_ℓ is block diagonal with m Vandermonde matrices of dimensions $(d_i + 1) \times (d_i + 1)$ $i = 1, m$ on the diagonal, which has full rank since s_j are distinct (in fact we only need groups of $d_i + 1$ values of s_j to be distinct).

Example 2.5 In Example 2.4 ($Q(s)$ 1x2, $\ell = 3$, $d_1 = 1$, $d_2 = 0$) take s_1, s_2, and s_3 distinct and let $a_1 = a_2 = e_1 = [1\;\;0]'$ and $a_3 = e_2 = [0\;\;1]'$. Then in $QS_3 = B_3$,

$$S_3 = \begin{bmatrix} 1 & 1 & 0 \\ s_1 & s_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which has a block diagonal form with 2(=m) Vandermonde matrices on the diagonal. Clearly S_3 has full rank since s_1 and s_2 are distinct; so there is a unique solution Q for any B_3. □

It is also important to know, especially in applications, what happens to the rank of S_ℓ for given a_j. It turns out that S_ℓ has full rank for almost any choice of a_j when s_j are distinct. In particular:

Lemma 2.4: Let s_j $j = 1, \ell$ with $\ell \leq \Sigma\, d_i + m$ be distinct complex scalars. Then the $(\Sigma d_i + m) x \ell$ matrix S_ℓ in (2.2) has full column rank ℓ for almost any set of nonzero a_j $j = 1, \ell$.

Proof: First note that S_ℓ has at least as many rows $(\Sigma\, d_i + m)$ as columns (ℓ). The structure of S(s) together with the fact that $a_j \neq 0$ and s_j distinct imply that the ℓth order minors of S_ℓ are nonzero multivariate polynomials in a_{ij}, the entries of a_j $j = 1, \ell$. These minors become zero only for values of a_{ij} on a hypersurface in the parameter space. Furthermore note that there always exists a set of a_j (see above) for which at least one ℓth order minor is nonzero. This implies that rank $S_\ell = \ell$ for almost any set of $a_j \neq 0$. □

Example 2.6 Let S(s) = blk diag{[1, s]', 1} and take $s_1 = 0, s_2 = 1$ (distinct). Then

$$S_2 = [S(s_1)a_1,\ S(s_2)a_2] = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where $a_1 = [a_{11}, a_{21}]'$ and $a_2 = [a_{12}, a_{22}]'$ $(\neq 0)$. Rank S_2 will be less than 2 $(=\ell)$ for values of a_{ij} which make zero all the second order minors: $a_{11}a_{12}$, $a_{11}a_{22} - a_{12}a_{21}$, $a_{12}a_{21}$. Such a case is, for example, when $a_{11} = a_{12} = 0$. □

Alternative Bases

Note that alternative polynomial bases, other than $[1, s, s^2, ...]'$, which might offer computational advantages in determining Q(s) from interpolation equations (2.5) can of course be used. Choices include Chebychev polynomials, among others, and they are discussed further later in this paper in relation to applications of the interpolation results.

Alternative Approaches to Matrix Interpolation

(i) Dual Version : In Theorem 2.1, a_j are column vectors which postmultiply $Q(s_j)$ in (2.3) to obtain the interpolation constraints $Q(s_j)a_j = b_j$; b_j are also column vectors. It is clear that one could also have interpolation constraints of the form

$$\underline{a}_j\, Q(s_j) = \underline{b}_j \quad j = 1, \ell \tag{2.13}$$

where $\underline{a}_j$ and $\underline{b}_j$ are row vectors. (2.13) gives rise to an alternative ("dual") matrix interpolation result which we include here for completeness.

Let $\underline{S}(s)$ = blk diag $\{[1,s, \ldots, s^{\underline{d}_i}]\}$ where $\underline{d}_i$ $i = 1, p$ are non-negative integers; let $\underline{a}_j \neq 0$ and $\underline{b}_j$ denote (1xp) and (1xm) complex vectors respectively and s_j complex scalars.

Corollary 2.5: Given $(s_j, \underline{a}_j, \underline{b}_j)$ $j = 1, \ell$ and nonnegative integers $\underline{d}_i$ with $\ell = \sum \underline{d}_i + p$ such that the $\ell x(\sum\underline{d}_i + p)$ matrix

$$\underline{S}_\ell := \begin{bmatrix} \underline{a}_1\underline{S}(s_1) \\ \cdot \\ \cdot \\ \cdot \\ \underline{a}_\ell\underline{S}(s_\ell) \end{bmatrix} \tag{2.14}$$

has full rank, there exists a unique (pxm) polynomial matrix Q(s), with ith row degree equal to $\underline{d}_i$ $i = 1, p$, for which (2.13) is true.

Proof: Similar to the proof of Theorem 2.1: Q(s) can be written as

$$Q(s) = \underline{S}(s)\underline{Q} \tag{2.15}$$

where $\underline{Q}$ $((\sum\underline{d}_i + p) \times m)$ contains the coefficients of the polynomial entries of Q(s). Substituting in (2.8) where $B_\ell = [\, \underline{b}_1', \ldots, \underline{b}_\ell' \,]'$, $\underline{Q}$ must satisfy

$$\underline{S}_\ell \underline{Q} = \underline{B}_\ell \tag{2.16}$$

Since $\underline{S}_\ell$ is nonsingular, $\underline{Q}$ and therefore Q(s) are uniquely determined. □

Example 2.7 Let Q(s) be a 1x2 (=pxm) polynomial matrix and let $\ell = 2$ interpolation points be specified: $\{(s_j, \underline{a}_j, \underline{b}_j)\ j = 1, 2\} = \{ (-1,1,[0\ \ 1]), (0,1,[1\ \ 1])\}$. Here $\ell = 2 = \sum\underline{d}_i + p$ from which $\underline{d}_1 = 1$; that is a matrix of row degree 1 may be uniquely determined. Note that $\underline{S}(s) = [1, s]$. Then

$$\underline{S}_\ell \underline{Q} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \underline{Q} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

from which

$$\underline{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and $Q(s) = \underline{S}(s)\underline{Q} = [s+1, 1]$ □

(ii) Q(s) as a matrix polynomial : The relation between representation (2.4) used in Theorem 2.1 and an alternative, also commonly used, representation of Q(s) is now shown, namely:

$$Q(s) = \hat{Q}S_d(s) = Q_0 + \ldots + Q_d s^d \tag{2.17}$$

where $S_d(s) := [I, \ldots, Is^d]'$ a $m(d+1)xm$ matrix with $d = \max(d_i)$ $i = 1, m$ and $\hat{Q} = [Q_0, \ldots, Q_d]$ the $(pxm(d+1))$ coefficient matrix. Notice that $S(s) = KS_d(s)$ where K $((\Sigma d_i + m) \times m(d + 1))$ describes the appropriate interchange of rows in $S_d(s)$ needed to extract $S(s)$ (of Theorem 2.1). Representation (2.17) can be used in matrix interpolation as the following corollary shows:

Corollary 2.6: Given (s_j, a_j, b_j) $j = 1, \ell$ and nonnegative integer d with $\ell = m(d+1)$ such that the $m(d+1)x\ell$ matrix

$$S_{d\ell} = [S_d(s_1)\, a_1, \ldots, S_d(s_\ell) a_\ell] \tag{2.18}$$

has full rank, there exists a unique (pxm) polynomial matrix Q(s) with highest degree d which satisfies (2.3).

Proof: Consider Theorem 2.1 with $d_i = d$; then

$$\hat{Q}\, S_{d\ell} = B_\ell \tag{2.19}$$

is to be solved. The result immediately follows in view of $S(s) = KS_d(s)$ which implies that $S_{d\ell}$ is nonsingular, since here K is nonsingular. □

Notice that in order to uniquely represent a matrix Q(s) with column degrees d_i $i = 1, m$, Corollary 2.6 requires more interpolation points (s_j, a_j, b_j) than Theorem 2.1 since $md \geq \Sigma d_i$. This is, however, to be expected as in this case less information about the matrix Q(s) is used (only the highest degree d), than in the case of the theorem where the individual column degrees are supposed to be known (d_i $i = 1, m$).

Example 2.8 Let Q(s) be 1x2 (= pxm), d = 1 and let the $\ell = m(d+1) = 4$ interpolation points (s_j, a_j, b_j) be as follows: let the first 3 be the same as in Example 2.1 and the fourth be (2, [0, 1]', 1). The equation (2.19) now becomes

$$\hat{Q}\, S_{d4} = \hat{Q}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = [0, 0, 1, 1] = B_4$$

from which $\hat{Q} = [1, 1, 1, 0]$ and $Q(s) = \hat{Q}S_d(s) = Q_1 s + Q_0 = [\ s+1,\ 1]$ as in Example 2.1 (i). If the fourth interpolation point is taken to be equal to (2, [0, 1]', 2) then $B_\ell = [0, 0, 1, 2]$ while $S_{d\ell}$ remains the same. Then $\hat{Q} = [0, 0, 0, 1]$ and $Q(s) = \hat{Q}S_d(s) = [\ 0, s]$ as in Example 2.1(ii) □

Similarly to the case of Theorem 2.1, if the number of interpolation points $\ell < m(d+1)$ then Q(s) of degree d is not uniquely specified. In this case one could satisfy in general $k := m(d+1) - \ell$ additional linear constraints by solving

$$\check{Q}\,[S_{d\ell}\,, C] = [B_\ell, D] \tag{2.20}$$

where $\check{Q}C = D$ represent the k linear constraints imposed on the coefficients Q. Constraints on Q other than linear can of course be imposed in the same way as in the case of Theorem 2.1.

(iii) Constraints of the form (z_k, R_k) $k = 1, q$: Interpolation constraints of the form

$$Q(z_k) = R_k \quad k = 1, q \tag{2.21}$$

have also appeared in the literature. These conditions are but a special case of (2.3). In fact for each k, (2.21) represents m special conditions of the form $Q(s_j)\, a_j = b_j$ $j = 1, \ell$ in (2.3). To see this, consider (2.3) and blocks of m interpolation points where $s_i = z_1$ $i = 1, m$ with $a_i = e_i$, $s_{m+i} = z_2$ $i = 1, m$ with $a_{m+i} = e_i$ and so on, where the entries of e_i are zero except the ith entry which is 1; then R_1 of (2.21) above is $R_1 = [b_1, ..., b_m]$, $R_2 = [b_{m+1}, ..., b_{2m}]$ and so on. In this case s_j are not distinct but they are m-multiple. This is illustrated in Example 2.9 below where: $m = 2$ and $s_1 = s_2 = 0$ with $a_1 = [1, 0]'$, $a_2 = [0, 1]'$ and $R_1 = [b_1, b_2] = [1,1]$; also $s_3 = s_4 = 1$ with $a_3 = [1, 0]'$, $a_4 = [0, 1]'$ and $R_2 = [b_3, b_4] = [2,1]$.

A simple comparison of the constraints (2.21) to the polynomial constraints (2.1) seems to suggest that this is an attempt to directly generalize the scalar results to the matrix case. As in the polynomial case, z_k $k = 1, q$ therefore should perhaps be distinct for Q(s) to be uniquely determined. Indeed this is the case as it is shown in the proof of the following corollary:

<u>Corollary 2.7:</u> Given (z_k, R_k) $k = 1, q$ with $q = d + 1$, and R_k pxm, such that the m(d+1)xmq matrix

$$S_{dk} := [S_d(z_1), ..., S_d(z_k)] \tag{2.22}$$

has full rank, there exists a unique (pxm) polynomial matrix Q(s) with highest degree d which satisfies (2.21).

<u>Proof</u>: Direct in view of Corollary 2.6; there are $\ell = mq$ interpolation points. Notice that here S_{dk} (after some reordering of rows and columns) is a block diagonal Vandermonde type matrix, and it is nonsingular if and only if z_k are distinct. □

Example 2.9 Let Q(s) be 1x2 (=pxm), d = 1 and let the q = d+1 = 2 interpolation points be $\{z_k, R_k\ k = 1, 2\} = \{(0, [1, 1]), (1, [2, 1])\}$. In view of $Q(s) = \bar{Q}\, S_d(s) = (Q_1 s + Q_0)$

$$\bar{Q}\,[S_d(z_1), S_d(z_2)] = [R_1, R_2] \text{ or}$$

$$\bar{Q}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [1, 1, 2, 1]$$

from which $\bar{Q}$ $((Q_0, Q_1) =) = [1, 1, 1, 0]$ and $Q(s) = \bar{Q}\, S_d(s) = [s+1, 1]$ as in Examples 2.1 and 2.8. □

Note that if, instead of the degree d, the column degrees d_i i = 1, m of Q(s) are known, then a result similar to Corollary 2.7 but based directly on Theorem 2.1 can be derived and used to determine Q(s) which satisfies (2.21) given (z_k, R_k) k = 1, q. In this case, for a unique solution, q is selected so that $mq \geq (\Sigma\, d_i + m)$.

In Corollaries 2.6 and 2.7 above, it is clear that the dual interpolation results of Corollary 2.5, instead of Theorem 2.1, could have been used to derive dual versions. These dual versions involve the row dimension p instead of m and they could lead in certain cases to requirements for fewer interpolation points, depending on the relative size of p and m. These alternative versions of the Corollaries can be easily derived and they are not presented here.

(iv) Using Derivatives : In the polynomial case, there are interpolation constraints which involve derivatives of q(s) with respect to s. In this way, one could use repeated values s_j and still have linearly independent equations to work with. In the matrix case it is not necessary to have derivatives to allow some repeated values for s_j, since the key condition in Theorem 2.1 is S_ℓ of (2.2) to be of full rank which, in general, does not imply that s_j must be distinct; see Example 2.4 and Corollary 2.7 above. Nevertheless it is quite easy to introduce derivatives of Q(s) in interpolation constraints and this is now done for generality and completeness.

Notice that the *kth* derivative of $S(s) := \text{blk diag}\ \{[\ 1, s, \ldots, s^{d_i}]'\}$ i = 1, m with respect to s, denoted by $S^{(k)}(s)$, is easily determined using the formula $(s^{d_i})^{(k)} = d_i\,(d_i - 1)\ldots(d_i - k + 1)s^{d_i - k}$ for k less or equal to d_i and $(s^{d_i})^{(k)} = 0$ for k larger than d_i. The interpolation constraints $Q(s_j)a_j = b_j$ in (2.3) now have a more general form

$$Q^{(k)}(s_j)a_{kj} = b_{kj} \quad k = 0, 1, \ldots \tag{2.23}$$

for each distinct value s_j. Clearly, $Q(s) = QS(s)$ implies $Q^{(k)}(s) = QS^{(k)}(s)$ and

$$QS^{(k)}(s_j)a_{kj} = b_{kj} \tag{2.24}$$

in view of (2.23). There is a total of ℓ relations of this type which can be written as $QS_\ell = B_\ell$, as in (2.5). To be able to uniquely determine $Q(s)$, the new matrix S_ℓ, which now contains columns of the form $S^{(k)}(s_j)a_{kj}$, must have full (column) rank. In particular, the following result can be shown:

<u>Theorem 2.8</u>: Consider interpolation triplets (s_j, a_{kj}, b_{kj}) where s_j $j = 1, \sigma$ distinct complex scalars and $a_{kj} \neq 0$ (mx1), b_{kj} (px1) complex vectors. If $k = 0, \ell_j - 1$, let the total number of interpolation points be $\ell = \sum_1^\sigma \ell_j$. For nonnegative integers d_i $i = 1, m$ and $\ell = \sum d_i + m$ assume that the $(\sum d_i + m)x\ell$ matrix S_ℓ with columns of the form $S^{(k)}(s_j)\, a_{kj}$ $j = 1, \sigma, k = 0, \ell_j - 1$ namely

$$S_\ell := [S^{(0)}(s_1)a_{01}, \ldots, S^{(\ell_1 - 1)}(s_1)a_{\ell_1 - 1, 1}, \ldots, S^{(0)}(s_\sigma)a_{0\sigma}, \ldots] \tag{2.25}$$

has full column rank. Then there exists a unique pxm polynomial matrix $Q(s)$ which satisfies (2.23).

<u>Proof</u> Similar to Theorem 2.1. Solve $QS_\ell = B_\ell$ to derive the unique Q and $Q(s) = QS(s)$. □

<u>Example 2.10</u>: Consider a 1x2 polynomial matrix Q with $d_1 = 1, d_2 = 0$ and let the $\ell = \Sigma d_i + m = 3$ interpolation points $\{(s_1, a_{01}, b_{01}), (s_1, a_{11}, b_{11}), (s_2, a_{02}, b_{02})\} = \{(-1,[1\ \ 0]',0),(-1,[1\ \ 0]',1), (0,[0\ \ 1]',1)\}$ satisfy $Q(s_1)a_{01} = b_{01}$, $Q^{(1)}(s_1)a_{11} = b_{11}$ and $Q(s_2)a_{02} = b_{02}$. Here $\sigma = 2, \ell_1 = 2, \ell_2 = 1$ and $\ell = \sum_1^\sigma \ell_j = 3$. Now

$$QS_3 = Q[S^{(0)}(s_1)a_{01}, S^{(1)}(s_1)a_{11}, S^{(0)}(s_2)a_{02}] = Q\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0\ 1\ 1] = [b_{01}, b_{11}, b_{02}] = B_3$$

from which $Q = [1\ 1\ 1]$ and $Q(s) = QS(s) = [s+1, 1]$. □

III SOLUTION OF POLYNOMIAL MATRIX EQUATIONS

In this section equations of the form $M(s)L(s) = Q(s)$ are studied. The main result is Theorem 3.1 where it is shown that all solutions $M(s)$ of degree r can be derived by solving an equation $MS_{r\ell} = B_\ell$ derived using interpolation. In this way, all solutions of degree r of the polynomial equation, if they exist, are characterized. The existence and uniqueness of solutions is discussed, as well as methods to impose constraints on the solutions. Alternative bases are examined in numerical considerations. The Diophantine equation is an important special case and it is examined at length. Lemma 3.2 and Corollary 3.3 establish some technical results necessary to prove the main result in Theorem 3.4 that shows the conditions under which a solution to the Diophantine equation of degree r does exist; a method based on the interpolation results to find all such solutions is also given. Using this method, it is quite easy to impose additional constraints the solutions must satisfy and this is shown.

Consider the equation

$$M(s)L(s) = Q(s) \tag{3.1}$$

where $L(s)$ (txm) and $Q(s)$ (kxm) are given polynomial matrices. The polynomial matrix interpolation theory developed above will now be used to solve this equation and determine the polynomial matrix solutions $M(s)$ (kxt) when one exists.

First consider the left hand side of equation (3.1). Let

$$M(s) := M_0 + \dots + M_r s^r \tag{3.2}$$

where r is a non-negative integer, and let $d_i := \deg_{ci}[L(s)]$ $i = 1, m$. If

$$\hat{Q}(s) := M(s)L(s) \tag{3.3}$$

then $\deg_{ci}[\hat{Q}(s)] = d_i + r$ for $i = 1, m$. According to the basic polynomial matrix interpolation Theorem 2.1, the matrix $\hat{Q}(s)$ can be uniquely specified using $\sum(d_i+r) + m = \sum d_i + m(r+1)$ interpolation points. Therefore consider ℓ interpolation points (s_j, a_j, b_j) $j = 1, \ell$ where

$$\ell = \sum d_i + m(r+1) \tag{3.4}$$

Let $S_r(s) :=$ blk diag$\{[1, s, \dots, s^{d_i+r}]'\}$ and assume that the $(\sum d_i + m(r+1))\mathrm{x}\ell$ matrix

$$S_{r\ell} := [S_r(s_1)\, a_1, \dots, S_r(s_\ell) a_\ell] \tag{3.5}$$

has full rank; that is the assumptions in Theorem 2.1 are satisfied. Note that in view of Lemma 2.2, for distinct s_j, $S_{r\ell}$ will have full column rank for almost any set of nonzero a_j. Now in view of Theorem 2.1 $\hat{Q}(s)$ which satisfies

$$\hat{Q}(s_j)a_j = b_j \quad j = 1, \ell \tag{3.6}$$

is uniquely specified given these ℓ interpolation points (s_j, a_j, b_j). To solve (3.1), these interpolation points must be appropriately chosen so that the equation $\hat{Q}(s)$ $(= M(s)L(s)) = Q(s)$ is satisfied:

Write (3.1) as

$$ML_r(s) = Q(s) \tag{3.7}$$

where

$$M := [M_0, ..., M_r] \quad (kxt(r+1))$$
$$L_r(s) := [L(s)', ..., s^r L(s)']' \quad (t(r+1)xm).$$

Let $s = s_j$ and postmultiply (3.7) by a_j $j = 1, \ell$; note that s_j and a_j $j = 1, \ell$ must be so that $S_{r\ell}$ above has full rank. Define

$$b_j := Q(s_j)a_j \quad j = 1, \ell \tag{3.8}$$

and combine the equations to obtain

$$ML_{r\ell} = B_\ell \tag{3.9}$$

where

$$L_{r\ell} := [L_r(s_1)\, a_1, ..., L_r(s_\ell) a_\ell] \quad (t(r+1)x\ell)$$
$$B_\ell := [b_1, ..., b_\ell] \quad (kx\ell).$$

Theorem 3.1: Given L(s), Q(s) in (3.1), let $d_i := \deg_{ci}[L(s)]$ $i = 1, m$ and select r to satisfy

$$\deg_{ci}[Q(s)] \le d_i + r \quad i = 1, m \tag{3.10}$$

Then a solution M(s) of degree r exists if and only if a solution M of (3.9) does exist; furthermore, $M(s) = M[I, sI, ..., s^r I]'$.

Proof: First note that (3.10) is a necessary condition for a solution M(s) in (3.1) of degree r to exist, since $\deg_{ci}[M(s)L(s)] = d_i + r$. Assume that such a solution does exist; clearly (3.9) also has a solution M. That is, all solutions of (3.1) of degree r map into solutions of (3.9). Suppose now that a solution to (3.9) does exist. Notice that the left hand side of (3.9) $ML_{r\ell} = \hat{Q}S_{r\ell}$ where $\hat{Q}(s) = M(s)L(s) = \hat{Q}S_r(s)$. Furthermore, the right hand side of (3.9) $B_\ell = QS_{r\ell}$, in view of (3.8); also note that Q(s) is uniquely represented by the ℓ interpolation points (s_j, a_j, b_j) in view of (3.10) and the interpolation theorem. Therefore (3.9) implies that $\hat{Q}S_{r\ell} = QS_{r\ell}$ or $\hat{Q} = Q$, since $S_{r\ell}$ is nonsingular, or that $M(s)L(s) = \hat{Q}(s)$ $= Q(s)$; that is $M(s) = M_0 + ..., + M_r s^r = M[I, sI, ..., s^r I]'$ is a solution of (3.1). □

Alternative Expression

It is not difficult to show that solving (3.9) is equivalent to solving

$$M(s_j)c_j = b_j \quad j = 1, \ell \tag{3.11}$$

where

$$c_j := L(s_j)a_j,\ b_j := Q(s_j)a_j \quad j = 1, \ell \tag{3.12}$$

In view now of Corollary 2.6, the matrices M(s) which satisfy (3.11) are obtained by solving

$$MS_{r\ell} = B_\ell \tag{3.13}$$

where $S_{r\ell} := [S_r(s_1)c_1, \ldots, S_r(s_\ell)c_\ell]$ $(t(r+1)x\ell)$, with $S_r(s) := [I, sI, \ldots, s^rI]'$ $(t(r+1)xt)$ and $B_\ell := [b_1, \ldots, b_\ell]$ $(kx\ell)$; M(s) is then $M(s) = M[I, sI, \ldots, s^rI]'$ where M $(kxt(r+1))$ satisfies (3.13). Solving (3.13) is an alternative to solving (3.9).

Discussion

Theorem 3.1 shows that there is a one-to-one mapping between the solutions of degree r of the polynomial matrix equation (3.1) and the solutions of the linear system of equations (3.9) (or of (3.13)). In other words, using (3.9) (or (3.13)), we can characterize all solutions of degree r of (3.1). Note that the conditions (3.10) of the theorem are not restrictive as they are necessary conditions for a solution M(s) in (3.1) of degree r to exist; that is, all solutions of M(s)L(s) = Q(s) of any degree can be found using Theorem 3.1. Also note that no assumptions were made regarding the polynomial matrices in (3.1), that is Theorem 3.1 is valid for any matrices L(s), Q(s) of appropriate dimensions.

To solve (3.1), first determine the column degrees d_i $i = 1, m$ of L(s) and select r to satisfy (3.10). Choose (s_j, a_j) $j = 1, \ell$ with $\ell = \sum d_i + m(r+1)$, so that $S_{r\ell} := [S_r(s_1)a_1, \ldots, S_r(s_\ell)a_\ell]$ has full rank; note that in view of Lemma 2.2, for s_j distinct $S_{r\ell}$ will have full rank for almost any a_j. Calculate $b_j := Q(s_j)a_j$ (B_ℓ) and $L_{r\ell}$ in (3.9), or S_ℓ in (3.13). Solving (3.9) (or (3.13)) is equivalent to solving (3.1) for solutions M(s) of degree $\leq r$; $M(s) = M[I, sI, \ldots, s^rI]'$. When applying this approach, it is not necessary to determine in advance a lower bound for r; it suffices to use a large enough r. Theorem 3.1 provides the theoretical guarantee that in this way all solutions of (3.1) can be obtained. Searching for solutions is straightforward in view of the availability of computer software packages to solve linear system of equations. Even when an exact solution does not exist, it can be approximated using, for example, least squares approximation.

Existence and Uniqueness of Solutions

A solution M(s) of degree $\leq$ r might not exist or, if it exists, might not be unique. A solution M to (3.9) exists if and only if

$$\text{rank}\begin{bmatrix} L_{r\ell} \\ B_{\ell} \end{bmatrix} = \text{rank}\, L_{r\ell} \tag{3.14}$$

If rank $L_{r\ell} = \ell$, full column rank, (3.14) is satisfied for any B_ℓ, which implies that the polynomial equation (3.1) has a solution for any Q(s) such that (3.10) is satisfied. Such would be the case, for example, when L(s) is unimodular (real or complex scalar in the polynomial case). In the case when $L_{r\ell}$ does not have full column rank, a solution M exists only when there is a similar column dependence in B_ℓ (see (3.14)), which implies certain relationship between L(s) and Q(s) for a solution to exist. Such would be the case, for example, when L(s) is a (right) factor of Q(s). A necessary condition for $L_{r\ell}$ to have full column rank is that it must have at least as many rows t(r+1), as columns $\ell = \sum d_i + m(r+1)$. It can be easily seen that if $t \leq m$, this is impossible to happen. This implies that if L(s) has more columns that rows, solutions M(s) exist only under certain conditions on L(s) and Q(s), a known fact. For example, when $|L(s)| \neq 0$ (t=m), solution exists if and only if, L(s) is unimodular. When t > m, more rows than columns in L(s), a necessary condition for $L_{r\ell}$ to have full column rank is:

$$r \geq \frac{1}{t - m} \sum d_i - 1 \tag{3.15}$$

In this case if (3.9) has a solution, then it has more than one solutions. Similar results can be derived if (3.13) is considered. This is the case in solving the Diophantine equation, which is considered in detail later in this section.

<u>Example 3.1</u> Consider the polynomial equation

$$M(s)L(s) = M(s)(s+1) = Q(s)$$

Here m=1 and $d_1 = \deg L(s) = 1$. Then $\ell = \sum d_i + m(r+1) = 2 + r$ interpolation points will be taken where r is to be decided upon. Note that since m=1, $a_j = 1$ and $S_{r\ell}$ will have full rank if s_j are taken to be distinct. Suppose $Q(s) = s^2 + 3s + 2$, a second degree polynomial. In view of Theorem 3.1, $\deg Q(s) = 2 \leq d_1 + r = 1 + r$ from which r = 1, 2, ... Let r = 1, and take $\{s_j, j = 1, 2, 3\} = \{0, 1, 2\}$. Then from (3.9)

$$\begin{aligned} ML_{r\ell} &= [M_0, M_1]\,[L_r(0), L_r(1), L_r(2)] \\ &= [M_0, M_1]\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \end{bmatrix} \\ &= [Q(0), Q(1), Q(2)] = [2, 6, 12] = B_\ell\,. \end{aligned}$$

Here rank$[L_{r\ell}', B_\ell']' = \text{rank}\, L_{r\ell} = 2$ so a solution exists. It is also unique: $[M_0, M_1] = [2, 1]$. That is M(s) = (s+2) is the unique solution of $M(s)(s+1) = s^2 + 3s + 2$.

It is perhaps of interest at this point to demonstrate the conditions for existence of solutions in the polynomial equation $M(s)(s+1) = Q(s)$ via (3.9) and the discussion above; note that the polynomial equation has a solution if and only if $Q(s)/(s+1)$ is a polynomial or equivalently $Q(-1) = 0$. From the above system of equations (r=1), for a solution to exist $Q(2) = -3Q(0) + 3Q(1)$ or $d_1 = d_2 + d_0$ if $Q(s) = d_2s^2 + d_1s + d_0$. But this is exactly the condition for $Q(-1) = 0$ as it should be. Similarly it can be shown that $Q(-1) = 0$ must be true for $r = 2,3, \dots$.

If now $\deg Q(s) = 0$ or 1 then $r = 0$ satisfies $\deg Q(s) \le d_1 + r$ and $\ell = 2$ interpolation points are needed. Let $\{s_j \; j = 1,2\} = \{0, 1\}$. Then

$$
\begin{aligned}
ML_{r\ell} &= [M_0, M_1]\,[L_r(0), L_r(1)] \\
&= [M_0, M_1][1, 2] = [Q(0), Q(1)] = B_\ell .
\end{aligned}
$$

Clearly a solution exists only when $Q(1) = 2Q(0)$. That is for $\deg Q(s) = 1$, and $Q(s) = d_1 s + d_0$ a solution exists only when $d_1 + d_0 = 2d_0$ or $d_1 = d_0$ or when $Q(s) = d_0(s+1)$ in which case $M(s) = d_0$. For $\deg Q(s) = 0$ and $Q(s) = d_0$, a constant, it is impossible to satisfy $Q(1) = 2Q(0)$, that is a solution does not exist in this case. □

It was demonstrated in the example that using the interpolation results in Theorem 3.1 one can derive the conditions for existence of solutions in polynomial equations. However the main use of Theorem 3.1 is in finding all solutions of polynomial matrix equation of certain degree when they exist.

Example 3.2 Consider

$$
M(s)L(s) = M(s)\begin{bmatrix} s & 1 \\ s-1 & 1 \end{bmatrix} = [s+1, 1] = Q(s)
$$

Here $m = 2$, $d_1 = 1$ and $d_2 = 0$; $\ell = \Sigma d_i + m(r+1) = 1 + 2(r+1) = 3 + 2r$. To select r, consider the conditions of Theorem 3.1 :

$$\deg_1 Q(s) = 1 \le d_1 + r = 1 + r$$

$$\deg_2 Q(s) = 0 \le d_2 + r = 0 + r$$

so $r = 0, 1, \dots$ satisfy the conditions. Let $r = 0$, then $\ell = 3$; take $\{s_j, a_j, j = 1, 2, 3\} = \{(0, [1, 0]'), (0, [0, 1]'), (1, [1, 0]')\}$ and note that $S_{r\ell}$ does have full rank. Then

$$
\begin{aligned}
ML_{r\ell} &= M_0[L(0)a_1, L(0)a_2, L(1)a_3] = M_0 \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \\
&= [Q(0)a_1, Q(1)a_2, Q(2)a_3] \\
&= [1, 1, 2] = B_\ell .
\end{aligned}
$$

This has a unique solution $M(s) = M_0 = [2, -1]$. Note that here $L(s)$ is unimodular and in fact the equation has a unique solution for any (1x2) $Q(s)$. $\square$

Constraints on Solutions

When there are more unknowns ($t(r+1)$) than equations ($\ell = \sum d_i + m(r+1)$) in (3.9) or (3.13), this additional freedom can be exploited so that $M(s)$ satisfies additional constraints. In particular, $k := t(r+1) - \ell$ additional linear constraints, expressed in terms of the coefficients of $M(s)$ (in M), can be satisfied in general. The equations describing the constraints can be used to augment the equations in (3.9). In this case the equations to be solved become

$$M[L_{r\ell}, C] = [B_\ell, D] \tag{3.16}$$

where $MC = D$ represents $k := t(r+1) - \ell$ linear constraints imposed on the coefficients M; C and D are matrices (real or complex) with k columns each. Similarly, if (3.13) is the equation to be solved, then to satisfy additional linear constraints one solves

$$M[S_\ell, C] = [B_\ell, D] \tag{3.17}$$

This formulation for additional constraints is used extensively in the following to obtain solutions of the Diophantine equation which have certain properties. It should also be noted that additional constraints on solutions which cannot be expressed as linear algebraic equations on the coefficients M can of course be handled directly. One could, for example, impose the condition that coefficients in M must minimize some suitable performance index.

Numerical Considerations

In $ML_{r\ell} = B_\ell$ (3.9), the matrix $L_{r\ell}$ ($t(r+1)\mathrm{x}\ell$) is constructed from $L_r(s) = [L(s)', \ldots, s^r L(s)']'$ and (s_j, a_j) $j = 1, \ell$. The choice of the interpolation points (s_j, a_j) certainly affects the condition number of $L_{r\ell}$. Typically, a random choice suffices to guarantee an adequate condition number. This condition number can many times be improved by using an alternative (other than $[1, s, \ldots]$) polynomial basis such as Chebychev polynomials. Similar comments apply to equation $MS_\ell = B_\ell$ (3.13). It is shown below how (3.9) and (3.13) change in this case.

Let $[p_0, \ldots, p_r]' = T[1, s, \ldots, s^r]'$ where $p_i(s)$ are the elements of the new polynomial basis and $T = [t_{ij}]$ $i,j = 1, r+1$ is the transformation matrix.

Then $M(s) = M[I, sI, \ldots, s^r I]' = \hat{M}[p_0 I, \ldots, p_r I]'$ from which

$$M = \hat{M}[T \otimes I_t] \tag{3.18}$$

where $\otimes$ denotes the Kronecker product. M and $\hat{M}$ are of course the representation of M(s) with respect to the different bases. (3.9) now becomes

$$\hat{M}\hat{L}_{r\ell} = B_\ell \tag{3.19}$$

where $\hat{L}_{r\ell}$ involves $\hat{L}_r(s) = [p_0L(s)', \ldots, p_rL(s)']'$ instead of $L_r(s)$. Here

$$\hat{L}_{r\ell} = [T \otimes I_t]L_{r\ell} \tag{3.20}$$

where $\hat{L}_{r\ell}$ will have improved condition number over $L_{r\ell}$ for appropriate choices of $p_i(s)$ or T. Similarly, (3.13) becomes in this case

$$\hat{M}\hat{S}_\ell = B_\ell \tag{3.21}$$

where

$$\hat{S}_\ell = [T \otimes I_t]S_\ell \tag{3.22}$$

The Diophantine Equation

An important case of (3.1) is the Diophantine equation:

$$X(s)D(s) + Y(s)N(s) = Q(s) \tag{3.23}$$

where the polynomial matrices D(s), N(s) and Q(s) are given and X(s), Y(s) are to be found. Rewrite as

$$[X(s),\ Y(s)]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = M(s)L(s) = Q(s) \tag{3.24}$$

from which it is immediately clear that the Diophantine equation is a polynomial equation of the form (3.1) with

$$M(s) = [X(s),\ Y(s)],\quad L(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \tag{3.25}$$

and all the previous results do apply. That is, Theorem 3.1 guarantees that all solutions of (3.24) of degree r are found by solving (3.9) (or (3.13)). In the theory of Systems and Control the Diophantine equation used involves a matrix $L(s) = [D'(s), N'(s)]'$ which has rather specific properties. These now will be exploited to solve the Diophantine equation and to derive results beyond the results of Theorem 3.1. In particular conditions are derived which, if satisfied, a solution to (3.24) of degree r does exist.

Consider N(s) (pxm) and D(s) (mxm) with $|D(s)| \neq 0$; $N(s)D^{-1}(s) = H(s)$ a proper transfer matrix, that is

$$\lim_{s \to \infty} H(s) < \infty$$

Then L(s) ((p+m)xm) in (3.25) has full column rank and, as it is known, the Diophantine equation (3.24) has a solution if and only if a greatest right divisor (grd) of the columns of L(s) is a right divisor (rd) of Q(s). Let (N,D) be right coprime (rc), a typical case. This implies that a solution M = [X, Y] of some degree r always exists. We shall now establish lower bounds for r, in addition to (3.10), for the system of linear equations (3.9) (or equivalently (3.13)) to have a solution for any B_ℓ; that is, in view of (3.14) we are interested in the conditions under which $L_{r\ell}$ ((p+m)(r+1)xℓ) has full column rankℓ. Clearly these equations can be used to search for solutions for lower degree than r, if desirable. Such solutions M(s) may exist for particular L(s) and Q(s), as discussed above; approximate solutions of certain degree may also be obtained using this approach.

$L_r(s)$ in (3.7) has column degrees d_i+r i = 1,m and it can be written as

$$L_r(s) = L_r S_r(s) \tag{3.26}$$

where $S_r(s) := \text{blk diag}[1, s, \ldots, s^{d_i+r}]'$. It will be shown that under certain conditions L_r ((p+m)(r+1)x$[\sum_1^m d_i + m(r+1)]$) has full column rank. Then in view of

$$\begin{aligned} L_{r\ell} &:= [L_r(s_1)\, a_1, \ldots, L_r(s_\ell)a_\ell] \\ &= L_r\, [S_r(s_1)\, a_1, \ldots, S_r(s_\ell)a_\ell] = L_r S_{r\ell} \end{aligned} \tag{3.27}$$

and the Sylvester's inequality it will be shown that $L_{r\ell}$ ((p+m)(r+1)xℓ) also has full column rank, thus guaranteeing that a solution to $ML_{r\ell} = B_\ell$ (3.9) does exist.

N(s), D(s) are right coprime with $N(s)D^{-1}(s) = H$ a proper transfer matrix. Let ν be the observability index and n := deg|D(s)|, the order of this system. Assume that D(s) is column reduced (column proper); note that $\deg_{ci}(L(s)) = d_i = \deg_{ci}D(s)$ since the transfer matrix is proper. Then $n = \sum d_i$.

Lemma 3.2 : Rank $L_r = n + m(r+1)$ for $r \geq \nu - 1$.

Proof : First note that L_r has more rows than columns when $r \geq n/p - 1$. It is now known that the observability index satisfies $\nu \geq n/p$. Therefore, for $r \geq \nu - 1$ L_r has more rows than columns and full column rank is possible. For $r = \nu - 1$, rank $L_r = n + m\nu = n + m(r+1)$, since L_r in this case is the eliminant matrix in [10] which has full rank when N, D are coprime. Let now $r = \nu$ and consider the system defined by $N_e(s) := L_{\nu-1}(s)$, $D_e(s) :=$

$s^{\nu} D(s)$ with $H_e(s) = N_e(s)D_e(s)^{-1}$. It can be quite easily shown that N_e and D_e are right coprime and D_e is column reduced; furthermore, the observability index of this system is $\nu_e = 1$. This is because there are $n + m\nu$ nonzero observability indices ≥ 1 since L_{r-1}, the output map of a state space realization of H(s), has $n + m\nu$ independent rows; in view of the fact that the order of the system is $\deg|s^{\nu} D(s)| = n + m\nu$, all these indices must be equal to 1. Now

$$\begin{bmatrix} N_e(s) \\ D_e(s) \end{bmatrix} = \begin{bmatrix} L_{\nu-1}(s) \\ s^{\nu} D(s) \end{bmatrix} = L_e S_{\nu}(s)$$

and rank $L_e = n + m\nu + m$ since N_e, D_e satisfy all the requirements of the eliminant matrix theorem [10]. This implies that for $r = \nu$ rank $L_r = n + m\nu + m$, since $L_r(s) = [N_e(s)', D_e(s)', s^{\nu} N(s)']'$ and addition of rows to L_e, to obtain L_{ν}, does not affect its full column rank. A similar proof can be used to show, in general, that if rank $L_r = n + m(r+1)$ for some $r = r_1 > \nu - 1$ then it is also true for $r = r_1 + 1$. In view of the fact that it is also true for $r = \nu - 1$ (also $r = \nu$), the statement of the lemma is true, by induction. □

The following corollary of the Lemma is now obtained. Assume that (s_j, a_j) are selected to satisfy the assumptions of Theorem 3.1, $S_{r\ell}$ full column rank, and let D(s) be column reduced:

<u>Corollary 3.3 :</u> Rank $L_{r\ell}$ = rank $S_{r\ell} = \ell \leq \sum d_i + m(r+1)$ for $r \geq \nu - 1$.

<u>Proof :</u> In (3.27), $L_{r\ell} = L_r S_{r\ell}$ where $L_{r\ell}$ $((p+m)(r+1)\times\ell)$, L_r $((p+m)(r+1)\times[\sum d_i + m(r+1)]$. Applying Sylvester's inequality,

$$\text{rank } L_r + \text{rank } S_{r\ell} - [\textstyle\sum d_i + m(r+1)] \leq \text{rank } L_{r\ell} \leq \min\,[\text{rank } L_r, \text{rank } S_{r\ell}].$$

For $r \geq \nu - 1$, rank $L_r = n + m(r+1)$ with $n = \sum d_i$ (D(s) column reduced) in view of Lemma 3.2. Therefore rank $L_{r\ell}$ = rank $S_{r\ell}$ which equals the number of columns ℓ, as it is assumed in Theorem 3.1. □

The main result of this section can now be stated: Consider the Diophantine equation of (3.24) where N(s)(pxm), D(s)(mxm) right coprime and $H(s) = N(s)D^{-1}(s)$ a proper transfer matrix. Let ν be the observability index of the system and let D(s) be column reduced with $d_i := \deg_{ci} D(s)$. Let $\ell = \sum d_i + m(r+1)$ interpolation points (s_j, a_j, b_j) $j = 1, \ell$ be taken such that $S_{r\ell}$ has full rank (condition of Theorem 3.1). Then

<u>Theorem 3.4 :</u> Let r satisfy

$$\deg_{ci}[Q(s)] \le d_i + r \quad i = 1, m \text{ and } r \ge v - 1. \tag{3.28}$$

Then the Diophantine equation (3.23) has solutions of degree r, which can be found by solving $ML_{r\ell} = B_\ell$ (3.9) (or (3.13)).

Proof : In view of Theorem 3.1 all solutions of degree r, if such solutions exist, can be found by solving (3.9). If, in addition $r \ge v - 1$, in view of Corollary 3.3 $L_{r\ell}$ has full column rank which implies that a solution exists for any B_ℓ, or that a solution of the Diophantine of degree $\le r$ exists for any Q(s). □

The theorem basically says that if the degree r of a solution to be found is taken large enough, in particular $r \ge v - 1$, then such a solution to the Diophantine does exist. All such solutions of degree r can be found by using the polynomial matrix interpolation results in Theorem 3.1 and solving (3.9) (or (3.13)). The fact that a solution of degree $r \ge v-1$ exists when D(s) is column reduced and certain constraints on the degrees of Q(s) has been known (see for example Theorem 9.17 in [12]). This same result was derived here using a novel formulation and a proof based on interpolation results.

Example 3.3 : Let

$$D(s) = \begin{bmatrix} s^2 & 0 \\ 1 & -s+1 \end{bmatrix}, \; N(s) = \begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and}$$

$$Q(s) = \begin{bmatrix} s^3 + 2s^2 - 3s - 5 & -5s - 5 \\ -2s^2 - 5s - 4 & -s^2 - 3s - 2 \end{bmatrix}$$

Here D(s) is column reduced with $d_1 = 2$, $d_2 = 1$ and $v = 2$. According to Theorem 3.1, $\deg_{ci}[Q(s)] \le d_i + r$ $i = 1,2$, implies $3 \le 2 + r$ and $2 \le 1 + r$ from which $r \ge 1$; $\ell = \sum_1^m d_i + m\,(r+1) = 5 + 2r$ interpolation points. For such r, all solutions of degree r are given by (3.9) or (3.13). Here $r \ge v - 1 = 1$, therefore in view of Theorem 3.4 a solution of degree $r = 1$ does exist. All such solutions are found using $ML_{r\ell} = B_\ell$ (3.9) or (3.13). For $r = 1$, $s_j = -3, -2, -1, 0, 1, 2, 3$ and

$$a_j = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

a solution is

$$M(s) = [X(s), Y(s)] = \begin{bmatrix} s+5 & 5 & -3s & -10 \\ 2 & s+4 & -4s-2 & -6 \end{bmatrix}.$$ □

If in D(s) the column reduced assumption is relaxed then:

Corollary 3.5 : Rank $L_{r\ell}$ = rank $L_r = n + m(r+1)$ for $\ell = \Sigma d_i + m(r+1)$ and $r \geq \nu - 1$.

Proof : First note that $S_{r\ell}$ in this case is square and nonsingular which in view of (3.27) implies that rank $L_{r\ell}$ = rank L_r . Since D(s) is not column reduced then $n < \Sigma d_i$. In general in this case, for $r \geq \nu - 1$ rank $L_r = n + m(r+1) \leq \Sigma d_i + m(r+1)$ (with equality holding when D(s) is column reduced); that is $n + m(r+1)$ is the highest rank L_r can achieve. This can be shown by reducing D(s) to a column reduced matrix by unimodular multiplication and using Sylvester's inequality together with Lemma 3.2. □

When D(s) is not column reduced, then, in view of Corollary 3.5, $L_{r\ell}$ in $ML_{r\ell} = B_\ell$ (3.9) will not have full column rank ℓ but rank $L_{r\ell} = n + m(r+1) < \Sigma d_i + m(r+1) = \ell$. In view of (3.14), solution will exist in this case if Q(s) is such that the rank condition in (3.14) is satisfied; this will happen when only n+m(r+1) equations in (3.9), out of ℓ , are independent. If r is chosen larger in this case, that is if it is selected to satisfy $\Sigma \deg_{ci} Q + m < n + m(r+1)$ or $\Sigma \deg_{ci} Q < n + mr$, instead of $\Sigma \deg_{ci} Q \leq \Sigma d_i + mr$ as required by Theorem 3.4, then in view to Theorem 2.1, there are $\ell - (\Sigma \deg_{ci} Q + m)$ more interpolation equations than needed to uniquely specify Q(s) and these additional columns in B_ℓ will be linearly dependent on the previous ones. If similar dependence exists between the corresponding columns of $L_{r\ell}$ then (3.14) is satisfied and a solution exists. In other words, if r is taken to be large enough, then the conditions of Theorem 3.4 on r will always be satisfied in this case (after D(s) is reduced to column reduced form by a multiplication of the Diophantine equation by an appropriate unimodular matrix). It should also be stressed at this point that numerically it is straightforward to try different values for r in solving $ML_{r\ell} = B_\ell$ (3.9).

Constraints on Solutions

In the equation $ML_{r\ell} = B_\ell$ (3.9) there are at each row $t(r+1) = (p+m)(r+1)$ unknowns (number of columns of $M = [M_0, ..., M_r] = [(X_0, Y_0), ..., (X_r, Y_r)]$) and $\ell = \Sigma d_i + m(r+1)$ linearly independent equations (number of columns of $L_{r\ell}$). Therefore, for r sufficiently large, there are $p(r+1) - \Sigma d_i$ more unknowns than equations and it is possible to satisfy, in general, an equal number of additional constraints on the coefficients M of M(s) = [X(s), Y(s)]. These constraints can be accommodated by selecting larger values for r and they are exceptionally easy to handle in this setting when they are linear. Then, the equation to be solved becomes

$$M\,[L_{r\ell}, C] = [B_\ell, D] \tag{3.29}$$

where M C = D are the, say k_d desired constraints on the coefficients of the solution; the matrices C and D have k_d columns each. The degree of the solution r should then be chosen so that

$$p(r+1) - \Sigma d_i \geq k_d \tag{3.30}$$

in addition to satisfying the conditions of Theorem 3.4.

Typically, we want solutions of the Diophantine with $|X(s)| \neq 0$. This can be satisfied by requiring for example that $X_r = I$ (or any other nonsingular matrix) which, in addition guarantees that $X^{-1}(s)Y(s)$ will be proper. Note that to guarantee that $X_r = I$ one needs to use m linear equations, that is in this case the number of columns of C and D will be at least m.

To gain some insight into this important technique, consider the scalar case which has been studied by a variety of methods. In particular, consider the polynomial Diophantine where p = m = 1. Let d_i = deg D(s) = n, n_q = degQ(s) and note that $\nu = n$. Therefore r, according to Theorem 3.4, must be chosen to satisfy $r \geq n_q - n$ and $r \geq n - 1$. Select Q(s) so that $n_q = 2n - 1$ then $r \geq n - 1$ satisfies all conditions, as it is well known. In view of the above, to guarantee that $X^{-1}Y$ will be proper, one needs to set an additional constraint such as $X_r = 1$ (m = 1), which in view of (3.30) implies that $X^{-1}(s)Y(s)$ proper can be guaranteed if r is chosen to satisfy $r \geq n$. In the case when $N(s)D^{-1}(s)$ is strictly proper (instead of proper), however, this additional constraint is not needed and $X^{-1}(s)Y(s)$ proper can be obtained for $r \geq n - 1$. This is because in this case a solution with $X_r = 0$ leading perhaps to a nonproper $X^{-1}(s)Y(s)$ is not possible. Notice that for r = n - 1 the solution is unique.

Example 3.4:

Consider Example 3.3, $p(r+1) - \Sigma d_i = 2(1+1) - (2+1) = 1$. From (3.30), one can add one extra constraint on the solution in the form of (3.16) or (3.17). Assume that in addition to solve for [X(s), Y(s)] in Example 3.3, it is desirable that X(s) has a zero at s=-10 and X(-10)[1 2]' = [0 0]'. This can be easily incorporated as an extra interpolation triplet using (3.17). The solution obtained is

$$M(s) = [X(s), Y(s)] = \begin{bmatrix} s-10 & 10 & 12s & -15(s+1) \\ 16 & -s-18 & -18s-2 & 16(s+1) \end{bmatrix}.$$

Note that X(s) has a zero at -10 and [X(s), Y(s)] is a solution of the Diophantine equation (3.23) with the D(s), N(s) and Q(s) given in Example 3.3. □

<u>Example 3.5</u>: Let

$$D(s) = \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \end{bmatrix},\ N(s) = \begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix},\ \text{and } Q(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From which $d_1 = d_2 = 1$; $\deg_{ci}Q(s) = 0$, $i=1, 2$; and $\ell = 2 + 2(r+1)$

For $r = 1$, $s_j = -2, -1, 0, 1, 2, 3$ and

$$a_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

a solution is

$$M(s) = [X(s), Y(s)] = \begin{bmatrix} s & -1 & -s & s+1 \\ 1/3 & 1/3 & 0 & -1/3s+2/3 \end{bmatrix}.$$ □

<u>Example 3.6</u>: Let

$$D(s) = \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \end{bmatrix},\ N(s) = \begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix},\ \text{and } Q(s) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

From which $d_1 = d_2 = 1$; $\deg_{ci}Q(s) = 0$, $i=1, 2$; and $\ell = 2 + 2(r+1)$

For $r = 1$, $s_j = -2, -1, 0, 1, 2, 3$ and

$$a_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

a solution is

$$\begin{aligned} M(s) &= [X(s), Y(s)] \\ &= \begin{bmatrix} -.4665s-.2954 & .3805 & .4665s+.2085 & -.3805(s+1) \\ .3401s-.4040 & .0320 & -.3401s+.7761 & -.0320(s+1) \end{bmatrix}. \end{aligned}$$

Note that, in this example, the rows of $[M_0, M_1]$ forms the basis for the left null space of $S_{d\ell}$ □

Note that in Example 3.5 and 3.6 we solved the problem

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

separately, where $X(s)$ and $Y(s)$ are the solution of the Bezout identity and $\tilde{D}^{-1}(s)\tilde{N}(s) = N(s)D^{-1}(s)$ gives the left coprime factorization.

IV. CHARACTERISTIC VALUES AND VECTORS

When all the n zeros of an nth degree polynomial q(s) are given, then q(s) is specified within a nonzero constant. In contrast, the zeros of the determinant of a polynomial matrix Q(s) do not adequately characterize Q(s); information about the structure of Q(s) is also necessary. This additional information is contained in the characteristic vectors of Q(s), which must also be given, together with the characteristic values, to characterize Q(s). The characteristic values and vectors of Q(s) are studied in this section.

We are interested in cases where the complex numbers s_j $j = 1, \ell$ used in interpolation, have special meaning. In particular, we are interested in cases where s_j are the roots of the nontrivial polynomial entries of the Smith form of the polynomial matrix Q(s) or, equivalently, roots of the minors of Q(s), or roots of the invariant polynomials of Q(s) (see Appendix A). Such results are useful in addressing a variety of control problems as it is shown later in this and the following sections. Here we first specialize certain interpolation results from Section II to the case when b_j, in the interpolation constraints (2.3), are zero and we derive Corollary 4.1. This corollary characterizes the structure of all nonsingular Q(s) if all of the roots of $|Q(s)|$ together with their associated directions, i.e. (s_j, a_j), are given. We then concentrate on the characteristic values and vectors of Q(s) and in Theorems 4.2, 4.3, 4.7 and in Appendix A, we completely characterize all matrices with given such characteristic values and vectors.

Note that here only the right characteristic vectors are discussed ($Q(s_j)a_j = 0$); similar results are of course valid for left characteristic vectors ($\underline{a}_j\, Q(s_j) = 0$; see also Corollary 2.5) and they can be easily derived in a manner completely analogous to the derivations of the results presented in this and the following sections. These dual results are omitted here.

Consider the interpolation constraints (2.3) with $b_j = 0$; that is

$$Q(s_j)a_j = 0 \quad j = 1, \ell \tag{4.1}$$

In this case one solves (2.5) with $B_\ell = 0$; that is

$$QS_\ell = B_\ell = 0 \tag{4.2}$$

where $S_\ell = [S(s_1)a_1, ..., S(s_\ell)a_\ell]$ $(\Sigma d_i + m) x \ell$ and Q $(px(\Sigma d_i + m))$. This case, $B_\ell = 0$, was briefly discussed in Section II, see (2.11); see also the discussion on eigenvalues and eigenvectors. We shall now start with the case when Q(s) is nonsingular. The following corollary is a direct consequence of Corollary 2.2:

<u>Corollary 4.1</u>: Let Q(s) be (mxm) and nonsingular with n = deg|Q(s)|. Let d_i i = 1, m be its column degrees and let $\Sigma d_i = n$. If (s_j, a_j) j = 1, ℓ with $\ell = n$ are given and they are such that $S_{1\ell}$ has full rank, then a Q(s) which satisfies (4.1) is uniquely specified within a premultiplication by an (mxm) nonsingular leading coefficient matrix C_c.
<u>Proof:</u> Since deg|Q(s)| = n = Σd_i the leading coefficient matrix C_c of Q(s) must be nonsingular. The rest follows directly from (2.7). □

This corollary says that if all the n zeros s_j of the determinant of Q(s) are given together with the corresponding vectors a_j which satisfy (4.1) then, under certain assumptions ($S_{1\ell}$ full rank), Q(s) is uniquely determined within a nonsingular leading coefficient matrix C_c provided that its column degrees d_i (given) satisfy $\Sigma d_i = n$. If d_i are not specified, there are many such matrices. One could relax some of the assumptions ($S_{1\ell}$ full rank) and further extend some of the results of Section II by using derivatives of Q(s) and Theorem 2.8. Instead, we start a new line of inquiry which concentrates on the meaning of (s_j, a_j) when they satisfy relations such as (4.1). We return to Corollary 4.1 later on in this section.

If a complex scalar z and vector a satisfy Q(z) a = 0, where Q(s) is a pxm matrix and the vector a ≠ 0, then under certain conditions z and a are called *characteristic value and vector of Q(s)* respectively. This is of course an extension of the well known concepts in the special case when Q(s) = sI-A; then z and a are an eigenvalue and the corresponding eigenvector of A respectively. Note that in the general matrix case, the fact that z and a satisfy Q(z) a = 0 does not necessarily imply that they do have special meaning; for example, for Q(s) = [1 0] and a = [0 1]', Q(z) a = 0 for any scalar z. On the other hand if Q(s) is square and nonsingular, Q(z) a = 0 would imply that z is a root of the determinant of Q(s); in fact in this case z and a are indeed characteristic value and vector of Q(s). Conditions of the form Q(z) a = 0 imposed so that to force Q(s) to have certain characteristic values and vectors are very important in applications. The definitions of characteristic values and vectors are given below.

Given a pxm polynomial matrix Q(s), its Smith form is uniquely defined; see Appendix A. The characteristic values (or zeroes) of Q(s) are defined using the invariant polynomials $\varepsilon_i(s)$ of Q(s).

Definition 4.1: The *characteristic values of $Q(s)$* are the roots of the invariant polynomials of $Q(s)$ taken all together. If a complex scalar s_j is a characteristic value of $Q(s)$, the mx1 complex nonzero vector a_j which satisfies

$$Q(s_j)a_j = 0 \tag{4.3}$$

is the corresponding *characteristic vector of $Q(s)$.*

$Q(s)$ may have repeated characteristic values and the algebraic and a geometric multiplicity of s_j are defined below for $Q(s)$ square and nonsingular; it is straightforward to extend these definitions to a pxm $Q(s)$. In the case of a real matrix A, if some of the eigenvalues are repeated one may have to use generalized eigenvectors. Here generalized characteristic vectors of $Q(s)$ are also defined. The general definition involves derivatives of $Q(s)$ and it is treated in the Appendix. In the results below, only characteristic vectors that satisfy relation (4.1), which does not contain derivatives of $Q(s)$, are considered for reasons of simplicity and clarity; a general version of these results can be found in the Appendix A.

Let $Q(s)$ be an (mxm) nonsingular matrix. If s_j is a zero of $|Q(s)|$ repeated n_j times, define n_j to be the *algebraic multiplicity* of s_j; define also the *geometric multiplicity* of s_j as the quantity $(m - \text{rank } Q(s_j))$.

Theorem 4.2: There exist complex scalar s_j and ℓ_j nonzero linearly independent (mx1) vectors a_{ij} $i = 1, \ell_j$ which satisfy

$$Q(s_j)a_{ij} = 0 \tag{4.4}$$

if and only if s_j is a zero of $|Q(s)|$ with algebraic multiplicity $(=n_j) \geq \ell_j$ and geometric multiplicity $(= m - \text{rank}Q(s_j)) \geq \ell_j$.

Proof: This is a special case of the Theorem A.1 of Appendix A for $k_{ij} = 1$ $i = 1, \ell_j$. □

The complex values s_j and vectors a_{ij} are characteristic values and vectors of $Q(s)$. In the case when $\ell_j = 1$, the theorem simply states that s_j is a zero of $|Q(s)|$ if and only if $\text{rank}Q(s_j) < m$, an obvious and well known result. The conditions of Theorem 4.2 imply certain structure for the Smith form of $Q(s)$, as it is shown in Corollary A.3 in Appendix A. In particular, if the conditions of Theorem 4.2 are satisfied then the Smith form of $Q(s)$ contains the factor $(s - s_j)$ in ℓ_j separate locations on the diagonal.

In the following it is assumed that $n = \deg|Q(s)|$ is known and the matrices $Q(s)$ with given characteristic values and vectors s_j and a_{ij} are characterized.

Theorem 4.3: Let $n = \deg|Q(s)|$. There exist σ distinct complex scalars s_j and (mx1) nonzero vectors a_{ij} $i = 1, \ell_j$ $j = 1, \sigma$ with $\sum_{1}^{\sigma} \ell_j = n$ and a_{ij} $i = 1, \ell_j$ linearly independent which satisfy (4.4) if and only if the zeros of $|Q(s)|$ have σ distinct values s_j $j = 1, \sigma$, each with algebraic multiplicity $(= n_j) = \ell_j$ and geometric multiplicity $(= m - \text{rank } Q(s_j)) = \ell_j$.

Proof: This is a special case of the Theorem A.4 in the Appendix. □

Note that the independence condition on the mx1 vectors $a_{1j}, a_{2j}, \ldots, a_{\ell_j j}$ implies that $\ell_j \leq m$; that is no characteristic value is repeated more that m times. One should use the general Theorem A.4 if this is not sufficient.

The following corollary of Theorem 4.3 formalizes the most familiar case:

Corollary 4.4: Let $n = \deg |Q(s)|$. There exist n distinct complex scalars s_j and (mx1) nonzero vectors a_j $j = 1, n$ which satisfy (4.1) if and only if the zeros of $|Q(s)|$ have n distinct values s_j. □

If a matrix $Q(s)$ satisfies the conditions of Theorem 4.3, its Smith form contains the factor $(s - s_j)$ in exactly ℓ_j different locations on the diagonal; see Corollary A.5 and (A.4). This is true for each distinct value s_j $j = 1, \sigma$. In view of the divisibility properties of the diagonal entries of the Smith form, this information specifies uniquely the Smith form; that is:

Corollary 4.5: All $Q(s)$ which satisfy the conditions of Theorem 4.3 have the same Smith form. □

If a Smith form with factors $(s - s_j)^{k_{ij}}$ $k_{ij} \neq 1$ in certain location is desired, one then must use Theorem A.4 and Corollary A.5 that utilize the derivatives of $Q(s)$.

Example 4.1: Suppose for some $Q(s)$, $\deg |Q(s)| = n = 2$ and, $Q(s_j)a_{ij} = 0$ is satisfied for $s_1 = 1$ and $a_{11} = [1, 0]'$ and $a_{21} = [0, 1]'$. Here $\ell_j = \ell_1 = 2$. Since $\ell_1 = 2 = n$, Theorem 3.3 implies that $\sigma = 1$, or that $s_1 = 1$ is the only distinct root of $|Q(s)|$ and it has an algebraic multiplicity (=n) = 2 = ℓ_1 and geometric multiplicity = 2 = ℓ_1. Its Smith form has s - 1 in ℓ_1 = 2 locations on the diagonal and it is uniquely determined. It is

$$E(s) = \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix}$$

(See also Example A.1). □

Additional structural information about matrices Q(s) which satisfy the conditions of Theorem 4.3 is given by applying Corollary 4.1. Corollary 4.1 has the condition that $S_{1\ell}$ must have full (column) rank. Notice that the repeated values s_j give rise to ℓ_j linearly independent columns $S(s_j)a_{ij}$ $i = 1, \ell_j$ in $S_{1\ell}$ because a_{ij} $i = 1, \ell_j$ are linearly independent; therefore $S_{1\ell}$ has full rank for almost any set of (s_j, a_{ij}) of Theorem 4.3. Corollary 4.1 then implies that the matrices Q(s) which satisfy the conditions of Theorem 4.3 are uniquely specified within a premultiplication by a nonsingular matrix C_c if the column degrees d_i are given and they satisfy $\Sigma d_i = n$; note that it is not possible to have $\Sigma d_i < n$ since $n = \deg|Q(s)|$. It should be pointed out that this result does not contradict the fact that if the eigenvalues and the eigenvectors of a matrix A are known, then sI-A = Q(s) is uniquely determined, since in this case the additional facts that $d_i = 1$ $i = 1,n$ and $C_c = I$ are being used; see Corollary 2.3. If $\Sigma d_i > n$ then Q(s) is underdefined and there are many such matrices Q(s) (note that C_c is singular in this case). To obtain such matrices in this case ($\Sigma d_i > n$) one could select a Q(s) with $\Sigma\, d_i = n$ and then premultiply Q(s) by an arbitrary unimodular matrix U(s); note that |Q(s)| and |U(s)Q(s)| have exactly the same zeros. Therefore, the conditions of Theorem 4.3 specify Q(s) within a unimodular premultiplication.

Lemma 4.6: Theorem 4.3 is satisfied by a matrix Q(s) if and only if it is satisfied by U(s)Q(s) where U(s) is any unimodular matrix.

Proof: Straight forward. Note that (4.3) is satisfied if and only if it is satisfied for U(s)Q(s) with the same s_j and a_{ij}; this is because $U(s_j)$ is nonsingular. □

It is of interest at this point to briefly summarize the results so far: Assume that, for an (mxm) polynomial matrix Q(s) yet to be chosen, we have decided upon the degree of |Q(s)| as well as its zero locations - that is about n, s_j and the algebraic multiplicities n_j. Clearly there are many matrices that satisfy these requirements; consider for example all the diagonal matrices that satisfy these requirements. If we specify the geometric multiplicities ℓ_j as well, then this implies that the matrices Q(s) must satisfy certain structural requirements so that m-rank$Q(s_j) = \ell_j$ is satisfied; in our example the diagonal matrix, the factors $(s-s_j)$ must be appropriately distributed on the diagonal. If k_{ij} are also chosen to be equal to 1 as it is the case studied here (see Appendix for $k_{ij} \neq 1$), then the Smith form of Q(s) is completely defined, that is Q(s) is defined within pre and post unimodular matrix multiplications. Note that this is equivalent to imposing the restriction that Q(s) must satisfy n relations of type (4.4), as in Theorem 4.3, without fixing the vectors a_{ij}^{k} . If in addition a_{ij}^{k}. are completely specified then Q(s) is determined within a unimodular premultiplication; see Lemma 4.6.

If an (mxm) nonsingular polynomial matrix Q(s) satisfies all conditions of Theorem 4.3 with the exception that deg|Q(s)| is not specified, then in view of Theorem 3.2 the following can be shown.

<u>Corollary 4.7</u>: Let $|Q(s)| \neq 0$. There exist σ distinct complex scalars s_j and (mx1) nonzero vectors a_{ij} $i = 1, \ell_j$ $j = 1, \sigma$ with $\sum_1^{\sigma} \ell_j = n$ and a_{ij} $i = 1, \ell_j$ linearly independent which satisfy (4.4) if and only if $\tilde{n} := \deg|Q(s)| \geq n$ with s_j $j = 1, \sigma$ roots of $|Q(s)|$, and with algebraic and geometric multiplicity of s_j in $Q(s) \geq \ell_j$. □

In view of this corollary, it can now be shown that the conditions of Theorem 4.3, with the exception that the deg|Q(s)| is not given, specify Q(s) within a premultiplication by a polynomial matrix. That is:

<u>Corollary 4.8</u>: Let $|Q(s)| \neq 0$ and let (4.4) be satisfied for (s_j, a_{ij}) $i = 1, \ell_j$ $j = 1, \sigma$ with $\sum_1^{\sigma} \ell_j = n$ with a_{ij} $i = 1, \ell_j$ linearly independent and s_j $j = 1, \sigma$ distinct. Then Q(s) is specified within a premultiplication by a polynomial matrix. This polynomial matrix is unimodular if $\deg|Q(s)| = n$. □

Note that if $\tilde{n} = n$, then the conditions of Corollary 4.7 are same as the ones in Theorem 4.3 and the fact that Q(s) is specified within a premultiplication by a unimodular matrix in Corollary 4.8 agrees with Lemma 4.6. Corollary 4.8 also agrees with Corollary 4.1 when it is applied with $\sum d_i > n$ (see discussion following Example 4.1).

The above Theorems and Corollaries show that the existence of appropriate (s_j, a_{ij}) which satisfy (4.4) implies (and it is implied by) the occurrence of certain roots in |Q(s)|

and certain directions associated with these roots. How does one go about selecting such a_{ij} and how does one go about finding an appropriate Q(s)? This can of course be done by Corollary 4.1. (s_j, a_{ij}) are chosen so that $S_{1\ell}$ has full rank as it was discussed following Example 4.1. Note that in view of Lemma 2.4, if s_j are distinct the corresponding (nonzero) a_j can be chosen almost arbitrarily as in this case $S_{1\ell}$ will have full rank for almost any set of nonzero a_j. Therefore if one is interested in determining a polynomial matrix Q(s) with |Q(s)| having n distinct zeros, one could (almost) arbitrarily choose n nonzero vectors a_j and apply Corollary 4.1 to determine such Q(s). If additional requirements are imposed, such as certain algebraic and geometric multiplicities for the zeros, then the results in this section and in the Appendix should be utilized.

In the following, the results in Corollaries 4.7 and 4.8 derived for Q(s) square and nonsingular are extended to the nonsquare case.

Given (mxm) Q(s), let n = deg|Q(s)| and assume that

$$Q(s_j)a_{ij} = 0 \tag{4.5}$$

is satisfied for σ distinct s_j $j = 1, \sigma$ with a_{ij} $i = 1, \ell_j$ linearly independent and $\sum \ell_j = n$. That is assume that s_j and a_{ij} and Q(s) satisfy Theorem 4.3.

Theorem 4.9: Q(s) is a right divisor (rd) of an (rxm) polynomial matrix M(s) if and only if M(s) satisfies

$$M(s_j)a_{ij} = 0 \tag{4.6}$$

with the same (s_j, a_{ij}) as in (4.5) above.

Proof: Necessity: If Q is a rd of M, M = $\hat{M}$Q. Premultiply (4.5) by $\hat{M}(s_j)$ to obtain (4.6). Sufficiency: Let M(s) satisfy (4.6) and let G(s) be a greatest rd of M and Q: Then there exist a unimodular matrix U such that $U\begin{bmatrix} Q \\ M \end{bmatrix} = \begin{bmatrix} G \\ 0 \end{bmatrix}$. This implies that G satisfies the same n relations as Q(s) and M(s) in (4.5) and (4.6) respectively. Therefore deg|G(s)| $\geq$ n in view of Corollary 4.6. Since G is a rd of Q, Q = $\hat{Q}$G which implies that $\hat{Q}$ is unimodular since deg|Q| = n. Therefore M = $\hat{M}$G = ($\hat{M}\hat{Q}^{-1}$)Q, that is Q is a rd of M. $\square$

Theorem 4.9 is very important; a more general version is given in Theorem A.7 in the Appendix. From the theoretical point of view, it generalizes the characteristic value and vector results to the nonsquare, nonfull rank case. In addition, from the practical point of view it provides a convenient way to impose the restriction on a rxm M(s) that can be written as

$$M = WQ \tag{4.7}$$

where the square and nonsingular Q has specific characteristic values and vectors and W is a " do not care" polynomial matrix.

In the polynomial case, Theorem 4.9 states that the polynomial m(s) has a factor q(s) if the (distinct) roots of q(s) are also roots of m(s). For repeated roots one should use Theorem A.7 in the Appendix.

In view of the above, it should be now clear that n relations of the form $M(s_j)a_j = 0$ $j = 1, n$ with s_j distinct and a_j nonzero (mx1) vectors will guarantee that the (rxm) M(s) has a rd Q(s) which has n distinct zeros of $|Q(s)|$ equal to s_j. Such M(s) can be determined using Corollary 4.1.

Corollary 4.10: An rxm polynomial matrix M(s) has a rd Q(s) with the property that the zeros of $|Q(s)|$ are equal to the n distinct values s_j $j = 1,n$ if and only if there exist nonzero vectors a_j such that

$$M(s_j)a_j = 0 \quad j = 1,n \tag{4.8}$$

Proof: There exists an mxm Q(s) with $\deg|Q(s)| = n$ which satisfies $Q(s_j)a_j = 0$. Then in view of Theorem 4.9, the result follows. □

V. POLE PLACEMENT AND OTHER APPLICATIONS

The results developed in the previous section on the characteristic values and vectors of a polynomial matrix Q(s) are useful in a wide range of problems in Systems and Control. Several of these problems and their solutions using interpolation are discussed in this section. The pole placement or pole assignment problem is discussed first.

Pole or eigenvalue assignment is a problem studied extensively in the literature. In the following it is shown how this problem can be addressed using interpolation, in a way which is perhaps more natural and effective. Dynamic (and static) output feedback is used first to arbitrarily shift the closed loop eigenvalues (also known as the poles of the system). Then state feedback is studied.

Output Feedback - Diophantine Equation

Dynamic Output Feedback

Here all proper output controllers of degree r (of order mr) that assign all the closed loop eigenvalues to arbitrary locations are characterized in a convenient way. This has not been done before.

We are interested in solutions [X(s), Y(s)] (mx(p+m)) of the Diophantine equation

$$X(s)\,D(s) + Y(s)N(s) = Q(s) \tag{5.1}$$

where only the roots of $|Q(s)|$ are specified; furthermore $X^{-1}(s)Y(s)$ should exist and be proper. This problem is known as the pole placement (eigenvalue assignment) problem where $N(s)D^{-1}(s)$ (pxm) is a description of the plant to be controlled and $C = X^{-1}(s)Y(s)$ (mxp) is the desired controller which assigns the closed-loop poles (eigenvalues) at desired locations.

Note the difference between the problem studied in Section III, where Q(s) is known, and the problem studied here where only the roots of $|Q(s)|$ (or $|Q(s)|$ within multiplication by some nonzero real scalar) are given. It is clear, especially in view of Section IV, that there are many (in fact an infinite number) of Q(s) with the desired roots in $|Q(s)|$. So if one selects in advance a Q(s) with desired roots in $|Q(s)|$ that does not satisfy any other design criteria (and there are usually additional control goals to be accomplished) as it is typically done, then one really solves a more restrictive problem than the eigenvalue assignment problem. In fact in this case one solves a problem where the methods of Section III are appropriate, as in this case Q(s) is given; note that this approach to the problem is closer to the characteristic value and vector assignment problem (eigenvalue / eigenvector problem) discussed below, than just the pole assignment problem. In the scalar polynomial case if Q(s) is selected so that the roots of $|Q(s)|$ are the desired ones then one really arbitrarily selects in addition only the leading coefficient of Q(s), which is not really restrictive. This perhaps explains the tendency to do something analogous in the multivariable case; this however clearly changes and restricts the original problem. It is shown here that one does not have to select Q(s) in advance. For the pole placement problem it is more natural to use the interpolation approach of Section IV, where the flexibility in selecting Q(s) is expressed in terms of selecting the characteristic vectors of Q(s); in general for almost any choice for the characteristic vectors, subject to some rather mild rank conditions (see Section IV) the pole assignment is accomplished. These vectors can then be seen as design parameters and they can be selected almost arbitrarily to satisfy requirements in addition to pole assignment. Note that this design approach is rather well known in the state feedback case as it is discussed later in this section.

Consider now the Diophantine equation (5.1). The results of Sections III and IV will be used to solve the pole assignment problem.

The Diophantine equation (5.1) has been studied at length in Section III and the notation developed there will also be used in this section. In particular, let $M(s) := [X(s), Y(s)]$ and $L(s) := [D'(s), N'(s)]'$ then (5.1) becomes $M(s)\,L(s) = Q(s)$. This equation can be written as $ML_r(s) = Q(s)$ (3.7) where $M := [M_0, ..., M_r]$ a real matrix with $M(s) := M_0 + ... + M_r s^r$ and $L_r(s) := [L(s)', ..., s^r L(s)']'$. If now $b_j := Q(s_j)a_j$ $j = 1, \ell$ and $B_\ell := [b_1, ..., b_\ell]$ then the equation to be solved,(see (3.9)) is

$$ML_{r\ell} = B_\ell = 0 \tag{5.2}$$

where $L_{r\ell} := [L_r(s_1)\,a_1, ..., L_r(s_\ell)a_\ell]$ $(p+m)(r+1)x\ell)$ (see also (3.37)); the unknown matrix M is $mx(p+m)(r+1)$.

If the the column degrees of $L(s) = [D'(s), N'(s)]'$ are d_i and the degree of $M(s) = [X(s), Y(s)]$ is r, then $\deg|X(s)\,D(s) + Y(s)N(s)| = \deg|M(s)L(s)| \le \sum d_i + mr$; the equality is satisfied when $X(s)\,D(s) + Y(s)N(s)$ is column reduced. In Corollary 3.3 the conditions under which $L_{r\ell}$ has full column rank were derived: if (s_j, a_j) are selected to satisfy the assumptions of Theorem 3.1, that is $S_{r\ell}$ to have full column rank, then rank $L_{r\ell}$ = rank $S_{r\ell}$

$= \ell \leq \Sigma d_i + m(r+1)$ for $r \geq \nu - 1$, where ν is the observability index [10] of the system; note that $L_{r\ell} := [L_r(s_1)\, a_1, ..., L_r(s_\ell) a_\ell] = L_r\, [S_r(s_1)\, a_1, ..., S_r(s_\ell) a_\ell] = L_r S_{r\ell}$ where $S_r(s) :=$ blk diag$[1, s, ..., s^{d_i + r}]'$. That is, under mild conditions on (s_j, a_j) and for $r \geq \nu - 1$, $L_{r\ell}$ has full column rank ℓ.

Suppose now that $X(s)\, D(s) + Y(s)N(s)$ is forced to satisfy

$$M[L_{r\ell}\, , C] = [0, D] \tag{5.3}$$

where $\ell = \Sigma d_i + mr$. Note that $ML_{r\ell} = 0$ imposes the condition that

$$(X(s_j)\, D(s_j) + Y(s_j)N(s_j))a_j = 0 \quad j = 1, \ell$$

$(= \Sigma d_i + mr)$; that is the $\Sigma d_i + mr$ roots of $|X(s)\, D(s) + Y(s)N(s)|$ are to take on the values s_j $j = 1, \ell$ (see Corollary 4.8 and Theorem 4.9 for the proof of this claim) . Here (s_j, a_j) must be such that $S_{r\ell}$ above has full column rank ℓ (see Corollaries 3.3, 3.5 and the discussion above); note that this is true for almost any a_j when s_j are distinct (Lemma 2.4). For $L_{r\ell}$ also to have full column rank ℓ, we need $r \geq \nu$-1 as it was shown in Corollary 3.3.

In the case when $N(s)D^{-1}(s)$ is proper with $|D(s)| = n$, n instead of Σd_i may be used in which case $\ell = n + mr$ poles are assigned. Note that n must be used when D(s) is not column reduced, as in this case deg $|X(s)\, D(s) + Y(s)N(s)| =$ deg $|X(s)\, D(s)| \leq n + mr < \Sigma d_i + mr$ since $X^{-1}(s)Y(s)$ is also proper; Corollary 3.5 shows that rank$L_{r\ell} = n+mr$ in this case and Corollary 4.8 shows that $|X(s)\, D(s) + Y(s)N(s)|$ will have the desired roots.

The equations MC = D can guarantee that the leading coefficient of X(s) is nonsingular so that $X^{-1}(s)$ exists and $X^{-1}(s)Y(s)$ is proper. This will add m more equations (or columns of C and D) for a total of $\Sigma d_i + m(r+1)$ equations. Thus the following theorem has been shown:

Let $N(s)D^{-1}(s)$ be proper with N, D right coprime and $|D(s)| = n$.

<u>Theorem 5.1</u> Let $r \geq \nu$-1. Then (X(s), Y(s)) exists such that all the n+mr zeros of $|X(s)\, D(s) + Y(s)N(s)|$ are arbitrarily assigned and $X^{-1}(s)Y(s)$ is proper. It is obtained by solving (5.3). □

In (5.3) there are (at each row) (p+m)(r+1) unknowns and n+m(r+1) equations; the fact that $r \geq \nu-1$ implies that there are more unknowns than independent equations as $p\nu \geq n$. Note that the Theorem was proved for the case when s_j are distinct or more generally the case when (s_j, a_j) exist so that $S_{r\ell}$ has full rank. The general case, where the desired values s_j and their multiplicities are not considered in Section IV, can be studied using the results in the Appendix which involve derivatives of the polynomial matrices and similar results can be derived.

Notice that the order of the compensator $C(s)=X^{-1}(s)Y(s)$ is mr with minimum order $m(\nu-1)$. By reducing the system to a single input controllable system and by using, if necessary, dual results it can be shown that the minimum order of the pole assigning compensator C(s) using this method is min $(\mu-1, \nu-1)$, where μ and ν are the controllability and observability indices of the system respectively. This agrees with the well known results in [13]. Furthermore, in certain cases lower order compensators which assign the desired poles can be determined. Our method makes it possible to easily search for such lower order compensators.

Example 5.1: Let $D(s) = s^2 - 1$, $N(s) = s+2$ and $|Q(s)| = (s+1)(s-1+j1)(s-1-j1)$, from which $n = \nu = 2$; $r \geq 1$ and $\deg|Q(s)| = 2+r$. For $r = 1$, $s_i = -1, 1\pm j1$ and $a_1 = a_2 = a_3 = 1$. Here

$$L(s) = \begin{bmatrix} s^2-1 \\ s+2 \end{bmatrix},\ L_T(s) = \begin{bmatrix} s^2-1 \\ s+2 \\ s(s^2-1) \\ s(s+2) \end{bmatrix},\ L_{T\ell} = \begin{bmatrix} 0 & -1+j2 & -1-j2 \\ 1 & 3+j1 & 3-j1 \\ 0 & -3+j1 & -3-j1 \\ -1 & 2+j4 & 2-j4 \end{bmatrix}$$

Notice that $L_{T\ell}$ is a complex matrix. To solve (5.2) only the real part of $L_{T\ell}$ needs to be considered. A solution is M = [4 -1 -3 -1], that is $X(s) = -3s+4$ and $Y(s) = -(s+1)$, where $X^{-1}(s)Y(s)$ is proper. □

Example 5.2: Let

$$D(s) = \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \end{bmatrix},\ N(s) = \begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix}$$

with $n = \deg|D(s)| = 2$. Here there are $\deg|X(s)D(s) + Y(s)N(s)| = n + mr = 2 + 2r$ number of closed-loop poles to be assigned. Note that $r \geq \nu - 1 = 1 - 1 = 0$.

i) For r = 0 and $\{(s_j, a_j), j = 1,2\} = \{(-1, [1\ 0]'), (-2, [0\ 1]')\}$,

$$L(s) = L_T(s) = \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \\ s-1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } L_{T\ell} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \\ -2 & 0 \\ 1 & 1 \end{bmatrix}$$

and a solution of (5.2) is

$$M = \begin{bmatrix} 2 & 0 & -3 & 0 \\ 0 & 2 & 1 & 2 \end{bmatrix}$$

For this case, $M = M(s) = [X(s)\ Y(s)]$.

ii) For $r = 1$, and

$$\{(s_j, a_j), j = 1,4\} = \{(-1, [1\ 0]^T), (-2, [0\ 1]^T), (-3, [-1\ 0]^T), (-4, [0\ -1]^T)\}$$

$$L_\tau(s) = \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \\ s-1 & 0 \\ 1 & 1 \\ s(s-2) & 0 \\ 0 & s(s+1) \\ s(s-1) & 0 \\ s & s \end{bmatrix}, \quad L_{\tau\ell} = \begin{bmatrix} -3 & 0 & 5 & 0 \\ 0 & -1 & 0 & 3 \\ -2 & 0 & 4 & 0 \\ 1 & 1 & -1 & -1 \\ 3 & 0 & -15 & 0 \\ 0 & 2 & 0 & -12 \\ 2 & 0 & -12 & 0 \\ -1 & -2 & 3 & 4 \end{bmatrix}$$

a solution of (5.2) yields

$$[X(s)\ Y(s)] = \begin{bmatrix} s-7 & -1 & 12 & s+1 \\ 5 & s+4 & -6 & s+4 \end{bmatrix}$$

Note that $X(s)^{-1}Y(s)$ exists and it is proper. □

<u>Example 5.3</u>: Consider the same problem in Example 5.2. Now we'd like to add the following two constraints. First, that the leading coefficient matrix of X(s) must be an identity matrix; second, that the first column of Y(s) must be zero, that is, only the second output is used in the feedback loop.

For $r = 1$, let $X(s) = X_0 + X_1 s$ and $Y(s) = Y_0 + Y_1 s$. From the above constraints, $X_1 = I$ and the first columns of Y_0 and Y_1 are zero vectors. Here $M = [X_0, Y_0, X_1, Y_1]$ and (5.2) is the same as

$$ML_{\tau\ell} = [X_0, Y_0, X_1, Y_1] \begin{bmatrix} -3 & 0 & 5 & 0 \\ 0 & -1 & 0 & 3 \\ -2 & 0 & 4 & 0 \\ 1 & 1 & -1 & -1 \\ 3 & 0 & -15 & 0 \\ 0 & 2 & 0 & -12 \\ 2 & 0 & -12 & 0 \\ -1 & -2 & 3 & 4 \end{bmatrix} = [0]$$

To find the solution M that satisfies the two extra constraints, $L_{\tau\ell}$ is first partitioned as

$$L_{\tau\ell} = \begin{bmatrix} L_{\tau\ell 1} \\ L_{\tau\ell 2} \\ L_{\tau\ell 3} \end{bmatrix}, \text{ where } L_{\tau\ell 1} = \begin{bmatrix} -3 & 0 & 5 & 0 \\ 0 & -1 & 0 & 3 \\ -2 & 0 & 4 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \ L_{\tau\ell 2} = \begin{bmatrix} 3 & 0 & -15 & 0 \\ 0 & 2 & 0 & -12 \end{bmatrix}, L_{\tau\ell 2} = \begin{bmatrix} 2 & 0 & -12 & 0 \\ -1 & -2 & 3 & 4 \end{bmatrix}$$

Since $X_1 = I$, the above equation can be rewritten as

$$[X_0, Y_0, Y_1] \begin{bmatrix} L_{\tau\ell 1} \\ L_{\tau\ell 3} \end{bmatrix} = L_{\tau\ell 2}$$

To zero the first columns of Y_0 and Y_1, two additional columns are added to the equation

$$[X_0, Y_0, Y_1]\ [L_{\tau\ell 13}, C] = [\ L_{\tau\ell 2}, D]$$

where $L_{\tau\ell 13} = \begin{bmatrix} L_{\tau\ell 1} \\ L_{\tau\ell 3} \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solving the last equation yields

$$M = \begin{bmatrix} 1 & -5 & 0 & 5 & 1 & 0 & 0 & 5 \\ 1 & 6 & 0 & 2 & 0 & 1 & 0 & -1 \end{bmatrix},$$

or,

$$X(s) = \begin{bmatrix} s+1 & -5 \\ 1 & s+6 \end{bmatrix}, \text{ and } Y(s) = \begin{bmatrix} 0 & 5(s+1) \\ 0 & -(s-2) \end{bmatrix}$$

Clearly $X^{-1}(s)Y(s)$ is proper. □

Q(s) = W(s)R(s)

There are cases when the equation to be solved has the form

$$X(s)D(s) + Y(s)N(s) = W(s)R(s) \tag{5.4}$$

where R(s) is a given mxm nonsingular matrix and W(s) is not specified; D(s), N(s) are right coprime. It is necessary to preserve the freedom in W(s) since X(s), Y(s) must satisfy additional constraints. An instance where this type of equation appears is the regulator problem with internal stability when the measured plant outputs may be different from the regulated outputs; in that case X(s),Y(s) must also satisfy another Diophantine equation (5.1) for pole assignment. The problem here in (5.4) is to select X(s), Y(s) so that R(s) is a right divisor of X(s)D(s) + Y(s)N(s). This problem can be easily solved using the approach presented here. The approach is based on Corollary 4.8 (Theorem 4.9 for the nonsquare case) and it is illustrated below:

<u>Example 5.4</u> Let

$$D(s) = \begin{bmatrix} s^2 & 0 \\ 1 & -s+1 \end{bmatrix}, N(s) = \begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix}$$

Solve (5.4) with

$$R(s) = \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}$$

To solve (5.4), determine first the appropriate (s_j, a_{ij}). In this case, $\deg|R(s)| = 2$ and $s_1 = -1$, $a_{11} = [1\ 0]'$, $a_{21} = [0\ 1]'$. Note that $R(s_j)a_{ij} = 0$ and the problem is reduced to solving (5.2) with $\ell = 2$ and $r = 1$. A solution can be found as

$$X(s) = \begin{bmatrix} s+3/2 & 1/2 \\ s+1/2 & s+1/2 \end{bmatrix},\ Y(s) = \begin{bmatrix} s+1 & s \\ s & 1 \end{bmatrix}$$

where $X^{-1}(s)Y(s)$ is proper and

$$W(s) = 1/2\begin{bmatrix} 2s^2+3s+3 & 1 \\ 2s^2+s+3 & -2s+3 \end{bmatrix}$$ □

$H(s) = N(s)D^{-1}(s)$

In the pole assignment problem, if the desired closed loop poles are different than the open loop poles (that is the poles of $H(s) = N(s)D^{-1}(s)$) then it is not necessary to use a coprime factorization $D(s)$, $N(s)$ as the transfer function matrix can be used directly. In particular, (5.1) can be written as $X(s) + Y(s)N(s)D^{-1}(s) = Q(s)D^{-1}(s)$. Substituting s_j and postmultiplying by a_j one obtains the equation to be solved

$$(X(s_j) + Y(s_j)\,H(s_j)\,)\,a_j = 0 \quad j = 1, \ell \tag{5.5}$$

Notice that the characteristic vector corresponding to s_j is in this case $D^{-1}(s_j)\,a_j$.

<u>Example 5.5</u> Let the open loop tranfer function be

$$H(s) = \frac{s+2}{s^2-1}$$

and $|Q(s)| = s(s+2)(s+3)(s+4)$. If $s_i = -2, -3, -4, 0$ and $a_i = 1$ $i = 1,4$, then a solution of (5.5) is

$$X(s) = s^2 + 9s + 14 \quad \text{and } Y(s) = 13s + 7$$ □

<u>Example 5.6</u> Let the open loop transfer funtion matrix be

$$H(s) = \begin{bmatrix} \frac{s-1}{s-2} & 0 \\ \frac{1}{s-2} & \frac{1}{s+1} \end{bmatrix} \quad \text{and } |Q(s)| = s(s+2)(s+3)(s+4)(s+5).$$

If $\{s_i, a_i\} = \{(-2, [1\ 0]'), (-3, [0\ 1]'), (-4, [-1\ 0]'), (-5, [0\ -1]'), (0, [1\ -1]')\}$, then a solution is

$$X(s) = \begin{bmatrix} 77.25s+1 & s \\ 76.25s & s+1 \end{bmatrix},\ Y(s) = \begin{bmatrix} -81s+43 & 7s+15 \\ -80s+44 & 6s+14 \end{bmatrix}$$

Note that $X^{-1}(s)Y(s)$ is proper. □

Static Output Feedback

This is a special case of the dynamic output feedback discussed above. Interpolation was first used to assign closed loop poles using static output feedback in [6,7]. It offers a convenient way to assign at least some of the poles arbitrarily and study the locations of the remaining poles. The equations to be solved here are

$$(D(s_j) + KN(s_j))a_j = 0 \quad j = 1, \ell \tag{5.6}$$

where K is a real matrix, the static output feedback gain matrix. Equivalently, it can also be written as

$$(I + KH(s_j))a_j = 0 \quad j = 1, \ell \tag{5.6a}$$

The example below illustrates the approach.

<u>Example 5.7</u> Let the open loop transfer matrix be

$$H(s) = \begin{bmatrix} \frac{s+1}{s^2} & \frac{s+2}{s^2+1} \\ \frac{2}{s} & \frac{2s+3}{s^2+2} \end{bmatrix}$$

and the desired poles are $s_1 = -1, -2$ with a_i = [-26.456 92.16]', [-0.4432 1]. From (5.6a), $KH(s_i)a_i = -a_i$. That is,

$$K[H(s_1)a_1, H(s_2)a_2] = -[a_1, a_2].$$

The solution is

$$K = \begin{bmatrix} -157.08 & 73.39 \\ 321.30 & -150.49 \end{bmatrix}$$

Note that by choosing a_i appropriately other poles can be affected as well. The above solution places the other two poles at -3 and -4. For details, see [7].

State Feedback

Given a state space description $\dot{x} = Ax + Bu$, the linear state feedback control law is defined by u=Fx. It is now known that if (A,B) is controllable then there exists F such that all the closed loop eigenvalues, that is the zeros of $|sI - (A+BF)|$ are arbitrarily assigned. It will now be shown that F which arbitrarily assigns all closed loop eigenvalures can be determined using interpolation.

Let A, B, F be nxn, nxm, mxn real matrices respectively. Note that $|sI - (A+BF)| = |sI - A|\cdot|I_n - (sI-A)^{-1}BF| = |sI - A|\cdot|I_m - F(sI-A)^{-1}B|$. If now the desired closed-loop eigenvalues s_j are different from the eigenvalues of A, then F will assign all n desired closed loop eigenvalues s_j if and only if

$$F[(s_jI-A)^{-1}Ba_j] = a_j \quad j = 1,n \tag{5.7}$$

The mx1 vectors a_j are selected so that $(s_jI-A)^{-1}Ba_j$ j = 1,n are linearly independent vectors.

Alternatively one could approach the problem as follows: let M(s) (nxm) D(s) (mxm) be right coprime polynomial matrices such that

$$[sI-A, B]\begin{bmatrix} M(s) \\ -D(s) \end{bmatrix} = 0 \tag{5.8}$$

That is $(sI-A)^{-1}B = M(s)D^{-1}(s)$. An internal representation equivalent to $\dot{x} = Ax + Bu$ in polynomial matrix form is $Dz = u$ with $x = Mz$. The eigenvalue assignment problem is then to assign all the roots of $|D(s) - FM(s)|$; or to determine F so that

$$FM(s_j)a_j = D(s_j)a_j \quad j = 1,n \tag{5.9}$$

Relation (5.9) was originally used in [6] to determine F. Note that this formulation does not require that s_j be different from the eigenvalues of A as in (5.7). The mx1 vectors a_j are selected so that $M(s_j)a_j$ j = 1,n are independent. Note that $M(s_j)$ has the same column rank as $S(s_j) = \text{block diag}\{[1,s,\ldots,s^{d_i-1}]'\}$ where d_i are the controllability indices of (A,B) [10,11]. Therefore, it is possible to select a_j so that $M(s_j)a_j$ j = 1,n are independent even when s_j are repeated. (see Section II; choice of interpolation points)

• In general, there is great flexibility in selecting the nonzero vectors a_j. Note for example that when s_j are distinct, a very common case, a_j can almost be arbitrarily selected in view of Lemma 2.4. For all the appropriate choices of a_j ($M(s_j)a_j$ j = 1,n linearly independent), the n eigenvalues of the closed-loop system will be at the desired locations s_j j = 1,n. Different a_j correspond to different F (via(5.9)) that produce, in general, different system behavior; this is a phenomenom unique to the multivariable case. This can be explained by the fact that the vectors a_j one selects in (5.9) are related to the eigenvectors of the closed-loop system and although the closed-loop eigenvalures are at s_j, for different a_j one assigns different eigenvectors, which lead to different behavior in closed-loop system.

The exact relation of the eigenvectors to the a_j can be found as follows:

$$[s_jI - (A+BF)]M(s_j)a_j = (s_j - A)M(s_j)a_j - BFM(s_j)a_j = BD(s_j)a_j - BD(s_j)a_j = 0$$

where (5.8) and (5.9) were used. Therefore $M(s_j)a_j = v_j$ are the closed-loop eigenvectors corresponding to s_j.

It is not difficult to see that the results in the Appedix can be used to assign generalized closed-loop eigenvectors and Jordan forms of certain type using this approach. This is of course related to the assignment of invariant polynomials of $sI - (A+BF)$ using state feedback, a problem originally studied by Rosenbrock. One may select a_j in (5.9) to impose constraints on the gain f_{ij} in F. For example one may select a_j so that a column of F is zero (take the corresponding row of all a_j to be nonzero), or an elements of F, $f_{ij} = 0$. This point is not elaborated further here.

In the next subsection on Assignment of Characteristic Values and Vectors, the problem of selecting a_j to achieve additional objective, beyond pole assignment is discussed. Now the relation to a similar approach for eigenvalues assignment via state feedback [14] is shown; note that this approach was developed in parallel but independently to the interpolation method described above:

Consider $s_jI - (A+BF)$ and postmultiply by the corresponding right eigenvector v_j to obtain

$$[sI-A,\ B]\begin{bmatrix} v_j \\ -Fv_j \end{bmatrix} = 0 \qquad (5.10)$$

In view of this, determine a basis for the right Kernel of [sI-A, B][14], namely

$$[sI-A,\ B]\begin{bmatrix} M_j \\ -D_j \end{bmatrix} = 0 \qquad (5.11)$$

where the basis has m (independent) columns; note that rank[sI-A, B] = n since (A,B) is controllable. Since it is a basis, there exists mx1 vector a_j so that $M_ja_j = v_j$ and $D_ja_j = Fv_j$. Combining, we obtain

$$FM_ja_j = D_ja_j \qquad (5.12)$$

which, for j=1,n determines F (for appropriate a_j). Note the similarity with (5.9); they are exactly the same in fact if we take $M(s_j) = M_j$ in (5.8) and (5.11). The difference between the two approaches in [6], [14] is that in [6] a polynomial basis for the kernel of [sI-A, B] is found first and then it is evaluated at $s=s_j$, while [14] a basis for the kernel of $[s_jI-A,\ B]$ is determined without involving polynomial bases and right factorizations.

<u>Example 5.8</u>

Consider

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -4 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and let the desired eigenvalues be $s_i = -0.1, -0.2, -2, -1\pm j1$. Take

$$a_i = \begin{bmatrix} 1.2648 \\ -0.3391 \end{bmatrix}, \begin{bmatrix} 1.67744 \\ -0.15072 \end{bmatrix}, \begin{bmatrix} 101 \\ -60 \end{bmatrix}, \begin{bmatrix} -7-j16 \\ 8+j10 \end{bmatrix}, \begin{bmatrix} -7+j16 \\ 8-j10 \end{bmatrix}$$

Then the state feedback matrix that assigns the eigenvalues of (sI-(A+BK)) to the desired locations is obtained by solving (5.7)

$$K = \begin{bmatrix} 1.16 & 0.64 & 17.76 & 9.44 & 6.6 \\ -0.08 & -1.32 & -8.88 & -3.22 & -3.3 \end{bmatrix}$$ □

Assignment of Characteristic Values and Vectors

In view of the discussion above on state feedback, the characteristic vectors a_j of $(D(s) - FM(s))$ or the eigenvectors $v_j = M(s_j)a_j$ of $sI - (A+BF)$ can be assigned so that additional design goals are attained, beyond the pole assignment at s_j $j = 1, n$. Two examples of such assignment follow:

Optimal Control: It is possible to select (s_j, a_j) so that the closed-loop system satisfies some optimality criteria. In fact it is straightforward to select (s_j, a_j) so that the resulting F calculated using the above interpolation method, is the unique solution of a Linear Quadratic Regulator (LQR) problem; see for example [11].

Unobservalble eigenvalues:

It is possible under certain conditions to select (s_j, a_j) so that s_j become an unobservable eigenvalue in the closed loop system. Suppose $\dot{x} = Ax + Bu$, $y = Cx$ is equivalent to $D(q)z = u$, $Y = N(q)z$; $H(s) = C(sI - A)^{-1}B = N(s)D^{-1}(s)$. Let M(s) be such that

$$(sI-A)M(s) = BD(s)$$

is satisfied, or,

$$M(s)D^{-1}(s) = (sI-A)^{-1}B.$$

Assume that it is possible to select (s_j, a_j) so that $CM(s_j)a_j = N(s_j)a_j = 0$. Now if (s_j, a_j) is used in (5.9) or (5.12) to determine F, then s_j will be an unobservable closed-loop eigenvalue. This is because of the fact that its eigenvectors $M(s_j)a_j$ satisfies $CM(s_j)a_j = 0$; see PBH test below. This can be used to derive solutions for problems such as diagonal decoupling and disturbance decoupling, among others.

Example 5.9

Let $H(s) = N(s)D^{-1}(s) = \dfrac{s+1}{s^2+2s+2}$, and the corresponding state space model is

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, B = [0\ 1]', \text{ and } C = [1\ 1]$$

Here, $CM(s) = N(s) = s+1$ and $CM(-1) = 0$. Obviously, if a desired closed-loop pole is chosen at -1, it will be unobservable. Indeed, if the desired closed-loop poles are -1 and

-2, a solution of (5.7) is F = [0 -1], which makes the eigenvalues of (A+BF) = {-1, -2}. The closed-loop transfer function is, however, 1/(s+2). Clearly, the eigenvalue at -1 is unobservable. □

Characteristic Value / Vector Tests for Controllability and Observability - PBH Test

It is known that s_j is an uncontrollable eigenvalue if and only if rank[s_jI-A, B] < rank[sI-A, B] or if and only if there exists a nonzero row vector v_j such that v_j[s_jI - A, B] = 0 (PBH controllability test [11]). The dual result is also true, namely that s_j is an unobservable eigenvalue if and only if rank[(s_jI-A)', C'] < rank[(sI-A)', C'] or if and only if there exists a nonzero column vector v_j such that [(s_jI - A)', C']' v_j = 0 (PBH observability test). These tests can be rather confusing when there are multiple eigenvalues in A; as it is not really clear which one of the multiple eigenvalues is the one that is uncontrollable or unobservable. So instead, many times the uncontrollable eigenvalues are defined by the roots of the determinant of a greatest left divisor of the polynomial matrices sI - A and B; this definition is applicable to polynomial matrix descriptions as well [9-11]. The exact relation between these two different approaches can now be derived. In particular, in view of the results in Section IV, (s_j, v_j) that satisfy [(s_jI - A)', C']' v_j = 0 define a square and nonsingular polynomial matrix that is a right divisor of the columns in [(sI - A)', C']' (see Theorem 4.9); one may have to use the results in the Appendix when the multiplicities of the eigenvalues in question cannot be handled by the results in Section IV. Based on this one can handle now cases of multiple eigenvalues using eigenvalue/eigenvector tests (characteristicvalue/vector tests) [(s_jI - A)', C']' v_j = 0 without confusion or difficulty.

Choosing an appropriate closed loop transfer function matrix

One of the challenging problems in practical control design is to choose an appropriate closed loop transfer function matrix that satisfies all the control specifications such as disturbance rejection, command following, etc. which can be obtained from the given plant by applying an internally stable feedback loop. For example, in the SISO system control design, if the plant has a RHP zero, then the desired close loop transfer function must have the same RHP zero, otherwise, the closed loop system will be internally unstable. Selecting appropriate closed loop transfer matrices is even more difficult for MIMO systems; note that in this case it is possible to have both a pole and a zero at the same location without cancelling each other. To prevent cancelling of the RHP zeros and to guarantee the internal stability of feedback control systems, both locations and directions of the RHP zeros must be considered. This can be best explained in the context of the Stable Model Matching Problem[15]:

Given proper rational matrices H(s) (pxm) and T(s) (pxq), find a proper and stable rational matrix M(s) such that the equation

$$H(s)M(s) = T(s) \tag{5.13}$$

holds. It is known that a stable solution for (5.13) exists if and only if T(s) has as it zeros all the RHP zeros of H(s) together with their directions. Let the coprime fraction representations of H(s) and T(s) be $H(s) = N(s)D^{-1}(s)$ and $T(s) = N_T(s)D_T^{-1}(s)$. The direction associated with a zero of H(s), z_j, is given by the vector a_j which satisfies

$$a_jN(s_j) = 0. \tag{5.14}$$

Furthermore, T(s) will have the same zero, z_i, together with its direction if T(s) satisfies

$$a_jN_T(s_j) = 0. \tag{5.15}$$

Thus, (5.15) must be taken into consideration when T(s) is selected.

Example 5.10

Consider a diagonal T(s); that is the control specificaitons demand diagonal decoupling of the system. Let

$$H(s) = \frac{1}{s+1}\begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix}$$

with a zero at s=1. Then aH(1)=0 gives a=[1 0] and T(s) must satisfy aT(1)=[1 0]T(1)=0. Since T(s) must be diagonal, $t_{11}(1) = 0$; that is the RHP zero of the plant should appear in the (1,1) entry of T(s) only. Certainly T(s) can be chosen to have 1 as a zero in both diagonal entries. However, the RHP zeros are undesirable in control and the minimum possible number should be included in T. □

VI. RATIONAL MATRIX INTERPOLATION - THEORY AND APPLICATIONS

In this section the results on polynomial matrix interpolation derived in previous sections are used to study rational matrix interpolation. In the first part, on theory, it is shown that rational matrix interpolation can be seen as a special case of polynomial matrix interpolation. This result is shown in Theorem 6.1, where the conditions under which a rational matrix H(s) is uniquely represented by interpolation triplets are derived. Theorem 6.1 is the rational interpolation theorem that corresponds to the main interpolation Theorem 2.1. Constraints are incorporated in (6.5) and an alternative form of the theorem is presented in Corollary 6.2. Theorem 6.3 shows the conditions under which the denominator of H(s) can be specified arbitrarily. These results are applied to rational matrix equations and results analogous to the results on polynomial matrix equations derived in the previous sections are obtained.

Theory

Similarly to the polynomial matrix case, the problem here is to represent a (pxm) rational matrix H(s) by interpolation triplets or points (s_j, a_j, b_j) $j = 1, \ell$ which satisfy

$$H(s_j)a_j = b_j \quad j = 1, \ell \tag{6.1}$$

where s_j are complex scalars and $a_j \neq 0$, b_j complex (mx1), (px1) vectors respectively.

It is now shown that interpolation of rational matrices can be studied via the polynomial matrix interpolation results developed above. In fact it is shown below that the rational matrix interpolation problem reduces to a special case of polynomial matrix interpolation.

Write $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ where $\tilde{D}(s)$ and $\tilde{N}(s)$ are (pxp) and (pxm) polynomial matrices respectively. Then (6.1) can be written as $\tilde{N}(s_j)a_j = \tilde{D}(s_j)b_j$ or as

$$[\tilde{N}(s_j), -\tilde{D}(s_j)]\begin{bmatrix} a_j \\ b_j \end{bmatrix} = Q(s_j)c_j = 0 \quad j = 1, \ell \tag{6.2}$$

That is the rational matrix interpolation problem for a pxm rational matrix H(s) can be seen as a polynomial interpolation problem for a px(p+m) polynomial matrix Q(s) := $[\tilde{N}(s), -\tilde{D}(s)]$ with interpolation points $(s_j, c_j, 0) = (s_j, [a_j', b_j']', 0)$ $j = 1, \ell$. There is also the additional constraint that $\tilde{D}^{-1}(s)$ exists. It should be pointed out here that this is a problem similar to the pole assignment problem studied in Section V, where the characteristic values and vectors of Q(s) defined in Section IV were used; the difference here is that Q(s) is not square and nonsingular, however results appropriate for such Q(s) have also been developed above, in Section IV. We shall now apply polynomial interpolation results to (6.2).

Let the column degrees of $Q(s) = [\tilde{N}(s), -\tilde{D}(s)]$ be d_i $i = 1, p+m$. By Corollary 2.2 $\ell = \Sigma d_i$ interpolation points $(s_j, [a_j', b_j']', 0)$ $j = 1, \ell$ together with a given px(p+m) leading coefficient matrix C_c uniquely specify Q(s). It is assumed here, (see Corollary 2.2) that the matrix $S_{1\ell}$ has full rank. Since C_c is chosen, the columns which corresponds to $\tilde{D}(s)$ can of course be arbitrarily selected; for example, they could be taken to be any pxp nonsingular matrix or simply the identity I_p thus guaranteeing that $\tilde{D}^{-1}(s)$ exists.

Alternatively, as it was done in (2.11) ($B_\ell = 0$ case) the additional constraints to be satisfied can be expressed as

$$[\tilde{N}, -\tilde{D}]\ [S_\ell, C] = [0, D] \tag{6.3}$$

where $[\tilde{N}(s), -\tilde{D}(s)] = [\tilde{N}, -\tilde{D}]\ S(s)$ with $S(s) = \text{blk diag}\{[1, s, ..., s^{d_i}]'\}$ $i = 1, p+m$

$$S_\ell := [S(s_1)c_1, ..., S(s_\ell)c_\ell]. \tag{6.4}$$

Here $c_j = [a_j', b_j']'$ and (s_j, c_j) are so that S_ℓ $(\Sigma d_i + (p+m))x\ell$ has full rank ℓ (see Theorem 2.1). Equations $[\tilde{N}, -\tilde{D}]C = D$ express the k additional constraints on the coefficients; k is the number of columns of C or D and it is taken to be $k = (\Sigma d_i + (p+m)) - \ell$. Furthermore C is selected so that rank $[S_\ell, C] = \ell$; in this way a unique solution exists for any D. Since $\tilde{D}(s)$ is a pxp matrix, it is possible to guarantee that the leading coefficient matrix of $\tilde{D}(s)$ is, say, I_p by using p equations (p columns of C). So the number ℓ of interpolation points can be $\ell = \Sigma d_i + m$. These ℓ interpolation points, together with the p constraints to guarantee that $\tilde{D}^{-1}(s)$ exists uniquely define $[\tilde{N}(s), -\tilde{D}(s)]$ and therefore H(s), assuming that $[S_\ell, C]$ has full rank; note that full rank can always be attained if S_ℓ has full column rank. The following theorem has been shown.

Theorem 6.1: Assume that interpolation triplets (s_j, a_j, b_j) $j = 1, \ell$ and nonnegative integers d_i $i = 1, p+m$ with $\ell = \Sigma d_i + m$ are given such that S_ℓ $(\Sigma d_i + (p+m))x\ell$ in (6.4) has full column rank. There exists a unique (pxm) rational matrix H(s) of the form $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ where the column degrees of the polynomial matrix $[\tilde{N}(s), -\tilde{D}(s)]$ are d_i $i = 1, p+m$, with the leading coefficient matrix of $\tilde{D}(s)$ being I_p (nonsingular), which satisfies (6.1). □

When the number of interpolation constraints ℓ on H(s) is less than $\Sigma d_i + m$, additional constraints can be used to impose other properties on H(s). For example, additional linear equations of the form $D(s_j)\alpha_j = 0$ can be added in (6.3) so that H(s) has poles in certain locations. Similarly for zeros of H(s) (see Example 6.2 below). In view of Corollary 2.6 an alternative form for (6.3) is

$$\hat{Q}[S_{d\ell}, C_d] = [0, D_d] \tag{6.5}$$

where d is the degree of $[\tilde{N}(s), -\tilde{D}(s)]$; see Corollary 2.6 and related discussion for details. Here $S_{d\ell}$ is a $((p+m)(d+1)x\ell)$ matrix. Similarly to the above, it is possible with p equations (p columns in C_d or D_d) to guarantee that $\tilde{D}^{-1}(s)$ exists. Therefore one could have $\ell = (p+m)d + m$ interpolation constraints together with the p additional equations to uniquely determine $\hat{Q}$ in (6.5) and therefore H(s). So, the following Corollary has been shown:

Corollary 6.2 Assume that interpolation triplets (s_j, a_j, b_j) $j = 1, \ell$ and nonnegative integers d with $\ell = (p+m)d + m$ are given such that $S_{d\ell}$ $((p+m)(d+1)x\ell)$ in (6.5) has full column rank. There exists a unique (pxm) rational matrix H(s) of the form $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ where the degree of the polynomial matrix $[\tilde{N}(s), -\tilde{D}(s)]$ is d, with the leading coefficient matrix of $\tilde{D}(s)$ being I_p (nonsingular), which satisfies (6.1). □

Example 6.1: Consider a scalar rational H(s) (p=m=1) with first degree numerator and denominator (d=1). Here we can have up to $\ell = (p+m)d + m = 2d + 1 = 3$ interpolation constraints and still guarantee that the denominator exists and it is of degree 1. Let

$\{(s_j, a_j, b_j)\ j = 1, 2, 3\} = \{(0,1,b_1), (1,1,b_2), (-1,1,b_3)\}$

Also let $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s) = (\alpha_1 s + \alpha_0)^{-1}(\beta_1 s + \beta_0)$. Here

$$[\tilde{N}(s), -\tilde{D}(s)] = [\tilde{N}, -\tilde{D}]\ S(s) = [\beta_0, \beta_1, -\alpha_0, -\alpha_1]\begin{bmatrix}1 & 0\\ 2 & 0\\ 0 & 1\\ 0 & s\end{bmatrix}$$

$$c_1 = \begin{bmatrix}1\\ b_1\end{bmatrix}, c_2 = \begin{bmatrix}1\\ b_2\end{bmatrix}, c_3 = \begin{bmatrix}1\\ b_3\end{bmatrix}$$

and

$$[N, -D]\ S_\ell = [\beta_0, \beta_1, -\alpha_0, -\alpha_1]\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -1\\ b_1 & b_2 & b_3\\ 0 & b_2 & -b_3\end{bmatrix} = [0\ 0\ 0]$$

A fourth equation representing additional constraints can be added (see (6.5)) to guarantee, say, $\alpha_1 = 1$. This is equivalent to solving

$$[\beta_0, \beta_1, -\alpha_0]\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -1\\ b_1 & b_2 & b_3\end{bmatrix} = [0\ b_2\ -b_3] \text{ from which}$$

$$[\beta_0, \beta_1, -\alpha_0] = \frac{-1}{2b_1 - b_2 - b_3}\ [b_1(b_3 - b_2), 2b_2b_3 - b_1(b_2+b_3), b_2 - b_3]$$ □

<u>Example 6.2</u>: Consider only the first two interpolation constraints of the previous example and require that $\alpha_1(-3) + \alpha_0 = 0$ or that H(s) has a pole at -3 and $\alpha_1 = 1$. Then

$$[\beta_0, \beta_1, -\alpha_0, -\alpha_1]\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ b_1 & b_2 & 1 & 0\\ 0 & b_2 & -3 & -1\end{bmatrix} = [0\ 0\ 0\ 1]$$

from which

$$[\beta_0, \beta_1, -\alpha_0] = [3b_1, -3b_1+4b_2, -3]$$

That is

$$H(s) = \frac{(-3b_1+4b_2)s + 3b_1}{s + 3}$$

satisfies all constraints. Namely, $H(0) = b_1$, $H(1) = b_2$ and the denominator of H(s) has a zero at -3 (pole of H(s)) with leading coefficient equal to 1. □

<u>Example 6.3</u>: Consider a 2x2 rational matrix $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$. Let $Q(s) = [\tilde{N}(s), -\tilde{D}(s)]$ and $\deg_c Q(s) = \{1\ 0\ 1\ 1\}$. For a solution Q(s) to exists, one needs $\ell \leq \sum d_i + p + m = 3 + 4 = 7$ interpolation triplets (s_j, a_j, b_j) $j = 1, \ell$. Suppose that two interpolation triplets of the form in (6.2) are given as: {(1, [0 1 1 0]', [0 0]'), (2,[1 1 4/3 -1/12]', [0 0]')}. In addition, it is required that H(s) has a zero at s=0 and poles at s=-1 and s=-2 with the their

directions specified as $\tilde{N}(0)[1\ 0]' = [0\ 0]'$, $\tilde{D}(-1)[1\ -1]' = [0\ 0]$ and $\tilde{D}(-2)[0\ 1]' = [0\ 0]$. These constraints can be equivalently expressed as interpolation triplets: $\{(0, [1\ 0\ 0\ 0]', [0\ 0]'), (-1, [0\ 0\ 1\ -1]', [0\ 0]'), (-2, [0\ 0\ 0\ 1]', [0\ 0]')\}$. Now the problem becomes a standard polynomial interpolation problem, i.e. to determine Q(s) s.t. $Q(s_j)c_j = b_j = [0, 0]'$ for j = 1, 5. Let $SJ = \{s_1, ..., s_\ell\}$, $C_\ell = [c_1, ..., c_\ell]$, $B_\ell = [b_1, ..., b_\ell]$. Then $QS_5 = B_5$ (2.5) is to be solved where

$$SJ = \{-2\ -1\ 0\ 1\ 2\ 3\ 4\},\quad B_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\quad C_5 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 4/3 \\ 1 & -1 & 0 & 0 & -1/12 \end{bmatrix}$$

The orthonormal basis of the left null space of S_5 is found to be

$$N_{S\ell} = \begin{bmatrix} 0.0000 & 0.3173 & 0.2522 & 0.0650 & -0.3173 & 0.7646 & 0.3823 \\ 0.0000 & -0.2726 & -0.7151 & 0.4425 & 0.2726 & 0.3396 & 0.1698 \end{bmatrix}.$$

Note that in general $N_{S\ell}$ is a $((\Sigma d_i+m)\text{-rank}\{S_\ell\}) \times (\Sigma d_i+m)$ matrix and all solutions of (2.5) with $B_\ell = 0$ can be characterized as $Q = M\, N_{S\ell}$ where M is any $p\text{x}((\Sigma d_i+m)\text{-rank}\{S_\ell\})$ real matrix. In this example M can be simply chosen as identity matrix, that is $Q = N_{S\ell}$, since $((\Sigma d_i+m)\text{-rank}\{S_\ell\}) = 7 - 5 = 2 = p$. Therefore,

$$Q(s) = QS(s) = \begin{bmatrix} 0.3173s & 0.2522 & -0.3173s+0.0650 & 0.3823s+0.7646 \\ -0.2726s & -0.7151 & 0.2726s+0.4425 & 0.1698s+0.3396 \end{bmatrix} = [\tilde{N}(s), -\tilde{D}(s)]$$

It can be easily verified that the resulting transfer matrix $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ has a zero at s=0 and poles at s=-1, -2.

To uniquely determine Q(s) in this example, two additional constraints in the form of (6.3) : $\{(3, [0\ 1\ 0\ 0]', [2\ 1]'), (4,[0\ 1\ 1\ 1]', [-3\ -6]')\}$ are imposed which lead to

$$SJ = \{-2\ -1\ 0\ 1\ 2\ 3\ 4\},\quad B_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -6 \end{bmatrix},\quad C_7 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 4/3 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1/12 & 0 & 1 \end{bmatrix}$$

by solving (2.5),

$$Q = \begin{bmatrix} 0 & 1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -2 & -1 \end{bmatrix},\ \text{and}\ Q(s) = QS(s) = \begin{bmatrix} s & 2 & -(s+1) & 0 \\ 0 & 1 & -1 & -(s+2) \end{bmatrix}$$

therefore,

$$H(s) = \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} s & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{s}{s+1} & \frac{2}{s+1} \\ \frac{-s}{(s+1)(s+2)} & \frac{s-1}{(s+1)(s+2)} \end{bmatrix}. \qquad \square$$

If it is desired that the denominator of H(s) be completely determined in advance, then this can be expressed in terms of equations (6.3) or (6.5). It is also possible to directly show this result based on Theorem 2.1. In particular

Theorem 6.3: Assume that interpolation triplets (s_j, c_j, b_j) $j = 1, \ell$ $c_j \neq 0$ and m nonnegative integers d_i $i = 1, m$ with $\ell = \sum d_i + m$ are given together with an (mxm) polynomial matrix D(s), $|D(s_j)| \neq 0$, such that the S_ℓ matrix in (2.2) with $a_j := [D(s_j)]^{-1} c_j$ has full rank. Then there exists a unique (pxm) rational matrix H(s) of the form $H(s) = N(s)D(s)^{-1}$, where the polynomial matrix N(s) has column degrees $\deg_{ci}[N(s)] = d_i$, $i = 1, m$ for which

$$H(s_j)c_j = b_j \qquad j = 1, \ell \tag{6.6}$$

Proof: Let N(s) = NS(s) as in Theorem 2.1. The proof is similar also. Notice that (6.6) implies $NS_\ell = B_\ell$ with $a_j = [D(s_j)]^{-1} c_j$ in S_ℓ of (2.2). $\square$

The mxm denominator matrix D(s) is arbitrarily chosen subject only to $|D(s_j)| \neq 0$. This offers great flexibility in rational interpolation. It should be pointed out that the matrix denominator D(s) is much more general than the commonly used scalar one d(s), since D(s) $\doteq$ d(s)I is clearly a special case of matrices D(s) with desired zeros of determinant; note that in this case $|D(s)| = d(s)^m$ that is, the zeros of $|D(s)|$ are all the zeros of d(s) each repeated m times.

As it was shown above, rational matrix interpolation results are directly derived from corresponding polynomial matrix interpolation results and all results of Section II (Sections III - V) can therefore be extended to the rational matrix case. One could of course use the results of Corollaries 2.5 to 2.7 and 2.8 to obtain alternative approaches to rational matrix interpolation.

Example 6.4: Consider the scalar rational example discussed above. Here $\ell = \sum d_i + m = 1 + 1 = 2$ and $S(s) = [1 \;\; s]'$. Consider interpolation points $(0,1,b_1)$ and $(1,1,b_2)$ as above and let the desired denominator be D(s) = s + 3. Then $c_1 = D^{-1}(0)a_1 = 1/3$, $c_2 = D^{-1}(1)a_2 = 1/4$ and

$$NS_\ell = [\beta_0, \beta_1]\ [S(0)c_1, S(1)c_2] = [\beta_0, \beta_1]\begin{bmatrix} 1/3 & 1/4 \\ 0 & 1/4 \end{bmatrix} = [b_1, b_2] = B_2$$

from which $[\beta_0, \beta_1] = [3b_1, -3b_1 + 4b_2]$. That is

$$H(s) = \frac{(-3b_1+4b_2)s + 3b_1}{s + 3}$$

satisfies all the constraints. Note that it is the same H(s) as in Example 6.2 even though the constraints were imposed via different approaches. □

Applications - Rational Matrix Equations

Now let's consider the rational matrix equation:

$$M(s)L(s) = Q(s) \tag{6.7}$$

where L(s) (txm) and Q(s) (kxm) are given rational matrices. The polynomial matrix interpolation theory developed above will now be used to solve this equation and determine the rational matrix solutions M(s) (kxt). Let $M(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$, a polynomial fraction form of M(s) to be determined. Then equation (6.7) can be written as:

$$[\tilde{N}(s) \quad -\tilde{D}(s)]\begin{bmatrix} L(s) \\ Q(s) \end{bmatrix} = 0 \tag{6.8}$$

Note that instead of solving (6.8) one could equivalently solve

$$[\tilde{N}(s) \quad -\tilde{D}(s)]\begin{bmatrix} L_p(s) \\ Q_p(s) \end{bmatrix} = 0 \tag{6.9}$$

where $[L_p(s)'\ Q_p(s)']' = [L(s)'\ Q(s)']'\phi(s)$ a polynomial matrix with $\phi(s)$ the least common denominator of all entries of L(s) and Q(s); in general, $\phi(s)$ could be any denominator in a right fractional representation of $[L(s)', Q(s)']'$. The problem to be solved is now (3.1), a polynomial matrix equation, where $L(s) = [L_p(s)'\ Q_p(s)']'$ and $Q(s) = 0$. Therefore, Theorem 3.1 does apply and all solutions $[\tilde{N}(s)\ -\tilde{D}(s)]$ of degree r can be determined by solving (3.9) or (3.13). Let $s = s_j$ and postmultiply (6.9) by a_j $j = 1, \ell$ with a_j and ℓ chosen properly (see below). Define

$$c_j := \begin{bmatrix} L_p(s) \\ Q_p(s) \end{bmatrix} a_j \qquad j = 1, \ell \tag{6.10}$$

The problem now is to find a polynomial matrix $[\tilde{N}(s) \;\; -\tilde{D}(s)]$ which satisfies

$$[\tilde{N}(s_j) \;\; -\tilde{D}(s_j)]\, c_j = 0 \qquad j = 1, \ell \tag{6.11}$$

as in (6.2). In fact (6.11) is of the form of (3.11) with $b_j = 0$.

Note that restrictions on the solutions can be easily imposed to guarantee that $\tilde{D}^{-1}(s)$ exists and/or that $M(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ is proper; see also above in this section, also Sections IV and V. The existence of solutions of (6.7) and their causality depends on the given rational matrices L(s) and Q(s) (see for example [12, 15] and references therein). Our approach here will find a proper rational matrix of order r when such solution exists. Additional interpolation type constraints can be added so the solution satisfies additional specifications.

Example 6.5: This is an example of solving the Model Matching Problem [15] using matrix interpolation techniques. Here L(s) and Q(s) are given as:

$$L(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 0 & -2 \\ \frac{-s}{s+1} & -1 \end{bmatrix} \qquad Q(s) = \begin{bmatrix} \frac{s}{s+3} & \frac{s+1}{s+3} \\ -\frac{s}{s+3} & -\frac{3s+7}{s+3} \end{bmatrix}$$

The monic least common denominator of all entries is $\phi(s) = s(s+1)(s+3)$ and therefore

$$\begin{bmatrix} L_p(s) \\ Q_p(s) \end{bmatrix} = \begin{bmatrix} s(s+3) & (s+1)(s+3) \\ 0 & -2s(s+1)(s+3) \\ -s^2(s+3) & -s(s+1)(s+3) \\ s^2(s+1) & s(s+1)^2 \\ -s^2(s+1) & -(3s+7)(s+1)s \end{bmatrix}$$

Let

$\{d_i = \deg_{ci} Q(s)\} = \{0, 0, 1, 1, 0\}$,

$$\ell = \sum d_i + t + k = 2 + 5 = 7,$$
$$\{s_j, j = 1, 5\} = \{-4, -2, 1, 2, 3\},$$
$$\{a_j, j = 1, 5\} = \{[0,1]', [1,0]', [1,1]', [0,-1]', [-1,0]'\}$$
$$\{b_j = [0\ 0]', j = 1, 5\}$$

from which c_j $j = 1, 5$ are obtained

$$[c_1, \ldots, c_5] = \begin{bmatrix} 3 & -2 & 12 & -15 & -18 \\ 24 & 0 & -16 & 60 & 0 \\ 12 & -4 & -12 & 30 & 54 \\ -36 & -4 & 6 & -18 & -36 \\ 60 & 4 & -22 & 78 & 36 \end{bmatrix}$$

Assume two additional constraints are introduced in the form of: $\{s_6, s_7\} = \{4, 5\}$, $\{c_6, c_7\} = \{[0\ 1\ 0\ 0\ 0]', [0\ 0\ 0\ 1\ 0]'\}$ and $\{b_6, b_7\} = \{[1\ 0]', [-1, -8]'\}$. Now, solving the polynomial matrix interpolation problem: $[\tilde{N}(s_j)\ \ -\tilde{D}(s_j)]c_j = b_j$ $j = 1, 7$, we obtained

$$[\tilde{N}(s)\ \ -\tilde{D}(s)] = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -(s+1) & -(s+3) & 0 \end{bmatrix}$$

which gives

$$M(s) = \begin{bmatrix} 1 & 1 \\ s+1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -(s+1) \end{bmatrix} \qquad \square$$

VII. CONCLUDING REMARKS

Some of the concepts and ideas presented here have appeared elsewhere. It is the first time however that the theory of polynomial and rational matrix interpolation in its complete form has appeared in the literature. The algorithms have been implemented in Matlab and are available upon request.

Interpolation is a very general and flexible way to deal with problems involving polynomial and rational matrices and the results presented here provide an appropriate theoretical setting and algorithms to deal effectively with such problems. At the same time it is also felt that the results presented here have only opened the way, as there are many more results that can and need be developed to handle the wide range of problems possible to study via polynomial and rational matrix interpolation theory.

Finally it should be noted that the rational interpolation results presented here compliment results that have appeared in the literature. The exact relationship is under investigation and new insight into the theory are certainly possible.

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APPENDIX A

In this Appendix, the general versions of the results in Section IV that are valid for repeated values of s_j, with multiplicities beyond those handled in Section IV, are stated. Detailed proofs of these results can be found in our main reference for characteristic values and vectors [1].

Let Q(s) be an (mxm) nonsingular matrix and let $Q^{(k)}(s_j)$ denote the kth derivative of Q(s) evaluated at $s = s_j$. If s_j is a zero of $|Q(s)|$ repeated n_j times, define n_j to be the algebraic multiplicity of s_j; define also the geometric multiplicity of s_j as the quantity (m-rank $Q(s_j)$).

<u>Theorem A.1</u> [1, Theorem 1]: There exist complex scalar s_j and $\sum_{i=1}^{\ell_j} k_{ij}$ mx1 nonzero vectors $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^{k_{ij}}$ $i = 1, \ell_j$ which satisfy

$$
\begin{aligned}
&Q(s_j)a_{ij}^1 = 0 \\
&Q(s_j)a_{ij}^2 = - Q^{(1)}(s_j)a_{ij}^1 \\
&\quad\vdots \\
&Q(s_j)a_{ij}^{k_{ij}} = - [Q^{(1)}(s_j)a_{ij}^{k_{ij}-1} + \ldots + \frac{1}{(k_{ij} - 1)!}Q^{(k_{ij}-1)}(s_j)a_{ij}^1]
\end{aligned}
\tag{A.1}
$$

with $a_{1j}^1, a_{2j}^1, \ldots, a_{\ell_j j}^1$ linearly independent if and only if s_j is a zero of $|Q(s)|$ with algebraic multiplicity $(=n_j) \geq \sum_{i=1}^{\ell_j} k_{ij}$ and geometric multiplicity (=(m-rank $Q(s_j)$)) $\geq \ell_j$. □

It is of interest to note that there are ℓ_j chains of (generalized) characteristic vectors corresponding to s_j., each of length k_{ij} Notice that Theorem 4.2 is a special case of this theorem; it involves only the top equation in (A.1) and it does not involve derivatives of Q(s). The proof of Theorem A.1 is based on the following lemma:

Lemma A.2 [1, Lemma 2]: Theorem A.1 is satisfied for given Q(s), s_j and $a_{\ell jj}^{k}$ if and only if it is satisfied for U(s)Q(s), s_j and $a_{\ell jj}^{k}$ where U(s) is any unimodular matrix (that is $|U(s)| = \alpha$, a nonzero scalar). □

This lemma allows one to carry on the proof of Theorem A.1 with a matrix Q(s) which is column proper (reduced). The proof of Theorem A.1 is rather involved and it involves the generalized eigenvectors of a real matrix associated with Q(s); it can of course be found in [1].

Given Q(s), if s_j and $a_{\ell jj}^{k}$ satisfy the conditions of Theorem A.1, then this implies certain structure for the Smith form of Q(s). First, let us define the (unique) Smith form of a polynomial matrix.

Smith Form of M(s) [9,11]

Given a pxm polynomial matrix M(s) with rankM(s) = r, there exist unimodular matrices U_1 , U_2 such that $U_1(s)M(s)U_2(s) = E(s)$ where

$$E(s) = \begin{bmatrix} \Lambda(s) & 0 \\ 0 & 0 \end{bmatrix} \qquad \Lambda(s) = \mathrm{diag}[\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_r(s)] \tag{A.2}$$

Each ε_i i = 1, r is a unique monic polynomial satisfying $\varepsilon_i(s) \mid \varepsilon_{i+1}(s)$ i = 1, r-1 where $p_2 \mid p_1$ means that there exists polynomial p_3 such that $p_1 = p_2 p_3$; that is ε_i divides ε_{i+1} . E(s) is the *Smith form of M(s)* and $\varepsilon_i(s)$ are the *invariant polynomials of M(s).* It can be shown that

$$\varepsilon_i(s) = D_i(s) / D_{i-1}(s) \quad i = 1, r \tag{A.3}$$

where $D_i(s)$ is the monic greatest common divisor of all the ith order minors of M(s); note that $D_0(s) = 0$. $D_i(s)$ are the *determinantal divisors of M(s).*

Corollary A.3 : [1, Corollary 3] Given Q(s), there exist a scalar s_j and nonzero vectors a_{ij}^{1}, a_{ij}^{2}, ..., $a_{ij}^{k_{ij}}$ i = 1, ℓ_j which satisfy the conditions of Theorem A.1 if and only if the Smith form of Q(s) contains the factors $(s - s_j)^{k_{ij}}$ i = 1, ℓ_j in ℓ_j separate locations on the diagonal; that is $(s - s_j)^{k_{ij}}$ is a factor in ℓ_j distinct invariant polynomials of Q(s). □

Theorem A.1 and Corollary A.3 refer to the value s_j, a root of $|Q(s)|$ which is repeated at least $\sum_{i=1}^{\ell_j} k_{ij}$ times. If σ distinct values s_j are given then the following result is derived. Note that the $\deg|Q(s)|$ is assumed to be known.

Theorem A.4 [1, Theorem 4]: Let $n = \deg|Q(s)|$. There exist σ distinct complex scalars s_j and n nonzero vectors $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^{k_{ij}}$ $i = 1, \ell_j$, $j = 1, \sigma$ with $\sum_{j=1}^{\sigma} \sum_{i=1}^{\ell_j} k_{ij} = n$ with each of the σ sets $\{a_{1j}^1, a_{2j}^1, \ldots, a_{\ell_j j}^1\}$ linearly independent for $j = 1, \sigma$ that satisfy (A.1) if and only if the zeros of $|Q(s)|$ have σ distinct values s_j $j = 1, \sigma$ each with algebraic multiplicity $(= n_j) = \sum_{i=1}^{\ell_j} k_{ij}$ and geometric multiplicity $(= m - \operatorname{rank} Q(s_j)) = \ell_j$. □

Note that to each distinct characteristic value s_j there correspond $\{ a_{1j}^1, a_{ij}^2, \ldots, a_{1j}^{k_{1j}} \}$ $\ldots, \{ a_{\ell_j j}^1, a_{\ell_j j}^2, \ldots, a_{\ell_j j}^{k_{\ell_j j}} \}$ characteristic vectors; there are ℓ_j $(= m - \operatorname{rank} Q(s_j) =$ geometric multiplicity) chains of length $k_{1j}, k_{2j}, \ldots, k_{\ell_j j}$ for a total of $\sum_{i=1}^{\ell_j} k_{ij}$ characteristic vectors equal to the algebraic multiplicity n_j.

Corollary A.5: [1, Corollary 5]. Given Q(s) with $n = \deg|Q(s)|$, there exist σ distinct complex scalars s_j and vectors a_{ij}^k $i = 1, \ell_j$ $k = 1, k_{ij}$ $j = 1, \sigma$ which satisfy the conditions of Theorem A.4 if and only the Smith form of Q(s) consists of factors $(s - s_j)^{k_{ij}}$ $i = 1, \ell_j$ in ℓ_j separate locations on the diagonal ($j = 1, \sigma$). □

Note that in view of the divisibility property of the invariant factors of Q(s), if the conditions of Corollary A.5 or similarly of Theorem A.4 are satisfied, the Smith form of Q(s) is uniquely determined. In particular, for $k_{1j} \le k_{2j} \le \ldots \le k_{\ell_j j}$, the Smith form of Q(s) in this case has the form

$$E(s) = \operatorname{diag} (\varepsilon_1 (s), \ldots, \varepsilon_m (s))$$

$$\varepsilon_m (s) = (s - s_j)^{k_{\ell_j j}} (\cdot), \varepsilon_{m-1} (s) = (s - s_j)^{k_{\ell_j - 1 j}} (\cdot), \ldots, \varepsilon_{m-(\ell_j - 1)} (s) = (s - s_j)^{k_{1j}} (\cdot) \quad (A.4)$$

with $\varepsilon_{m-\ell_j}(s) = ... = \varepsilon_1(s) = 1$. This is repeated for each distinct value of s_j $j = 1, \sigma$ until the Smith form is completely determined.

<u>Example A.1</u> To illustrate the above results consider

$$Q(s) = \begin{bmatrix} s^2 & -1 \\ 0 & s \end{bmatrix}.$$

Notice that

$$Q^{(1)}(s) = \begin{bmatrix} 2s & 0 \\ 0 & 1 \end{bmatrix}, \; Q^{(1)}(s) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \; Q^{(k)}(s) = 0 \text{ for } k > 2.$$

For $s_1 = 0$ ($j = 1$), relations (A.1) become:

Let $i = 1$. $Q(0)a_{11}^1 = 0$ implies $a_{11}^1 = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ ($\alpha \neq 0$); Note that no other linearly independent a_{i1}^1 exists, so $\ell_j = 1$.

$$Q(0)a_{11}^2 = -Q^{(1)}(0)a_{11}^1 \text{ implies } a_{11}^2 = \begin{bmatrix} \beta \\ 0 \end{bmatrix} (\beta \neq 0)$$

$$Q(0)a_{11}^{k_{11}} = -[Q^{(1)}(0)a_{11}^2 + \frac{1}{(2)!}Q^{(2)}(0)a_{11}^1] \text{ implies } a_{11}^3 = \begin{bmatrix} \gamma \\ \alpha \end{bmatrix} (\alpha\gamma \neq 0).$$

It can be verified that a_{11}^4 etc are zero. So $k_{11} = 3$. Note that m-rank$Q(0) = 2-1 = 1 = \ell_1$, that is the geometric multiplicity of $s_1 = 0$ is 1 and so no other chain of characteristic vectors associated with $s_1 = 0$ exists.

Assume that $Q(s)$ is not known and it is given that $s_1 = 0$ and a_{11}^k $k = 1, 2, 3$ satisfy (A.1). Then according to Theorem A.1, the algebraic multiplicity of $s_1 = 0$ is at least 3 ($=k_{11}$) and the geometric multiplicity is at least 1 ($=\ell_1$). Furthermore, in view of Corollary A.3 the factor $s^3(= (s-s_1)^{k_{11}})$ appears in 1 ($=\ell_1$) location in the Smith form of $Q(s)$.

Assume now that $n = \deg|Q(s)| = 3$ is also given together with $s_1 = 0$ and a_{11}^k $k = 1, 2, 3$ which satisfy (A.1). Notice that here $\ell_1 = 1$, $k_{11} = 3$ (see above) so $k_{11} = 3 = n$ which implies that $\sigma = 1$, or $s_1 = 0$ is the only distinct root of $|Q(s)|$. Theorem A.4 can now be applied to show that $s_1 = 0$ has algebraic multiplicity exactly equal to $k_{11} = 3$ and geometric multiplicity exactly equal to $\ell_1 = 1$. These can be easily verified from the given $Q(s)$. In view of Corollary A.5 and (A.4) the Smith form of $Q(s)$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix}$$

which can also be derived from Q(s) via pre and post multiplication by unimodular matrices. □

The following lemma highlights the fact that the conditions of Theorem A.4 specify Q(s) within a unimodular premultiplication; see also Lemma 4.6.

Lemma A.6: Theorem A.4 is satisfied by a matrix Q(s) if and only if it is satisfied by U(s)Q(s) where U(s) is any unimodular matrix. □

It is important at this point to briefly discuss and illustrate the results so far: Assume that, for an (mxm) polynomial matrix Q(s) yet to be chosen, we have decided upon the degree of $|Q(s)|$ as well as its zero locations - that is about n, s_j and the algebraic multiplicities n_j. Clearly there are many matrices that satisfy these requirements; consider for example all the diagonal matrices that satisfy these requirements. If we specify the geometric multiplicities ℓ_j as well, then this implies that the matrices Q(s) must satisfy certain structural requirements so that m-rank$Q(s_j) = \ell_j$ is satisfied; in our example the diagonal matrix, the factors $(s-s_j)$ must be appropriately distributed on the diagonal. If k_{ij} are also chosen, then the Smith form of Q(s) is completely defined, that is Q(s) is defined within pre and post unimodular matrix multiplications. Note that this is equivalent to imposing the restriction that Q(s) must satisfy n relations of type (A.1), as in Theorem A.4, without fixing the vectors a_{ij}^k (see Example A.1). If in addition a_{ij}^k are completely specified then Q(s) is determined within a unimodular premultiplication; see Lemma A.6.

Given (mxm) Q(s), let $n = \deg|Q(s)|$ and assume that Q(s) and s_j, a_{ij}^k satisfy the conditions of Theorem A.4; that is they satisfy (A.1) for σ distinct s_j $j = 1, \sigma$.

Theorem A.7 [1, Theorem 6]: Q(s) is a right divisor (rd) of an (rxm) polynomial matrix M(s) if and only if M(s) satisfies the conditions of Theorem A.4 with the same s_j and a_{ij}^k; that is M(s) also satisfies the conditions (A.1) with the same s_j, a_{ij}^k for σ distinct s_j $j = 1,\sigma$.

Proof: Necessity: If Q is a rd of M, $M = \hat{M}Q$. then it can be shown directly that (A.1) are also satisfied by M(s) with the same s_j and a_{ij}^k. Sufficiency: Same as the sufficiency proof of Theorem 4.9. □

In the proof of Theorem A.1 [1], the Jordan form of a real matrix A derived from Q(s) was used. Later in the Appendix results concerning the Smith form of Q(s) were described. It is of interest to outline here the exact relations between the Jordan form of A and the Smith form of sI-A and of Q(s). This is done in the following:

Relations Between The Smith and Jordan Forms

Given an mxm nonsingular polynomial matrix Q(s) and a real nxn matrix A, assume that there exist matrices B (nxm) and S(s) (nxm) so that

$$(sI-A)\, S(s) = B\, Q(s) \tag{A.5}$$

where (sI-A), B are left and S(s), Q(s) right coprime. Then there is a direct relation between the Smith forms of (sI-A) and Q(s) as it will be shown. First the relation between the Jordan form of A and the Smith form of (sI-A) is described.

Let A (nxn) have σ distinct eigenvalues s_j each repeated n_j times ($\sum n_j = n$); n_j is the algebraic multiplicityof s_j . The geometric multiplicity of s_j , ℓ_j, is defined as $\ell_j = n-\text{rank}(s_jI-A)$, that is thereduction in rank in sI-A when $s = s_j$. There exists a similarity transformation matrix P such that PA = JP where J is the Jordan canonical form of A.

$$J = \text{diag}[J_j],\ J_j = \text{diag}[J_{ij}] \tag{A.6}$$

where J_j (n_jxn_j) j = 1, σ is the block diagonal matrix associated with s_j; J_j has ℓ_j ($\leq n_j$) matrices J_{ij} (k_{ij}xk_{ij}) i = 1,ℓ_j on the diagonal each of the form

$$J_{ij} = \begin{bmatrix} s_j & 1 & 0 & \dots & 0 \\ 0 & s_j & 1 & \dots & 0 \\ \dots & & & & \\ & & & & 1 \\ 0 & 0 & \dots & & s_j \end{bmatrix} \tag{A.7}$$

where $\sum_{1}^{\ell_j} k_{ij} = n_j$.

The structure of J is determined by the generalized eigenvectors v_{ij}^{k} of A; they are used to construct P. To each distinct eigenvalue s_j correspond ℓ_j chains of generalized eigenvectors eachof length k_{ij} i = 1, ℓ_j for a total of n_j linearly independent generaliezed eigenvectors.

Note that the characteristic polynomial of A, $\alpha(s)$, is

$$\alpha(s) = \prod_{i=1}^{\sigma} (s-s_j)^{n_j} \qquad (= |sI-A|)$$

while the minimal polynomial of A, $\alpha_m(s)$, is $\prod_{i=1}^{\sigma} (s\text{-}s_j)^{\bar{n}_j}$ where $\bar{n} := \max_i k_{ij}$, that is the dimension of the largest block in J associated with s_j.

The Smith form of a polynomial matrix was defined above. It is not difficult to show the following result about the Smith form of sI-A, $E_A(s)$ [1]: Without loss of generality, assume that $k_{1j} \le k_{2j} \le \ldots \le k_{\ell_j j}$ $(=\bar{n})$, see also (A.4). If $E_A(s) = \text{diag}[\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_r(s)]$, then

$$\varepsilon_n(s) = (s - s_j)^{k_{\ell_j j}}(\,.\,),\ \varepsilon_{n-1}(s) = (s - s_j)^{k_{\ell_{(j-1)} j}}(\,.\,), \ldots, \varepsilon_{n-(\ell_j - 1)}(s) = (s - s_j)^{k_{1j}}(\,.\,) \qquad \text{(A.8)}$$

with $\varepsilon_{n-\ell_j}(s) = \ldots = \varepsilon_1(s) = 1$. That is the n_j factor $(s\text{-}s_j)$ are factors of the ℓ_j invariant polynomials $\varepsilon_{n-(\ell_j-1)}(s), \ldots, \varepsilon_n(s)$; the exponents k_{ij} of $(s\text{-}s_j)$ are the dimensions of the matrices J_{ij} $i = 1, \ell_j$ of the Jordan canonical form, or equivalently they are the lengths of the chains of the generalized eigenvectors of A corresponding to s_j. The relations in (A.8) are of course repeated for each distinct value of s_j $j = 1, \sigma$ until the Smith form $E_A(s)$ is completely determined.

<u>Example A.2</u> Let

$$A = J = \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix} = \begin{bmatrix} J_{11} & & \\ & J_{21} & \\ & & J_{12} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

that is $s_1 = -3$, $n_1 = 3$, $\ell_1 = 2$ with $k_{11} = 2$, $k_{21} = 1$; $s_2 = -1$, $n_2 = \ell_2 = k_{12} = 1$. In view of (A.8), the Smith form of sI-A is

$$E_A(s) = \begin{bmatrix} \varepsilon_1(s) & & & \\ & \varepsilon_2(s) & & \\ & & \varepsilon_3(s) & \\ & & & \varepsilon_4(s) \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & s-3 & \\ & & & (s-3)^2(s-1) \end{bmatrix}$$

Here $\alpha(s) = |sI - A| = (s\text{-}3)^3(s\text{-}1)$ and $\alpha_m(s) = (s\text{-}3)^3(s\text{-}1)$. □

It is can be shown [9-11] that if sI-A and Q(s) satisfy relation (A.5), then the matrices

$$\begin{bmatrix} sI\text{-}A & B \\ -I_n & 0 \end{bmatrix}, \qquad \begin{bmatrix} I_{n-m} & 0 & 0 \\ 0 & Q(s) & I_m \\ 0 & -S(s) & 0 \end{bmatrix}$$

are unimodularly equivalent and they have the same Smith forms. That is, if $E_Q(s)$ is the Smith form of Q(s), then

$$E_A(s) = \begin{bmatrix} I_{n-m} & 0 \\ 0 & E_Q(s) \end{bmatrix} \qquad \text{(A.9)}$$

It is now easy to show that Q(s) has σ distinct roots s_j of $|Q(s)|$ each repeated n_j times (= algebraic multiplicity as defined before Theorem A.1); the geometric multiplicity of s_j defined by m - rank $Q(s_j)$ equals ℓ_j since $\ell_j = m$ - rank $E_Q(s)$. If

$E_Q(s) = \text{diag}(\bar{\varepsilon}_1(s), \ldots, \bar{\varepsilon}_m(s))$, then (see also (A.4)) for $k_{1j} \leq k_{2j} \leq \ldots \leq k_{\ell_j j}$ $(=\bar{n})$

$$\bar{\varepsilon}_m(s) = (s - s_j)^{k_{\ell_j j}}(\,.\,),\ \bar{\varepsilon}_{m-1}(s) = (s - s_j)^{k_{\ell(j-1) j}}(\,.\,), \ldots, \bar{\varepsilon}_{m-(\ell_j-1)}(s) = (s - s_j)^{k_{1j}}(\,.\,) \quad (A.10)$$

with $\bar{\varepsilon}_{n-\ell_j}(s) = \ldots = \bar{\varepsilon}_1(s) = 1$. Compare with the Smith form $E_A(s)$ in (A.8). It is clear that $E_Q(s)$ and $E_A(s)$ or Q(s) and (sI-A) have the same nonunity invariant polynomials as it is of course clear in view of (A.9). Note that the characteristic polynomial of Q(s) is in this case $\delta(s) = |Q(s)| = \prod_{j=1}^{\delta}(s - s_j)^{n_j}$ $(= \alpha(s) = |sI-A|)$ while the minimal polynomial of Q(s) is

$$\delta_m(s) = |Q(s)| = \prod_{j=1}^{\delta}(s - s_j)^{\bar{n}_j} \quad (= \alpha_m(s)).$$

<u>Example A.3</u> Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $Q(s) = \begin{bmatrix} s^2 & -1 \\ 0 & s \end{bmatrix}$. Note that if $S(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, (sI-A)S(s) = BQ(s) as in (A.5) with (sI- A), B left coprime and S(s), Q(s) right coprime. Notice that A is already in Jordan canonical form. In fact, $A = J = J_1$ with $s_1 = 0$, $\ell_1 = 1$, $k_{11} = 3$ and $n_1 = 3$. The Smith form of sI-A is then (A.8)

$$E_A(s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & s-3 & \\ & & & s^3 \end{bmatrix}$$

In view of (A.10), the Smith form of Q(s) is

$$E_Q(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix}$$

Note that this Q(s) was also studied in Example A.1 □

APPENDIX B

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Polynomial Matrix Characterization Using Characteristic Values and Vectors.

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Introduction

In control problems like pole allocation and regulation with stability the desired compensator is the solution of an equation involving polynomial matrices. To solve these equations one can use the coefficients of the polynomial entries of the matrices after reducing them to some canonical form. Many times this is difficult or even impossible. When one works with equations involving just polynomials one can work either with the coefficients or with the values obtained when certain values of the indeterminant s are "plugged in" ; the latter corresponds to polynomial representation by a number of points (using interpolation the coefficients can be determined). The motivation of this work is exactly this. To establish the necessary theoretical background so that equations involving polynomial matrices can be solved using "plug in" values.

Note that the zeros of the determinant of a polynomial matrix $P(s)$ alone do not fully characterize $P(s)$. Information about the structure is also necessary. Characteristic vectors or latent vectors or simply vectors $a_i^{k,j}$ are introduced to accommodate this in Theorems 1 and 4. The relation between $a_i^{k,j}$ and the Smith form of $P(s)$ is established in Corollaries 3 and 5. Theorem 6, Corollary 7 and Lemma 8 establish the relations between two polynomial matrices when they both satify relations of certain type involving $a_i^{k,j}$. Actually, when the two matrices satisfy exactly the same relations, then they are related by a unimodular premultiplication, which is the generalization of the case when two polynomials of the same degree have the same roots; they are equal within a constant multiplication. Finally the relation between $a_i^{k,j}$ and the generalized eigenvectors of an equivalent to $\{P,Q,R,W\}$ system $\{A,B,C,E\}$ is established. The Appendix contains a detailed account of relations between several canonical forms of A (and $P(s)$) as well as a number of definitions.

It should be noted that the results presented here are of interest not only because of their relation to control applications or their relation to the solution of equations involving polynomial matrices. They are also of interest in their own right because they rigorously establish the relation between $P(s)$ and its "characteristic" values and vectors and by doing so, they generalize and combine results known from the eigenvalue-eigenvector theory of real matrices $(P(s) = sI - A)$ and the theory of polynomials $(P(s) = p(s))$.

Main Results

Let $P(s)$ be an $(m \times m)$ nonsingular matrix and let $P^{(k)}(s_i)$ denote the k^{th} derivative of $P(s)$ evaluated at $s = s_i$. If s_i is a zero of $|P(s)|$ repeated n_i times, define n_i to be the algebraic multiplicity of s_i; define also the **geometric multiplicity** of s_i as the quantity m-rank $P(s_i)$ (see Appendix).

Theorem 1 There exist $(m \times 1)$ nonzero vectors $a_i^{1,j}, a_i^{2,j}, \ldots, a_i^{k_i^j,j}$ $j = 1,2,\ldots,\ell_i$ which satisfy

$$
\begin{aligned}
P(s_i)a_i^{1,j} &= 0 \\
P(s_i)a_i^{2,j} &= -P^{(1)}(s_i)a_i^{1,j} \\
&\vdots \\
P(s_i)a_i^{k_i^j,j} &= -[P^{(1)}(s_i)a_i^{k_i^j-1,j} + \ldots + \frac{1}{(k_i^j-1)!}P^{(k_i^j-1)}(s_i)a_i^{1,j}]
\end{aligned}
\tag{1}
$$

with $a_i^{1,1}, a_i^{1,2}, \ldots, a_i^{1,\ell_i}$ linearly independent if and only if s_i is a zero of $|P(s)|$ with algebraic multiplicity $n_i \geq \sum_{j=1}^{\ell_i} k_i^j$ and geometric multiplicity m-rank $P(s_i) \geq \ell_i$.

Lemma 2 Theorem 1 is satisfied for $P(s)$ and $a_i^{k,j}$ if and only if it is satified for $U(s)P(s)$ and $a_i^{k,j}$ where $U(s)$ is any unimodular matrix.

Proof First note that P and UP have exactly the same zeros of determinant with the same algebraic and geometric multiplicities.

Assume that P and $a_i^{k,j}$ satisfy (1). Then $U(s_i)P(s_i)a_i^{1,j} = UP(s_i)a_i^{1,j} = 0$, $-(UP)^{(1)}(s_i)a_i^{1.j} = -[U^{(1)}(s_i)P(s_i) + U(s_i)P^{(1)}(s_i)]a_i^{1,j} = -U(s_i)P^{(1)}(s_i)a_i^{1,j} = U(s_i)P(s_i)a_i^{2,j} = UP(s_i)a_i^{2,j}$ etc.
That is UP and $a_i^{k,j}$ also satisfy (1). The sufficiency proof is similar since $U^{-1}(s)$ is also a unimodular matrix QED.

It is known [3] that given $P(s)$ there exists a unimodular matrix $U(s)$ such that UP is column proper i.e. $C_c[UP(s)]$, the matrix with entries the coefficients of the highest power of s in each column of UP is of full rank. It is therefore clear, in view of Lemma 2, that without loss of generality we can assume in the proof of Theorem 1 that $P(s)$ is a column proper matrix.

<u>Proof of Theorem 1</u> Let $d_1, d_2, .., d_m$ denote the column degrees of (the comumn proper matrix) $P(s)$. Write

$$P(s) = B_m^{-1} [\text{diag } (s^{d_i}) - A_m S(s)]$$

where $S(s) \triangleq \text{diag } ([1, s, .., s^{d_i - 1}]^T)$ and A_m, B_m † appropriate real matrices and observe that

$$B_c P(s) = (s - A_c) S(s) \tag{2}$$

where A_c and B_c are given by (A6); the column degrees d_i are the controllability indices of (A_c, B_c) while the above defined A_m and B_m make up the "nontrivial" rows of A_c, B_c. Repeated differentiations of (2) give

$$B_c P^{(k)}(s) = (s - A_c) S^{(k)}(s) + k\, S^{(k-1)}(s) \qquad k=1,2,.., \tag{3}$$

---

† $B_m^{-1} = C_c[P(s)]$; without loss of generality it is assumed that $C_c[P(s)]$ is in upper triangular form with 1s on the diagonal.

Assume that vectors $a_i^{k,j}$ which satisfy (1) have been found. Premultiply the relations in (1) by B_c and use (2) and (3) to substitute $P(s)$ and its derivatives.

Then

$$
\begin{aligned}
(s_i - A_c)v_i^{1,j} &= 0 \\
(s_i - A_c)v_i^{2,j} &= -v_i^{1,j} \\
&\vdots \\
(s_i - A_c)v_i^{k_i^j,j} &= -v_i^{k_i^j-1,j}
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
v_i^{1,j} &= S(s_i)a_i^{1,j} \\
v_i^{2,j} &= [S(s_i)a_i^{2,j} + S^{(1)}(s_i)a_i^{1,j}] \\
&\vdots \\
v_i^{k_i^j,j} &= [S(s_i)a_i^{k_i^j,j} + \ldots + \frac{1}{(k_i^j - 1)!}\, S^{(k_i^j-1)}(s_i)a_i^{1,j}]
\end{aligned}
\tag{5}
$$

(4) implies that $(s_i - A_c)^{k_i^j} v_i^{k_i^j,j} = 0$ and $(s_i - A_c)^{k_i^j-1} v_i^{k_i^j,j} = v_i^{1,j} = S(s_i)a_i^{1,j} \neq 0$; furthermore $v_i^{1,1}, \ldots, v_i^{1,\ell_i}$ are linearly independent since $a_i^{1,1}, \ldots, a_i^{1,\ell_i}$ are linearly independent. In view of (A2) $v_i^{1,j}, \ldots, v_i^{k_i^j,j}$ $j=1,2,\ldots,\ell_i$ are ℓ_i chains of generalized eigenvectors of A_c corresponding to an eigenvalue s_i , each chain of length k_i^j $j=1,2,\ldots,\ell_i$. Therefore $|s - A_c|$ (or $|P(s)|$ in view of (2) and the Appendix)has at least $\sum_{j=1}^{\ell_i} k_i^j$ zeros at s_i , which implies that the algebraic multiplicity $n_i \geq \sum_{j=1}^{\ell_i} k_i^j$; furthermore the geometric multiplicity of s_i is at least ℓ_i which implies that $\ell_i \leq n\text{-rank}(s_i - A_c)$ or, in view of the Appendix that $\ell_i \leq m\text{-rank}\, P(s_i)$.

Conversely, assume that s_i is a zero of $|P(s)|$ with algebraic and geometric multiplicities n_i and m-rank $P(s_i)$ respectively. The matrix A_c defined in (2) has an eigenvalue s_i with the same algebraic and geometric multiplicities. Out of the m-rank $P(s_i)$ chains of generalized eigenvectors of A_c which correspond to s_i, one can always choose ℓ_i ($\le$ m-rank $P(s_i)$) distinct chains each of some length k_i^j (less than or equal to the actual lengths) with $\sum_{i=1}^{\ell_i} k_i^j \le n_i$; call them $v_i^{1,j},\ldots,v_i^{k_i^j,j}$ $j=1,2,..,\ell_i$. Note that these eigenvectors satisfy (4); furthermore, because of the special structure of A_c and (2) it is straight forward to show that they are actually given by (5) where $a_i^{1,j},\ldots,a_i^{k_i^j,j}$ satisfy (1). Q.E.D.

<u>Corollary 3</u> There exist $(m \times 1)$ nonzero vectors $a_i^{1,j},\ldots,a_i^{k_i^j,j}$ $j=1,2,..,\ell_i$ which satisfy (1) with $a_i^{1,1},a_i^{1,2},\ldots,a_i^{1,\ell_i}$ linearly independent if and only if the Smith form of $P(s)$, $E_P(s)$, contains the factors $(s-s_i)^{k_i^j}$ $j=1,2,..,\ell_i$ in ℓ_i separate locations on the diagonal.

<u>Proof</u> In view of the proof of Theorem 1,(1) are satisfied iff A_c has a Jordan form of certain structure, or in view of the Appendix, iff $E_P(s)$ has the factors $(s-s_i)^{k_i^j}$ $j=1,2,..,\ell_i$ on the diagonal (see (A5),(A9) and (A11)). Q.E.D.

Theorem 1 implies that given $P(s)$ the maximum number of nonzero vectors $a_i^{k,j}$ which satisfy (1) is n_i , the algebraic multiplicity of s_i . If this maximum number of $a_i^{k,j}$ has been found, then it is clear from the proof of Theorem 1 that ℓ_i will be equal to the geometric multiplicity m-rank $P(s_i)$ of s_i ; if it were less, then A_c would have its generalized eigenvectors corresponding to s_i distributed among less than m-rank $P(s_i)$ chains which is impossible. In this particular case, the numbers k_i^j $j=1,2,..,\ell_i$ are the lengths of the chains of the eigenvectors and appear as the exponents of the $(s-s_i)$ factors in ℓ_i locations in the Smith form of $P(s)$ (Corollary 3).

<u>Theorem 4</u> Let n be the degree of $|P(s)|$. There exist $n(m \times 1)$ nonzero vectors $a_i^{1,j}, a_i^{2,j}, \ldots, a_i^{k_i^j,j}$ $j=1,2,..,\ell_i$ $i=1,2,\ldots,\sigma$ ($\sum_{i=1}^{\sigma} \sum_{j=1}^{\ell_i} k_i^j = n$) which satisfy (1) with $a_i^{1,1}, a_i^{1,2}, .., a_i^{1,\ell_i}$ linearly independent if and only if the zeros of $|P(s)|$ have σ distinct values s_i $i=1,2,..,\sigma$ each with algebraic multiplicity $n_i = \sum_{j=1}^{\ell_i} k_i^j$ and geometric multiplicity m-rank $P(s_i) = \ell_i$.

<u>Proof</u> If (1) is satisfied for an s_i , according to the necessity proof of Theorem 1, there exist at least $\sum_{j=1}^{\ell_i} k_i^j$ linearly independent generalized eigenvectors of A_c corresponding to s_i in at least ℓ_i distinct chains. Since (1) is satisfied for $i=1,2,..,\sigma$, there exist $\sum_{i=1}^{\sigma} \sum_{j=1}^{\ell_i} k_i^j = n$ linearly independent [1] generalized eigenvectors. This implies that A_c has exactly $\sum_{j=1}^{\ell_i} k_i^j$ eigenvectors corresponding to s_i distributed in exactly ℓ_i chains (if A_c had more generalized eigenvectors corresponding to s_i , then, since the total is n , $\sum_{j=1}^{\ell_i} k_i^j$ for some other i must have been larger than the corresponding n_i which is impossible by Theorem 1);

therefore the algebraic and geometric multiplicities of s_i are exactly $\sum_{j=1}^{\ell_i} k_i^j$ and ℓ_i respectively. Sufficiency can be easily shown in manner analoguous to the sufficiency proof of Theorem 1.

Q.E.D.

<u>Corollary 5</u> Let n be the degree of $|P(s)|$. There exist $n(m \times 1)$ nonzero vectors $a_i^{1,j},\dots,a_i^{k_i^j,j}$ $j=1,2,..,\ell_i$ $i=1,2,..,\sigma$ $\left(\sum_{i=1}^{\sigma}\sum_{j=1}^{\ell_i} k_i^j = n\right)$ which satisfy (1) with $a_i^{1,1},\dots,a_i^{1,\ell_i}$ linearly independent if and only if the Smith form of $P(s)$, $E_P(s)$, consists of factors $(s-s_i)^{k_i^j}$ $j=1,2,..,\ell_i$ in ℓ_i locations on the diagonal $(i=1,2,..,\sigma)$.

<u>Proof</u> Clear in view of the proof of Theorem 4 and the Appendix (see (A5), (A9) and (A11)). Q.E.D.

Theorem 4 implies that given $P(s)$ the maximum number of nonzero vectors $a_i^{k,j}$ which satisfy (1) for all possible s_i is n, the degree of $|P(s)|$. If this maximum number of $a_i^{k,j}$ have been found then the algebraic and geometric multiplicities are determined as well as the distribution of the factors $(s-s_i)$ in $E_P(s)$. In particular, in view of the divisibility property of the invariant factors in the Smith form of $P(s)$, $E_P(s)$ is completely determined in this case as the following example shows.

<u>EX</u>

Let $P(s) = \begin{bmatrix} s^2 & -1 \\ 0 & s \end{bmatrix}$ $\quad P^{(1)}(s) = \begin{bmatrix} 2s & 0 \\ 0 & 1 \end{bmatrix}$; $P^{(2)}(s) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$P^{(3)}(s) = 0$. (1) implies $(s_1 = 0)$: $P(0)a_1^{1,j} = 0$, $a_1^{1,1} = \begin{bmatrix}\alpha\\0\end{bmatrix}$ $(\alpha \neq 0)$

$P(0)a_1^{2,1} = -P^{(1)}(0)a_1^{1,1}$, $a_1^{2,1} = \begin{bmatrix}\beta\\0\end{bmatrix}(\beta \neq 0)$;

$P(0)a_1^{3,1} = -[P^{(1)}(0)a_1^{2,1} + \frac{1}{2!}P^{(2)}(0)a_1^{1,1}]$, $a_1^{3,1} = \begin{bmatrix}\gamma\\\alpha\end{bmatrix}$.

We stop here since $a_1^{4,1}$ etc. are zero i.e. $k_1^1 = 3$. Note that $\ell_1 = 1 =$ m-rank $P(0)$; this was also seen in the solution of $P(0)a_1^{1,j} = 0$ where dim{Null space $P(0)\} = 1$ ($=$ m-rank $P(0)$) which implies that there is only one vector $a_1^{1,j}$ i.e. $\ell_1 = 1$. Observe that $\sum_{j=1}^{\ell_i} k_i^j = 3 = n_1$ the algebraic multiplicity of $s_1 = 0$ and that the total number of $a_i^{k,j}$ is $3 = n$ the degree of $|P(s)|$. The Smith form of $P(s)$ is $E_P(s) = \begin{bmatrix}1 & 0\\0 & s^3\end{bmatrix}$ since $(s - s_1) = s$ appears in $\ell_1 = 1$ locations with exponent $k_1^1 = 3$. Finally note that the generalized eigenvectors of the corresponding $A_c = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$ (see (2)) are $v_1^{1,1} = [\alpha,0,0]^T$, $v_1^{2,1} = [\beta,\alpha, 0]^T$ and $v_1^{3,1} = [\gamma,\beta,\alpha]^T$ (see (5)) .

In view of the above, given two polynomial matrices, (1) can be used to find the relation between their Smith forms. Note that the relations between polynomial matrices which satisfy (1) are examined in detail in the following.

Assume that $P(s)$ is a given $(m \times m)$ polynomial matrix and let n be the degree of $|P(s)|$. $|P(s)|$ has σ distinct zeros s_i with algebraic and geometric multiplicities n_i $(\sum_{i=1}^{\sigma} n_i = n)$ and

$\ell_i (= m - \text{rank } P(s_i))$ respectively. In view of Theorem 4, there exist n $(m \times 1)$ nonzero vectors $a_i^{1,j}, .., a_i^{k_i^j, j}$ $j = 1, 2, .., \ell_i$ $i = 1, 2, .., \sigma$ which satisfy (1) with $a_i^{1,1}, .., a_i^{1,\ell_i}$ linearly independent $\left(\sum_{j=1}^{\ell_i} k_i^j = n_i \right)$.

<u>Theorem 6</u> $P(s)$ is a right divisor (rd) of a $(r \times m)$ polynomial matrix $M(s)$ if and only if $M(s)$ satisfies (1) with the same s_i and $a_i^{k,j}$.

<u>Proof</u> $P(s)$ is a rd of $M(s)$ iff $P(s)$ is a greatest common right divisor (gcrd) of $P(s)$ and $M(s)$ or iff there exists a unimodular matrix U such that $U \begin{bmatrix} P \\ M \end{bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix}$ [3].

Assume that such U exists i.e. P is a rd of M. Since P satisfies (1), $U \begin{bmatrix} P \\ M \end{bmatrix}$ also satisfies (1) which implies that $\begin{bmatrix} P \\ M \end{bmatrix}$ satisfies (1) because a premultiplication by a unimodular matrix does not affect these relations (see Lemma 2). Therefore M satisfies (1) with the same s_i and $a_i^{k,j}$.

Assume now that M satisfies (1). Let G be a gcrd of M and P i.e. $\hat{U} \begin{bmatrix} P \\ M \end{bmatrix} = \begin{bmatrix} G \\ 0 \end{bmatrix}$ with $\hat{U}$ a unimodular matrix. This implies that G satisfies the same n relations as M; the degree of $|G|$ is therefore at least n in view of Theorems 1 and 4. Note however that since G is a rd of P, $P = \tilde{P}G$ which implies that $\tilde{P}$ is a unimodular matrix. Therefore $M = \tilde{M}G = (\tilde{M}\tilde{P}^{-1})P$ and P is a rd of M. Q.E.D.

If $M(s)$ is a square matrix, the above proof can be used to show the following.

Corollary 7 $M(s) = U(s)P(s)$ with $U(s)$ unimodular if and only if the degree of $|M(s)|$ is n and $M(s)$ satisfies (1) with the same s_i and $a_i^{k,j}$.

An extension of Corollary 7, which gives insight into the relation between the vectors $a_i^{k,j}$ and the structure of $P(s)$, is the follwing lemma.

Consider the matrix $P(s)$ of Theorem 6 together with the corresponding s_i and $a_i^{k,j}$ which satisfy (1). Then

Lemma 8 Given an $(m \times m)$ polynomial matrix $\bar{P}(s)$, there exist unimodular matrices $U_1(s)$, $U_2(s)$ such that

$$P(s) = U_1(s)\bar{P}(s)U_2(s)$$

if and only if the degree of $|\bar{P}(s)|$ is n and there exist n $(m \times 1)$ nonzero vectors $\bar{a}_i^{k,j}$ which satisfy together with $\bar{P}(s)$ and s_i the same relations (1) satisfied by $a_i^{k,j}$, $P(s)$ and s_i . Furthermore if such U_1 , U_2 exist

$$\begin{aligned}
\bar{a}_i^{1,j} &= U_2(s_i)a_i^{1,j} \\
\bar{a}_i^{2,j} &= U_2(s_i)a_i^{2,j} + U_2^{(1)}(s_i)a_i^{1,j} \qquad (6) \\
&\vdots \\
\bar{a}_i^{k_i^j,j} &= U_2(s_i)a_i^{k_i^j,j} + \dots + \frac{1}{(k_i^j - 1)!} U_2^{(k_i^j - 1)}(s_i)a_i^{1,j}
\end{aligned}$$

Proof Assume that there exist U_1, U_2 such that $P = U_1 \bar{P} U_2$ with P, s_i and $a_i^{k,j}$ satisfying (1). Substitute P by $\bar{P} U_2$ in the relations since U_1 cancels out (see Lemma 2). Straight calculations show that vectors $\bar{a}_i^{k,j}$ given by (6) together with $\bar{P}$ and s_i satisfy the same relations (1).

Assume now that the degree of $|\bar{P}|$ is n and $\bar{P}$, $\bar{a}_i^{k,j}$ and s_i satisfy the same n relations (1), P, $a_i^{k,j}$ and s_i satisfy. In view of Corollary 5 and the remark following the corollary, P and $\bar{P}$ have exactly the same Smith forms; therefore, there exist unimodular matrices $\tilde{U}_1, \tilde{U}_2$ [2] such that $P = \tilde{U}_1 \bar{P} \tilde{U}_2$. Q.E.D.

Note that Lemma 8 applies to nonsquare matrices as well if n is taken to be the degree of the greatest common divisor of all highest order minors; e.g. given R, Q, $RU_3 = [\tilde{R}, 0]$, $QU_4 = [\tilde{Q}, 0]$; then $\tilde{R} = U_1 \tilde{Q} U_2$ (Lemma 8 applies here) and

$$RU_3 = U_1[\tilde{Q}, 0] \begin{bmatrix} U_2 & 0 \\ 0 & I \end{bmatrix} = U_1 Q\, U_4 \begin{bmatrix} U_2 & 0 \\ 0 & I \end{bmatrix} \quad (U_i \text{ unimodular}).$$

Remark If P, $a_i^{k,j}$, s_i and $\bar{P}$, $\bar{a}_i^{k,j}$, s_i satisfy the same n relations (1) where, in addition, $a_i^{k,j}$ and $\bar{a}_i^{k,j}$ satisfy (6) for some unimodular matrix $U_2(s)$, then P and $\bar{P}$ not only have the same Smith form, as it was shown in the sufficiency proof of Lemma 8, but they are related by: $P = \bar{U}_1 \bar{P} U_2$. This is shown as follows: Assume that $P = \tilde{U}_1 \bar{P} \tilde{U}_2$ where $\tilde{U}_2 \neq U_2$. Lemma 8 implies that $a_i^{k,j}$, $\bar{a}_i^{k,j}$ and $\tilde{U}_2$ will also satisfy relations similar to (6). Equating $\bar{a}_i^{k,j}$ in the relations involving U_2 and $\tilde{U}_2$, we derive n relations of type (1) satisfied by $a_i^{k,j}$, s_i and the polynomial matrix $U_2 - \tilde{U}_2$. According to Theorem 4, $|U_2 - \tilde{U}_2|$ must be of at least degree n which is false. Therefore $U_2 = \tilde{U}_2$. Note that this result agrees with Corollary 7 since if $U_2 = I$ i.e. $\bar{a}_i^{k,j} = a_i^{k,j}$, P and $\bar{P}$ are related by $P = U\bar{P}$.

Assume that, for an $(m \times m)$ polynomial matrix $P(s)$ yet to be chosen, we have decided upon the degree of $|P(s)|$ as well as its zeros i.e. n, s_i and the algebraic multiplicities n_i . Clearly there are many matrices which satisfy these requirements e.g. diagonal matrices. If we specify the geometric multiplicities ℓ_i , our matrix has more structure e.g. the factors $(s - s_i)$ are appropriately distributed in our diagonal matrix. If k_i^j are also chosen, then, the Smith form of $P(s)$ is completely defined (see Appendix) i.e. $P(s)$ is defined within pre and post unimodular multiplication. This is equivalent to imposing the restriction that $P(s)$ has to specify n relations of type (1) ($P(s)$ of Theorem 6) without though restricting $a_i^{k,j}$ (other than being nonzero and $a_i^{1,1}, \ldots, a_i^{1,\ell_i}$ being linearly independent). If $a_i^{k,j}$ are also specified then $P(s)$ is determined within a unimodular premultiplication (Corollary 7) .

If an $(r \times m)$ polynomial matrix $M(s)$ satisfies n relations of type (1) ($M(s)$ of Theorem 6) then, there exists an $(m \times m)$ polynomial matrix $P(s)$ with specified structure which is a rd of $M(s)$ i.e. $M = \tilde{M}P$. This is because in view of the above, there exists a matrix $P(s)$ with degree of $|P(s)|$ equal to n which satisfies exactly the same n relations with the same $a_i^{k,j}$ and s_i (this $P(s)$ is specified within a unimodular premultiplication); in view of Theorem 6 this $P(s)$ is a rd of $M(s)$. If in the n relations which are satisfied by $M(s)$, the vectors $a_i^{k,j}$ are not specified, then the Smith form of $M(s)$ has factors $(s-s_i)^{k_i^j}$ in ℓ_i locations on the diagonal. In other words a rd $P(s)$ is not specified in this case, but it is only known that there exists a rd of the form $E_p U_2$ where E_p is a completely specified Smith form (it consists of the $(s-s_i)^{k_i^j}$) and U_2 an arbitrary unimodular matrix.

In view of the above, if it is known that an $(m \times m)$ polynomial matrix $P(s)$ satisfies n relations of type (1) ($P(s)$ of Theorem 6) i.e. $n, s_i, n_i, k_i^j, \ell_i, a_i^{k,j}$ are given, then $P(s)$ is specified within a unimodular premultiplication. In the following, some methods are outlined which can be used to determine an appropriate $P(s)$ matrix.[†]

(a) Let $$[a_o, a_1, \ldots, a_r] \begin{bmatrix} I_m \\ I_m s \\ \vdots \\ I_m s^r \end{bmatrix} = P(s)$$ where a_k are $m \times m$

matrices to be determined. If the n given relations (1) are written in terms of a_i, a linear system of equations is obtained

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† These methods can be used to arbitrarily assign the poles of a system via feedback compensation; a detailed account of these techniques will be given in a future publication. Note that special cases have already appeared in [5] and [6] .

with a_i as the unknowns. In order to have more unknowns than equations $m(r+1) \geq n$ i.e. $r \geq \frac{n}{m} - 1$ e.g. For s_i all distinct, solve $[a_o, \ldots, a_r] \begin{bmatrix} I_m \\ I_m s_i \\ \vdots \\ I_m s_i^r \end{bmatrix} . \; a_i^{1,1} = 0 \quad i=1,2,\ldots,n$.

(b) Let $\operatorname{diag}(s^{d_i}) - A_m S(s) = P(s)$ where $S(s) = \operatorname{diag}[(1, s, \ldots, s^{d_i - 1})^T]$ and A_m an $m \times n$ matrix to be determined. d_i are the column degrees of the (column proper) $P(s)$ ($\Sigma d_i = n$). The n relations (1) are written in terms of A_m and a linear system of equations is obtained with A_m as unknown. Note however that d_i are not completely free; they must satisfy certain inequalities involving k_i^j [2] .

(c) One can also use the Smith form of $P(s)$ (specified by n, s_i, n_i, k_i^j and ℓ_i) and the relations (6) of Lemma 8.

Assume that $P(s)$ is associated with a polynomial matrix description of a linear, time-invariant system; that is $P(D)z(t) = Q(D)u(t)$, $y(t) = R(D)z(t) + W(D)u(t)$ where u, y and z are the input, output and partial state respectively. Clearly in this case the roots s_i of $|P(s)|$ are the poles of the system; it will be shown that the nonzero vectors $a_i^{k,j}$ are closely related to the eigenvectors of an equivalent to $\{P,Q,R,W\}$ state-space description $\{A,B,C,E\}$ of the given system.

__Def [4]__ $\{P_1,Q_1,R_1,W_1\}$ and $\{P_2,Q_2,R_2,W_2\}$ are equivalent iff there exist M_1,M_2,X_2,Y_1 polynomial matrices such that

$$\begin{bmatrix} M_2 & 0 \\ X_2 & I \end{bmatrix} \cdot \begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix} = \begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix} \cdot \begin{bmatrix} M_1 & -Y_1 \\ 0 & I \end{bmatrix} \tag{7}$$

with (M_2,P_2) left prime and (P_1,M_1) right prime.

From (7), $M_2P_1 = P_2M_1$. If $P_1 = s - A,\ P_2 = P$

then $\quad M_2(s)(s - A) = P(s)M_1(s)$ (8a)

with (M_2,P) left prime and $(s - A,M_1)$ right prime.

If $P_1 = P\ ,\ \ P_2 = s - A$, then

$$\bar{M}_2(s)P(s) = (s - A)\bar{M}_1(s) \tag{8b}$$

with $(\bar{M}_2,\ s - A)$ left prime and $(P,\bar{M}_1)$ right prime. Note that a special case of (8b) is (2) where $\bar{M}_2 = B_c$, $s - A = s - A_c$, $\bar{M}_1 = S(s)$.

Assume that $P(s)$ satisfies (1) together with s_i and $a_i^{k,j}$. Then $\bar{M}_2P$ also satisfies (1) with the same s_i and $a_i^{k,j}$. This can be shown as follows: $P(s_i)a_i^{1,j}=0$ implies that $\bar{M}_2P(s_i)a_i^{1,j}=0$; $[\bar{M}_2P(s_i)]^{(1)}a_i^{1,j} = \bar{M}_2^{(1)}(s_i)P(s_i)a_i^{1,j} + \bar{M}_2(s_i)P^{(1)}(s_i)a_i^{1,j} = \bar{M}_2(s_i)P^{(1)}(s_i)a_i^{1,j} = -\bar{M}_2(s_i)P(s_i)a_i^{2,j} = -[\bar{M}_2P(s_i)]a_i^{2,j}$ etc. Substitute now in all the relations (1) $\bar{M}_2P$ by $(s - A)\bar{M}_1$ (see (8b)). Then

$$\begin{aligned} (s_i - A)\ v_i^{1,j} &= 0 \\ (s_i - A)\ v_i^{2,j} &= -\,v_i^{1,j} \\ &\ \vdots \\ (s_i - A)\ v^{k_i^j,j} &= -\,v_i^{k_i^j-1,j} \end{aligned} \tag{9b}$$

where

$$
\begin{aligned}
v_i^{1,j} &= \bar{M}_1(s_i)a_i^{1,j} \\
v_i^{2,j} &= [\bar{M}_1(s_i)a_i^{2,j} + \bar{M}_1^{(1)}(s_i)a_i^{1,j}] \\
&\vdots \\
v_i^{k_i^j,j} &= [\bar{M}_1(s_i)a_i^{k_i^j,j} + \dots + \frac{1}{(k_i^j - 1)!}\bar{M}_1^{(k_i^j-1)}(s_i)a_i^{1,j}]
\end{aligned}
\tag{10b}
$$

which implies that $v_i^{k,j}$ are the generalized eigenvectors of A corresponding to the eigenvalue s_i (compare with (4) and (5)).

That is, the vectors $a_i^{k,j}$ which satisfy (1) with $P(s)$ and s_i, determine the generalized eigenvectors $v_i^{k,j}$ of A of the equivalent state-space description via (10b).

Assume that $v_i^{k,j}$ are the generalized eigenvectors of A corresponding to the eigenvalue s_i i.e. they satisfy relations (9b). Then relation (8a) implies that there exist $a_i^{k,j}$ such that $a_i^{k,j}$, s_i together with $P(s)$ satisfy (1) where $a_i^{k,j}$ are given by

$$
\begin{aligned}
a_i^{1,j} &= M_1(s_i)\, v_i^{1,j} \\
a_i^{2,j} &= [M_1(s_i)\, v_i^{2,j} + M_1^{(1)}(s_i)\, v_i^{1,j}] \\
&\vdots \\
a_i^{k_i^j,j} &= [M_1(s_i)\, v_i^{k_i^j,j} + \frac{1}{(k_i^j - 1)!} M_1^{(k_i^j - 1)}(s_i)v_i^{1,j}]
\end{aligned}
\tag{10a}
$$

This can be shown as follows:

$(s_i - A)\, v_i^{1,j} = 0$ implies that $P(s_i)M_1(s_i)\, v_i^{1,j} = 0$; let $a_i^{1,j} = M_1(s_i)\, v_i^{1,j}$. Differentiate (8a) and postmultiply by $v_i^{1,j}$:

$$M_2^{(1)}(s_i)(s_i - A)\ v_i^{1,j} + M_2(s_i)\ v_i^{1,j} = M_2(s_i)\ v_i^{1,j} = - M_2(s_i)(s_i - A)\ v_i^{2,j}$$

$$= - P(s_i)M_1(s_i)\ v_i^{2,j} = P^{(1)}(s_i)M_1(s_i)\ v_i^{1,j} + P(s_i)M_1^{(1)}(s_i)\ v_i^{1,j}$$

from which $P(s_i)\ [M_1(s_i)v_i^{2,j} + M_1^{(1)}(s_i)v_i^{1,j}] = - P^{(1)}(s_i)[M_1(s_i)v_i^{1,j}]$;

let $a_i^{2,j} = M_1(s_i)v_i^{2,j} + M_1^{(1)}(s_i)v_i^{1,j}$ etc.

That is, the generalized eigenvectors $v_i^{k,j}$ of A which satisfy (9b) determine vectors $a_i^{k,j}$ via (10a) which satisfy (1) with s_i and $P(s)$ where $P(s)$ is the corresponding to A matrix (see (8a)) of an equivalent polynomial matrix description.

<u>Remark</u> The above analysis can be used in the feedback compensation of systems described by polynomial matrices, not only to assign the closed loop poles but also the closed loop eigenvectors.

Finally note that all the results in this paper reduce to well known results when special cases are considered e.g. $P(s) = p(s)$ a polynomial $(m=1)$ and $P(s) = sI - A\ (m=n)$.

Conclusion

In this report several basic theorems were given, which establish the relations between a polynomial matrix $P(s)$ and its "characteristic" vectors $a_i^{k,j}$ and "characteristic" values s_i . This account is by no means complete. Extensions together with applications to control problems will be given in a future publication.

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APPENDIX

Canonical Forms (Jordan, Smith and Controllable companion forms).

Let $(n \times n)A$ have σ distinct eigenvalues s_i each repeated n_i times $(\sum_{i=1}^{\sigma} n_i = n)$; n_i is the algebraic multiplicity of s_i. The geometric multiplicity of s_i, ℓ_i, is defined as $\ell_i = n - \mathrm{rank}(s_i - A)$ i.e. the reduction in rank in $s-A$ when $s=s_i$.

There exists a similarity transformation matrix Q such that

$$AQ = QJ$$

where J is the Jordan canonical form of A.

$$J = \begin{bmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots A_\sigma \end{bmatrix} , \quad A_i = \begin{bmatrix} A_{i1} & & 0 \\ & A_{i2} & \\ 0 & & \ddots A_{i\ell_i} \end{bmatrix} \tag{A1}$$

where A_i $i=1,\ldots\sigma$ is an $(n_i \times n_i)$ matrix with eigenvalues s_i; A_i is a block diagonal matrix with ℓ_i $(n_i \geq \ell_i)$ matrices A_{ij} $j=1,\ldots,\ell_i$ on the diagonal, of the form

$$(k_i^j \times k_i^j) \quad A_{ij} = \begin{bmatrix} s_i & 1 & 0\ldots 0 \\ 0 & s_i & 1\ldots 0 \\ \vdots & & \ddots \quad \vdots \\ \vdots & & \ddots \quad 1 \\ 0 & 0 & \ldots \; s_i \end{bmatrix} \quad \left(\sum_{j=1}^{\ell_i} k_i^j = n_i\right) .$$

The structure of the Jordan canonical form of A is determined by the generalized eigenvalues of A (they are used to construct Q). To each eigenvalue s_i correspond ℓ_i chains of generalized eigenvectors each chain of length k_i^j i.e. $v_i^{1,j}, v_i^{2,j}, \ldots, v_i^{k_i^j,j}$ $j=1,\ldots,\ell_i$ a total of n_i linearly independent generalized eigenvectors. The eigenvectors of a particular chain can be determined from

$$\begin{aligned}(s_i - A)v_i^{1,j} &= 0\\ (s_i - A)v_i^{2,j} &= -v_i^{1,j}\\ &\vdots\\ (s_i - A)v_i^{k_i^j,j} &= -v_i^{k_i^j-1,j}\end{aligned} \qquad \text{(A2)}$$

where $(s_i - A)^{k_i^j} v_i^{k_i^j,j} = 0$ and $(s_i - A)^{k_i^j-1} v_i^{k_i^j,j} \neq 0$. [1].

If $\bar{n}_i \triangleq \max_j k_i^j$ (the dimension of the largest block associated with s_i) then

<u>Minimal Polynomial</u> $\Delta_m(s) = \prod_{i=1}^{\sigma} (s-s_i)^{\bar{n}_i}$ while the

<u>Characteristic Polynomial</u> $\Delta(s) = \prod_{i=1}^{\sigma} (s-s_i)^{n_i}$ ($= |s-A|$) .

Note that A is <u>cyclic</u> iff there exists a vector b such that rank $[b,Ab,\ldots,A^{n-1}b] = n$

$\Leftrightarrow$ $\Delta(s) = \Delta_m(s)$ $\Leftrightarrow$ $n_i = \bar{n}_i$ $i=1,\ldots,\sigma$ i.e. only one block is associated with each distinct eigenvalue

$\Leftrightarrow$ $\ell_i = 1$ $i=1,\ldots,\sigma$ i.e. only one chain of generalized eigenvectors is associated with each distinct egenvalue. (A3)

$\Leftrightarrow$ $\text{rank}(s_i - A) = n-1$.

There exist unimodular matrices $U_1(s)$, $U_2(s)$ such that

$$U_1(s)(s-A)U_2(s) = E_A(s)$$

where

$$E_A(s) = \begin{bmatrix} \varepsilon_1(s) & & 0 \\ & \varepsilon_2(s) & \\ & & \ddots \\ 0 & & \varepsilon_n(s) \end{bmatrix} \tag{A4}$$

is the Smith form of $s-A$; $\varepsilon_i(s)$ are the (monic) invariant polynomials of $s-A$ defined by $\varepsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)}$ where the determinantal divisor $D_i(s)$ is the (monic) greatest common divisor of all the ith order minors of $s-A$. ($D_o(s) \triangleq 1$). Note that $\varepsilon_k(s)$ divides $\varepsilon_{k+1}(s)$ $k=1,\dots,n-1$ [2] .

Clearly, there are n_i factors $(s-s_i)$ distributed among the n invariant polynomials $\varepsilon_i(s)$. Note that the rank reduction in $E_A(s_i)$ equals the number of entries on the diagonal containing $(s-s_i)$ and it is also equal to the rank reduction in s_i-A , that is ℓ_i . Therefore, the n_i factors $(s-s_i)$ are present only in ℓ_i invariant polynomials on the diagonal of $E_A(s)$. Assume, without loss of generality that

$$k_i^1 \le k_i^2 \le \dots \le k_i^{\ell_i}(=\bar{n}_i)$$

Then it can be shown using the minors of J (or A) that

$$\begin{aligned} \varepsilon_n(s) &= (s-s_i)^{k_i^{\ell_i}} \quad (\ . \ ) \\ \varepsilon_{n-1}(s) &= (s-s_i)^{k_i^{\ell_i-1}} \quad (\ . \ ) \\ &\vdots \\ \varepsilon_{n-(\ell_i-1)}(s) &= (s-s_i)^{k_i^1} \quad (\ . \ ) \end{aligned} \tag{A5}$$

i.e. the (n_i) factors $(s-s_i)$ are factors of the ℓ_i invariant polynomials $\varepsilon_{n-(\ell_i-1)}(s) \dots \varepsilon_n(s)$; the exponents k_i^j of $(s-s_i)$

are the dimensions of the matrices A_{ij} $j=1,\ldots,\ell_i$ of the Jordan canonical form or equivalently, they are the lengths of the chains of the generalized eigenvectors corresponding to s_i .

In view of the divisibility property of $\varepsilon_i(s)$ it is therefore clear that if s_i and the dimensions of the submatrices of J are known, the Smith form of $s-A$ is uniquely determined. Furthermore note that the characteristic polynomial of $s-A$, is $\Delta(s) = \varepsilon_1(s)\,\varepsilon_2(s)\,\ldots\,\varepsilon_n(s)$, the minimal polynomial is $\Delta_m(s) = \varepsilon_n(s)$ and A is cyclic if $(s-s_i)$ is a factor only of $\varepsilon_n(s)$.

<u>Ex</u>

Let
$$J = \left[\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & & \\ & A_{12} & \\ \hline & & A_{21} \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

that is
$$\left\{\begin{array}{lll} s_1 = -3 \ , & n_1 = 3 \ , & \ell_1 = 2 \\ k_1^1 = 2 \ , & k_1^2 = 1 \ \text{ and } \ s_2 = -1 \ , & n_2 = \ell_2 = k_2^1 = 1 \end{array}\right\}$$

Then the Smith form is

$$E_A(s) = \begin{bmatrix} \varepsilon_1(s) & 0 & 0 & 0 \\ 0 & \varepsilon_2(s) & 0 & 0 \\ 0 & 0 & \varepsilon_3(s) & 0 \\ 0 & 0 & 0 & \varepsilon_4(s) \end{bmatrix}$$

where
$$\varepsilon_1(s) = \varepsilon_2(s) = 1$$
$$\varepsilon_3(s) = (s-3)$$
$$\varepsilon_4(s) = (s-3)^2(s-1)$$

Clearly $\Delta(s) = (s-3)^3(s-1)$ and $\Delta_m(s) = (s-3)^2(s-1)$.

Assume that (A,B), where $(n \times m)B$ has full rank $m(\leq n)$ is a controllable pair. There exists an equivalence transformation matrix Q such that

$$AQ = QA_c \quad , \qquad B = QB_c$$

with (A_c, B_c) in <u>controllable companion form</u>,

$$A_c = [A_{ij}] \qquad (d_i \times d_j)A_{ij} \begin{cases} \begin{bmatrix} & 0 & \\ x & x \; \ldots & x \end{bmatrix} & \text{for} \quad i \neq j \\ \\ \begin{bmatrix} 0 & I_{d_{i-1}} \\ \vdots & \\ 0 & \\ x \; x & \ldots \; x \end{bmatrix} & \text{for} \quad i = j \end{cases} \tag{A6}$$

i and $j=1,2,..,m$

$$B_c = [\, B_i \,] \quad i=1,2,..,m \quad (d_i \times m)B_i = \begin{bmatrix} & & 0 & & \\ 0 \; .. \; 0 & 1 & x & \ldots & x \end{bmatrix}$$

ith column

d_i $i=1,2,..,m$ are the <u>controllability indices</u> of (A,B). The m nontrivial $(\sum_{k=1}^{j} d_k)$th $j=1,2,..,m$ rows of A_c and B_c define the matrices $(m \times n)A_m$ and $(m \times m)B_m$ respectively where B_m is in upper triangular form with 1s on the diagonal [3].

The structure of A_c implies that $\text{rank}(s_i - A) = \text{rank}(s_i - A_c) \geq n-m$ for any s_i, which in turn implies that the geometric multiplicity ℓ_i of s_i satisfies

$$\ell_i (\triangleq n\text{-rank}(s_i - A)) \leq m \;. \quad i=1,2,..,\sigma \;. \tag{A7}$$

There exist two polynomial matrices $(n \times m) S(s)$ and $(m \times m) P(s)$, closely related to the controllable pair (A_c, B_c), which satisfy the identity

$$B_c P(s) = (s - A_c) S(s) \tag{A8}$$

These matrices are defined by:

$$S(s) = \mathrm{diag}([1, s, \ldots, s^{d_i - 1}]^T), \quad P(s) \triangleq B_m^{-1}[\mathrm{diag}(s^{di}) - A_m S(s)] [3]$$

It can be known that the system matrices

$$\left[\begin{array}{c|c} s - A_c & B_c \\ \hline -I_n & 0 \end{array}\right] \qquad \text{and}$$

$$\left[\begin{array}{cc|c} I_{n-m} & 0 & 0 \\ 0 & P(s) & I_m \\ \hline 0 & -S(s) & 0 \end{array}\right]$$ are unimodularly equivalent [2][†];

this implies that the Smith forms $E_A(s)$ $E_P(s)$ of $s - A_c$ (or $s - A$) and $P(s)$ respectively satisfy the relation:

$$E_A(s) = \begin{bmatrix} I_{n-m} & 0 \\ 0 & E_P(s) \end{bmatrix} \tag{A9}$$

---

† therefore the system representations $\dot{x} = A_c x + B_c u$, $y = x$ and $Pz = u$, $y = Sz$ are equivalent.

It is now clear that $|s - A| = |P(s)|$ which implies that $|P(s)|$ has σ distinct roots s_i each repeated n_i times (n_i is the <u>algebraic multiplicity</u> of s_i). Furthermore the <u>geometric multiplicity</u> of s_i, ℓ_i, is given by $\ell_i = m\text{-rank } P(s_i)^{\dagger}$ since $\ell_i = n\text{-rank } E_A(s_i) = n - [(n-m) + \text{rank } E_P(s_i)] = m\text{-rank } E_P(s_i)$.

If

$$E_P(s) = \begin{bmatrix} \bar{\varepsilon}_1(s) & & \\ & \ddots & \\ & & \bar{\varepsilon}_m(s) \end{bmatrix} \tag{A10}$$

where $\bar{\varepsilon}_k$ divides $\bar{\varepsilon}_{k+1}$ $k=1,2,..,m-1$ then assuming that

$k_i^1 \le k_i^2 \le \ldots \le k_i^{\ell_i} (=\bar{n}_i)$ we have

$$\begin{aligned} \bar{\varepsilon}_m(s) &= (s - s_i)^{k_i^{\ell_i}} \quad (\quad . \quad) \\ \bar{\varepsilon}_{m-1}(s) &\quad (s-s_i)^{k_i^{\ell_i-1}} \quad (\quad . \quad) \\ &\vdots \\ \bar{\varepsilon}_{m-(\ell_i-1)}(s) &= (s - s_i)^{k_i^1} \quad (\quad . \quad) \end{aligned} \tag{A11}$$

which are completely analoguous to (A5).

The <u>characteristic polynomial</u> of $P(s)$ is $\Delta(s) = |P(s)| = \prod_{i=1}^{\sigma} (s - s_i)^{n_i}$ while the <u>minimal polynomial</u> is

$$\Delta_m(s) = \prod_{i=1}^{\sigma} (s - s_i)^{\bar{n}_i}$$

$P(s)$ is <u>cyclic</u> (or <u>simple</u>) iff there exists a vector g so that $P(s), g$ are relatively left prime

$$\Leftrightarrow \quad \Delta(s) = \Delta_m(s) \qquad (n_i = \bar{n}_i \quad i=1,2,..,\sigma) \tag{A12}$$

$$\Leftrightarrow \quad \ell_i = m\text{-rank } P(s_i) = 1 \quad i=1,2,..,\sigma .$$

APPENDIX C

Computer Code - Matlab Routines

1. The basic program:

```
% The function name is pmi_basic.m. It solves the basic polynomial interpolation
% problem: given {sj,aj,bj} j = 1,L, find a Q(s) s.t. Q(sj)aj=bj for all j.
% To use this program, a data file called AL_BL_SJ_DI has to be created, where AL
contains aj as column vectors and BL is similarly defined as in Theorem 2.1 in the paper;
SJ is a vector containing the interpolation points sj.'

AL_BL_SJ_DI; %Input the matrices AL,BL,SJ and DI from the file AL_BL_SJ_DI.m

SL = SL_BUILD(AL,SJ,DI); %find SL

Q = BL/(SL), DI,  %Calculate the solution of QSL = BL in equation (2.5)

%if Q = BL/inv(SL) is used instead of Q = BL/(SL), it can cause large
% error in some cases.
```

```
function SL = SL_BUILD(AL,SJ,DI)
% Find the matrix SL

[m,L] = size(AL);
DIM = sum(DI) + m;

org(1) = 1;
for j = 2:m,org(j) = org(j-1)+DI(j-1)+1; end

for n = 1:L
        S = zeros(DIM,m);
    for j = 1:m
        for i = 0:DI(j)
                S(org(j)+i,j) = (SJ(n))^i;
        end
    end

    S = S*AL(:,n);
    if n == 1, SL = S;
    else
        SL = [SL,S];
    end
end
```

```
% This is a data file for pmi_basic.m
% This is the data that gives the correct Q(s)=D-1(s)N(s).
AL = [ 0  0  1  0   1
                0  0  0  1   1
   0  1  0  1  4/3
   1 -1  0  0 -1/12 ];

BL = [ 0  0  0  0  0
   0  0  0  0  0 ];
```

```
SJ = [-2 -1  0  1   2 ];
DI = [1 0 1 1]; %contains the column degrees of Q(s).
```

2.Solving M(s)L(s) = Q(s)

```
% filename: Solve_M.m
% This is a program to solve M(s)L(s)=Q(s).
% To use this program, a data file called L_Q_ALPHA_SJ_DI.m has to be
% created,where L=[L0,...,Ldl] Q=[Q0,...,Qdq] completely specify L(s) and Q(s)
% as L(s) = L0+sL1+..., Q(s) = Q0+sQ1+... .  SJ is a vector containing the
% interpolation points sj. Also contained in this are L_DIM and Q_DIM which are
% dimensions of L(s) and Q(s) repectively. DI contains the column degrees of M.

L_Q_ALPHA_SJ_DI; %Input the matrices L, Q, ALPHA, SJ and DI from the file
AL = find_XL(L,L_DIM,ALPHA,SJ); %find AL=[L(s1)alpha1, ..., L(sL)alphaL]
BL = find_XL(Q,Q_DIM,ALPHA,SJ); %find BL=[Q(s1)alpha1, ..., Q(sL)alphaL]

% now the problem becomes a standard PMI problem: find M(s) such that
% M(si)AL(i) = BL(i) for i = 1, ... L.

[M,SL] = solve_pmi_standard(AL,BL,SJ,DI);

% M(s) = M*S(s), where S(s)=blk diag[1, s, ...sdi]'
% Since M(s) is obtained using the standard PMI approach, M ≠ M0+sM1+...
% Again, one has to use M(s) = M*S(s) to find M(s) by hand.


M, DI, %display M, Di on screen.


function XL = find_XL (X,X_DIM,ALPHA,SJ);
% Given a polynomial matrix X(s), SJ=[s1,s2,...,sL],
% and ALPHA=[alpha1,...,alphaL], find the matrix XL as
% XL = [X(s1)alpha1,...,X(sL)alphaL];

[m,L] = size(ALPHA);
p = X_DIM(1); q = X_DIM(2);
[u,v]=size(X);
d = v/q - 1; % d is the highest degree of X(s).
        for n = 1:L
                        Xsj = X(:,1:q);
                                for i = 1:d
                                        bg = q*i+1; en = q*(i+1);
                                 Xsj = Xsj + X(:,bg:en)*SJ(n)^i;   % build X(sj)
                                end
                        if n == 1,XL = Xsj*ALPHA(:,1);
                        else
                        XL = [XL,Xsj*ALPHA(:,n)];
                        end

        end


function [Q,SL] = solve_pmi_standard(AL,BL,SJ,DI)
```

```
% This function solves basic polynomial matrix interpolation problems: Given {sj,aj,bj}
and column degrees di, find the unique Q(s) of degree d such that Q(sj)aj=bj for all j.

SL = SL_BUILD(AL,SJ,DI); %find SDL
SL_cond = cond(SL),
Q = BL/(SL);  %Calculate the solution of QSL = BL in equation (2.5)
```

3. Solving Diophantine Equation

```
% filename: Solve_Diophantine.m

% This is a program to solve X(s)D(s)+Y(s)N(s)=M(s)L(s)=Q(s). where
%               M(s)=[X(s),Y(s)], L(s)=[D(s)',N(s)']'.
%Everything is the same as in M(s)L(s)=Q(s) problem except we want
%to have X(s) nonsingular and inv(X(s))Y(s) proper.

% To use this program, a data file called DATA.m has to
%be created,where L=[L0,...,Ldl] Q=[Q0,...,Qdq] completely specify L(s)
%and Q(s) as L(s) = L0+sL1+..., Q(s) = Q0+sQ1+... .  SJ is a vector
%containing the interpolation points sj. Also contained in this are
%L_DIM and Q_DIM which are dimensions of L(s) and Q(s) repectively.
%DI contains the column degrees of M.

Data; %data file

AL = find_XL(L,L_DIM,ALPHA,SJ); %find AL=[L(s1)alpha1, ..., L(sL)alphaL]
BL = find_XL(Q,Q_DIM,ALPHA,SJ); %find BL=[Q(s1)alpha1, ..., Q(sL)alphaL]

% Here the diference from M(s)L(s)=Q(s) comes in.
% Let X(s)=X0+...+s^rXr. By setting Xr=Identity will ensure X(s) nonsigular
% inv(X(s))Y(s) proper. This can be achieved by formulating the the
% problem as follows: let AL=[a1,...,aL] BL=[b1,...,bL], X(s)=X1(s)+s^rXr
% find [X1(s),Y(s)] s.t. [X1(sj),Y(sj)]aj=bj-[sj^rXr,0]aj=cj, j=1,L
% where Xr is chosen as a identity matrix.

X_DIM=Q_DIM(1); %# of rows in X(s) is the same as in Q(s)
Y_DIM=[X_DIM,(L_DIM(1)-X_DIM)]; % dimension of Y(s)
Xr = eye(X_DIM(1));

% Now we need to construct CL=[c1,...,cL]

CL = find_CL(AL,BL,Xr,Y_DIM,SJ,r);

% now the problem becomes a standard PMI problem: find M(s) such that
% M1(si)AL(i) = CL(i) for i = 1, ... L.

d = r;

% The SDL is m(d+1)xL. For solution to exist, L must be s.t. L≤m(d+1).
% Therefore we force:

[M,SDL,ERROR] = solve_pmi(AL,CL,SJ,d);
```

```
% M1(s) = M0+sM1+...s^r-1Mr-1. M(s) = M1(s)+[s^rXr,0]

M, ERROR%display M and error on screen.

function CL = find_CL (AL,BL,Xr,Y_DIM,SJ,r)

Z = 0*ones(Y_DIM(1),Y_DIM(2)); %A zero matrix of the same size of Y(s)

L = size(SJ); L = L(2);
%cj = bj-[sj^rXr,0]aj

    for j = 1:L
                                CL(:,j)=BL(:,j)-[SJ(1,j)^r*Xr,Z]*AL(:,j);
                                    end;

end

function [M,SDL,ERROR] = solve_pmi(AL,BL,SJ,d)

% This is a function to solve the problem defined in Corollary 2.6:
% Given {sj,aj,bj,d}, j=1,L, find Qd such that Q(s)=QdSd(sj)aj = bj.
% The relation of this formulation and the standard one is that Sd(s) = KS(s).
% see eqation (2.11).

SDL = SDL_BUILD(AL,SJ,d); %find SDL
%Calculate the solution of MSDL = BL.M=[M0,M1,...] and M(s)=M0+sM1+...
M = BL/(SDL);
ERROR=norm((M*SDL-BL),'fro');

function SDL = SDL_BUILD(AL,SJ,d)
% Find the matrix SDL defined in Corrollary 2.6

[m,L] = size(AL);

I = eye(m);
        for n = 1:L
                        SD = I;
                                for i = 1:d
                                        SD = [SD;I*SJ(n)^i];   % build Sd(sj)
                                end
                        if n == 1, SDL = SD*AL(:,1);
                        else
                        SDL = [SDL,SD*AL(:,n)];
                        end

        end
```