ON OUTPUT REGULATION WITH STABILITY IN MULTIVARIABLE SYSTEMS

by

P. J. Antsaklis

Technical Report No. 810
April, 1981

Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

On Output Regulation with Stability in Multivariable Systems

by

P. J. Antsaklis*

ABSTRACT

A general version of the tracking and regulation problem in linear multivariable systems is considered and the class of output dynamic compensators C which regulate and at the same time internally stabilize the closed loop system is parametrically characterized. The solution is obtained by first considering the class of all stabilizing compensators and then restricting this class to achieve regulation as well. It is shown that if a solution exists, C must be of the form $G_d^{-1}\,\hat{C}\,G_n$ where G_d, G_n depend on the system parameters and the exogenous signal to be rejected while $\hat{C}$ depends on the desired closed loop characteristic polynomial.

*The author is with the Dept. of Electrical Engineering, Univ. of Notre Dame, Notre Dame, IN 46556. Parts of this work were carried out while the author was at the Dept. of Electrical Engineering, Rice Univ. and with the Dept. of Computing and Control, Imperial College, England.

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

I. INTRODUCTION

A general version of the regulator problem with internal stability in multivariable systems is studied in this paper. Specifically it is assumed that the system is described by (1) where y and z are the measured and regulated outputs respectively. The objective is to determine a suitable controller C so that the transfer matrix from d to z, T_{zd} is stable, i.e., regulation, while internal stability is attained, i.e., the closed loop is stable and no unstable cancellations take place. Note that d represents the bounded part of the exogenous signal $w = Q_d^{-1}d$ whose effect must be eliminated from z exponentially.

Less general versions of this problem have been studied by a number of authors. In particular, the state-space representation and the geometric approach were used in [1],[2] (solution via state-variable feedback). In [3] a slightly more general model was used to directly derive an output compensator using polynomial matrix factorizations of transfer matrices under the assumption that the system is stabilizable and detectable. Clearly this assumption is not restrictive since if it is not satisfied, the system can't even be stabilized, let alone regulated. The model used in these papers represents a special case of the formulation used here as it implies specific interelations among H_1, H_2, G_1, and G_2. In particular, $-H_1=C_1(s-A_1)^{-1}B_1+C_3$, $H_2=D_1(s-A_1)^{-1}B_1+D_3$, $G_1=[C_1(s-A)^{-1}A_3+C_2](s-A_2)^{-1}$, $G_2=[D_1(s-A_1)^{-1}A_3+D_2](s-A_2)^{-1}$ ($d=x_2(0)$) to use the notation of [3]. In [4] the case where the measured outputs are those to be regulated (y=-z in our formulation) was studied using for the first time polynomial matrix factorizations although polynomial matrices had been used earlier [5] to study this problem (y=-z) for step disturbances only. The characterization, via a stable rational matrix K, of all stabilizing output compensators introduced in [17] was used in [6]

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

to solve the regulation problem with internal stability when $H_1=FP$ and $H_2=P$. In [7] the case when $y=T_s z$, where T_s a stable sensor, was considered ($-H_1=T_sH_2$, $G_1=T_sG_2$) while in [8] and [9] the case $-H_1=V_1T^{-1}U_1$, $H_2=V_2T^{-1}U_1$, $G_1=V_1T^{-1}U_2R^{-1}M$ and $G_2=V_2T^{-1}U_2R^{-1}M$, where all new symbols represent matrices, was studied using the transformation $\lambda=1/(s-a)$. It should also be noted that a number of authors have studied the problem of regulation with internal stability under small plant perturbations (state-space model or $y=-z$ using transfer matrices), i.e., with robusteness (see for example [10],[11],[12],[13]). The problem of robusteness is not addressed here although a number of relevant observations are made in the form of brief comments.

In this paper given (1), necessary and sufficient conditions are derived for regulation with internal stability. The class of all appropriate compensators C is parametrically characterized and it is shown that C must be of the form $G_d^{-1}\hat{C}G_n$ where G_d,G_n depend on the system parameters and the exogenous signal to be rejected while $\hat{C}$ depends on the desired closed loop characteristic polynomial. This structure is imposed on C because of the requirement for regulation. The problem is solved using the characterization, via two polynomial matrices A and B, of all internally stabilizing output compensators introduced in [14]. Then the requirement for internal stability is replaced by $|A|$ being a stable polynomial. A new problem RPS1 is defined in terms of A and B which is equivalent to the regulation problem with internal stability, RPS, except the properness requirement for C. RPS1 is first solved and then it is shown that one can always obtain a proper compensator C from the class of solutions A, B of RPS1. The necessary and sufficient conditions for the solution of RPS are therefore the conditions for the solution of RPS1. Note that while internal stability implies B arbitrary and $|A|$ stable, if regulation in addition to internal stability is required then B should be-

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

long to an appropriate class of matrices. It should be noted that the method used in the proofs resembles that of [6] while results involving polynomial matrices are drawn heavily from [14].

If the present work is to be compared with other published studies of the regulator problem with internal stability the following should be pointed out: A complete parametric characterization of all compensators C which solve a more general version of the problem is given. It is shown that the objective of regulation is translated into restrictions only on the matrix B where A($|A|$ stable) and B fully characterize all internally stabilizing compensators. The dual role of the compensator C then becomes quite apparent. If regulation, in addition to internal stability is required, C must introduce in the loop characteristics of the exogenous signal so that appropriate signals are generated to cancel the exogenous ones while internal stability is maintained. In particular C must be of the form $G_d^{-1}\hat{C}\,G_n$, but the existence of G_d, G_n in C is only a necessary condition for regulation in this general case (when y = -z it is necessary and sufficient); it is sufficient as well only when certain conditions are satisfied ($|G_d||G_n|=\alpha|\tilde{Q}_1|$). The properness of C is also studied. It had been intuitively known for quite sometime that if the order of C is "high enough", there will be enough coefficients to be evaluated so that the design objectives can be achieved with a proper C. This is formally shown here to be true. In addition, comments and remarks throughout this paper are included to clarify the meaning and stress the significance of internal stability, the necessary and sufficient conditions and the structure of the compensator.

This paper is organized as follows: The problem is formulated in section III while in IV internal stability is discussed and the internally stabilizing output compensators are characterized via the polynomial matrices B and A ($|A|$ stable). In Section V a new problem, RPS1, is introduced and solved (Theorem 1, and Corollaries 2 and 3). RPS1 is then shown to be equivalent to the regulator problem with internal stability, RPS, in Theorem 4. This is done by proving that a proper compensator C can always be derived from the solution A, B of RPS1. The structure of C is discussed in Section VI and an illustrative example is given. In Section VII the important special case $y = -z$ is studied while in VIII the necessary and sufficient conditions for the solution of RPS are discussed. Note that certain results appearing in this paper have been presented in [15].

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

II. PRELIMINARIES

A rational matrix H_1 can be written as $H_1=N_1D_1^{-1}=\tilde{D}_1^{-1}\tilde{N}_1$ where (N_1,D_1) are relatively right prime (rrp) and $(\tilde{N}_1,\tilde{D}_1)$ relatively left prime (rlp) polynomial matrices [16]. Then, there exist polynomial matrices $X_1,Y_1,\tilde{X}_1$ and $\tilde{Y}_1$ such that [14]

$$\begin{aligned} X_1D_1 + Y_1N_1 &= I & D_1X_1 + \tilde{Y}_1\tilde{N}_1 &= I \\ -X_1\tilde{Y}_1 + Y_1\tilde{X}_1 &= 0 & D_1Y_1 - \tilde{Y}_1\tilde{D}_1 &= 0 \\ -\tilde{N}_1D_1 + \tilde{D}_1N_1 &= 0 & N_1X_1 - \tilde{X}_1\tilde{N}_1 &= 0 \\ \tilde{N}_1\tilde{Y}_1 + \tilde{D}_1\tilde{X}_1 &= I & N_1Y_1 + \tilde{X}_1\tilde{D}_1 &= I \end{aligned} \tag{P1}$$

Notice that $|A|$ denotes the determinant of the square matrix A and $\sigma(A)$ the collection of roots of the polynomial $|A|$. $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the closed right half and open left half of the complex plane. Stable polynomial p means that all the roots of p are in $\mathbb{C}^-$ and stable rational matrix H means that the characteristic polynomial of H is stable (the poles of H are in $\mathbb{C}^-$). $(H)_+$ represents the part of a partial fraction expansion of the rational matrix H with poles in $\mathbb{C}^+$ [4]. H can always be written as $H = (H)_+ + (H)_-$.

It should be noted that the results of this paper are still valid if $\mathbb{C}^+$ and $\mathbb{C}^-$ are taken to denote instead any disjoint partition of the complex plane i.e., $\mathbb{C}^+\cup\mathbb{C}^-=\mathbb{C}$, $\mathbb{C}^+\cap\mathbb{C}^-=\phi$ where $\mathbb{C}^-$ is symmetric with respect to the real axis and contains at least one real point. In this case, $\mathbb{C}^-$ and $\mathbb{C}^+$ are the "good" and the "bad" regions respectively instead of the stable and the unstable regions. Appropriate charges should then be made throughout.

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

III. PROBLEM FORMULATION

Consider the system

$$
\begin{aligned}
y &= -H_1 u + G_1 d \\
z &= H_2 u + G_2 d \qquad (1)\\
u &= Cy
\end{aligned}
$$

where y is the (px1) vector of the measured outputs, z is the (rx1) vector of the outputs to be regulated, u is the (mx1) vector of the control inputs and d a (qx1) vector representing the disturbances; H_1, H_2, G_1 and G_2 are transfer matrices of appropriate dimensions. u=Cy is the control law and C the compensator to be determined. Substituting we obtain

$$z = [H_2 C [I + H_1 C]^{-1} G_1 + G_2] d = T_{zd} d \qquad (2)$$

The objective is to determine a suitable compensator C such that T_{zd} is stable <u>(regulation)</u> while <u>internal stability</u> in the closed loop is also attained. Note that d corresponds to the bounded part of the known exogenous signal w; w has been expressed as $w = Q_d^{-1} d$ and $Q_d^{-1}(\sigma(Q_d) \subset \mathbb{C}^+)$ has already been included in G_1 and G_2 in (1).

Let

$$H_1 = N_1 D_1^{-1} = \tilde{D}_1^{-1} \tilde{N}_1 \qquad (3)$$

$$H_2 D_1 = \tilde{D}_2^{-1} \tilde{N}_2 \qquad (4)$$

where (N_1, D_1) are rrp and (N_1, D_1), $(\tilde{N}_2, \tilde{D}_2)$ are rlp polynomial matrices. Let also C be represented by a fraction of two rlp polynomial matrices.

$$C = \tilde{D}_c^{-1} \tilde{N}_c \qquad (5)$$

Using the identity $C(I + H_1 C)^{-1} = (I + C H_1)^{-1} C$ and substituting the above in (2), we obtain

$$T_{zd} = \tilde{D}_2^{-1} \tilde{N}_2 [\tilde{D}_c D_1 + \tilde{N}_c N_1]^{-1} \tilde{N}_c G_1 + G_2 \qquad (6)$$

It will be assumed in the following that $|\tilde{D}_2|$ is a stable polynomial†. This

---

† If m=1 this assumption is not necessary as the techniques presented in this paper can still be applied directly.

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

assumption, which has been implicitly or explicitly made in all other published studies of the regulator problem, greatly simplifies the analysis. It simply means that all the unstable poles of H_2 also appear as poles of H_1 (together with some similarity in structure). Considering that one must be able to observe through y the unstable modes which affect the trajectory of z in order to regulate z, it is clear that this assumption is not unreasonable. If the problem were solved without this assumption it can be conjectured in view of the conditions derived in this paper, that the new necessary and sufficient conditions would involve, in general, more intricate structural relations among H_1, H_2, G_1 and G_2 than at present.

In the following sections, the internal stability of the system is discussed and the regulator problem with internal stability is precisely stated and solved.

IV. INTERNAL STABILITY

The system will be called internally stable if $|\tilde{D}_c D_1 + \tilde{N}_c N_1|$ is a stable polynomial i.e. a polynomial with roots in $\mathbb{C}^-$. Therefore, for internal stability

$$\tilde{D}_c D_1 + \tilde{N}_c N_1 = A \tag{7}$$

where A is any square polynomial matrix of appropriate dimensions with $|A|$ a stable polynomial.

To explain this, let d=0 in (1) and consider $y=-H_1u$, $u=Cy$ i.e., H_1 compensated by output feedback. The loop will be stable to an outside observer if the numerator of $|I+H_1C|$ is a stable polynomial (loop stability). It is possible though, to have unstable cancellations in H_1C with loop stability present. If no such unstable cancellations take place and loop stability is present, the system is called internally stable. Internal stability is therefore the property to aim for in practice since if the loop is stable but unstable cancellations take place, system components will saturate (because of imprecise cancellations) and the system will behave in a manner other than intended. A number of internal stability criteria have appeared in the literature. If, for example, $|D_1||\tilde{D}_c||I+H_1C|$ is a stable polynomial or if $H=[H_{ij}]$ where $H_{11}=(I+H_1C)^{-1}$, $H_{21}=CH_{11}$, $H_{22}=(I+CH_1)^{-1}$, $H_{12}=-H_1H_{22}$ is a stable transfer matrix then, as it can be easily seen using (3) and (5), no unstable cancellations take place and the system is internally stable. Clearly the danger of unstable cancellations remaining undetected arises from the use of transfer matrices or external descriptions in the analysis. An internal description, the polynomial matrix representation, was used to derive (7) in [14] in a straightforward manner. Note also that criterion (7) directly leads to the full characterization of all stabilizing compensators. In particular, since (N_1,D_1) are rrp, there exist polynomial matrices X_1,Y_1 such that

$$X_1 D_1 + Y_1 N_1 = I \tag{8}$$

In view now of [14], (7) becomes

$$\begin{aligned} \tilde{D}_c &= AX_1 - B\tilde{N}_1 \\ \tilde{N}_c &= AY_1 + B\tilde{D}_1 \end{aligned} \qquad \text{or} \qquad [\tilde{D}_c, \tilde{N}_c] = [A, B]U \tag{9}$$

where $(\tilde{N}_1, \tilde{D}_1)$ are rlp (see (3)), B is an arbitrary polynomial matrix and $U \triangleq \begin{bmatrix} X_1 & Y_1 \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix}$ is a unimodular matrix. (9) can also be written as

$$\begin{aligned} \tilde{D}_c D_1 + \tilde{N}_c N_1 &= A \\ -\tilde{D}_c \tilde{Y}_1 + \tilde{N}_c \tilde{X}_1 &= B \end{aligned} \qquad \text{or} \qquad [A, B] = [\tilde{D}_c, \tilde{N}_c]U^{-1} \tag{10}$$

where the unimodular matrix $U^{-1} = \begin{bmatrix} D_1 & -\tilde{Y}_1 \\ N_1 & \tilde{X}_1 \end{bmatrix}$ with $\tilde{Y}_1, \tilde{X}_1$ appropriate polynomial matrices satisfying

$$\tilde{D}_1 \tilde{X}_1 + \tilde{N}_1 \tilde{Y}_1 = I \tag{11}$$

as well as (P1). As it was shown in [14], $C = \tilde{D}_c^{-1} \tilde{N}_c$ where $\tilde{D}_c, \tilde{N}_c$ from (9) with B and A($|A|$stable) any polynomial matrices such that $|\tilde{D}_c| \neq 0$, fully characterizes all internally stabilizing compensators, proper and nonproper (a method to obtain proper compensators by choosing B and A is given in a later section).

It should be noted that the characterization of all compensators achieving internal stability was first given by [17] and it involved both polynomial and rational matrices ($C = (X_1 - K\tilde{N}_1)^{-1}(Y_1 + K\tilde{D}_1)$ where K any stable rational matrix). Later, in [14], the equivalent characterization (9) (and (10)) which involves polynomial matrices only, was derived in a natural way from the polynomial matrix representation of the system. Recently, in [18], it has been shown that this latter characterization is also valid when more general rings, than the ring of polynomials, are considered.

In the above, all stabilizing compensators were effectively parametrized by the matrices B and A ($|A|$ stable) via (9) and (10). Note that $|A|$ is the characteristic polynomial of the closed loop. This parametrization will now

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

problem by the restriction that $|A|$ is stable, thus simplifying the analysis. In this setting, it will be shown that the regulation requirement implies restrictions only on the matrix B. That is, it will be shown that if regulation in addition to internal stability is required, (9) and (10), still characterize all appropriate compensators, where A is any matrix such that $|A|$ stable; B is not arbitrary any longer but it must satisfy certain conditions imposed by the regulation requirement (see (18)).

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

V. THE REGULATOR PROBLEM WITH INTERNAL STABILITY (RPS)

In view of the above, RPS can now be precisely stated:

RPS Determine $C = \tilde{D}_c^{-1}\tilde{N}_c$ proper such that

a) $T_{zd} = \tilde{D}_2^{-1}\tilde{N}_2[\tilde{D}_c D_1 + \tilde{N}_c N_1]^{-1}\tilde{N}_c G_1 + G_2$ is stable (regulation) (6)

and b) $\tilde{D}_c D_1 + \tilde{N}_c N_1 = A$ where $|A|$ stable (internal stability) (7)

Using (7) and (9), (6) becomes

$$T_{zd} = \tilde{D}_2^{-1}[\tilde{N}_2 A^{-1} B \tilde{D}_1 G_1 + (\tilde{N}_2 Y_1 G_1 + \tilde{D}_2 G_2)] \tag{12}$$

Let

$$\begin{aligned} \tilde{D}_1(G_1)_+ &= P_1 Q_1^{-1} = \tilde{Q}_1^{-1}\tilde{P}_1 \\ (\tilde{N}_2 Y_1 G_1 + \tilde{D}_2 G_2)_+ &= P_2 Q_2^{-1} \end{aligned} \tag{13}$$

where (P_1,Q_1), (P_2,Q_2) are rrp and $(\tilde{P}_1,\tilde{Q}_1)$ rlp polynomial matrices. Note that $\sigma(Q_1) = \sigma(\tilde{Q}_1)$ and $\sigma(Q_2)$ are in $\mathbb{C}^+$. Since $|\tilde{D}_2|$ is stable by assumption, T_{zd} is stable if and only if

$$T_1 = \tilde{N}_2 A^{-1} B P_1 Q_1^{-1} + P_2 Q_2^{-1} \tag{14}$$

is stable. Define now a new problem, RPS1, related to RPS.

RPS1 Determine polynomial matrices B and A, $|A|$ stable, so that T_1 is stable.

Note that if RPS has a solution so does RPS1. If RPS1 has a solution A,B then RPS also has a solution provided that a proper C can always be determined from A and B. In the following, RPS will be solved by first solving RPS1 and then showing that a proper C always exists.

Since (P_1,Q_1) and $(\tilde{P}_1,\tilde{Q}_1)$ in (15) are rrp and rlp respectively, there exist polynomial matrices X_G, Y_G, $\tilde{X}_G$ and $\tilde{Y}_G$ which satisfy

$$\begin{aligned} X_G Q_1 + Y_G P_1 &= I \\ \tilde{Q}_1 \tilde{X}_G + \tilde{P}_1 \tilde{Y}_G &= I \end{aligned} \tag{15}$$

and relations (P1).

Theorem 1: RPS1 has a solution if and only if

a) $Q_2^{-1} Q_1 = Q$ a polynomial matrix (16)

and b) there exist polynomial matrices a_1 and a_2 so that

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

$$\tilde{N}_2 a_1 + a_2 \tilde{Q}_1 = P_2 Q Y_G \tag{17}$$

If solution exists then

$$A = \text{any polynomial matrix } (|A| \text{ stable}) \tag{18}$$

and $B = -Aa_1 + m_1$

where (a_1, a_2) is a solution of (17) and (m_1, m_2) any solution of

$$\hat{N}_2 m_1 + m_2 \tilde{Q}_1 = 0 \tag{19}$$

where $\tilde{N}_2 A^{-1} = \hat{A}^{-1}\hat{N}_2$ with $(\hat{N}_2, \hat{A})$ rlp polynomial matrices.

<u>Proof</u>: Assume that RPS1 has a solution, that is B and A ($|A|$ stable) have been found so that T_1 is stable. Let $\tilde{N}_2 A^{-1} = \hat{A}^{-1}\hat{N}_2$ where $(\hat{A}, \hat{N}_2)$ rlp. Then (14) implies that

$$\hat{N}_2 B P_1 Q_1^{-1} + \hat{A} P_2 Q_2^{-1} = M \tag{20}$$

a polynomial matrix. This is because if M were not a polynomial matrix, it would have been a rational matrix with all of its poles in $\mathbb{C}^+$ in which case a premultiplication by $\hat{A}^{-1}$ would have shown that $T_1 = \hat{A}^{-1}M$ is not stable contrary to the assumption. Postmultiplying (20) by Q_1

$$\hat{N}_2 B P_1 + \hat{A} P_2 Q_2^{-1} Q_1 = M Q_1 \tag{21}$$

which, in view of the fact that (P_2, Q_2) are rrp and $\sigma(\hat{A}) \subset \mathbb{C}^-$, $\sigma(Q_2) \subset \mathbb{C}^+$, implies that $Q_2^{-1}Q_1 = Q$ a polynomial matrix i.e., (18). Substituting and rearranging

$$(-\hat{N}_2 B) P_1 + (M) Q_1 = \hat{A} P_2 Q \tag{22}$$

Since (P_1, Q_1) are rrp, in view of (15) and [14], (22) implies

$$-\hat{N}_2 B = (\hat{A} P_2 Q) Y_G + W \tilde{Q}_1 \tag{23a}$$

$$M = (\hat{A} P_2 Q) X_G - W \tilde{P}_1 \tag{23b}$$

where W an appropriate polynomial matrix .(23a) can be rewritten as

$$\hat{N}_2 (B) + (W) \tilde{Q}_1 = -\hat{A}(P_2 Q Y_G) \tag{24}$$

or

$$\hat{N}_2 (B) + \hat{A}(P_2 Q Y_G) = -W \tilde{Q}_1 \tag{25}$$

Note that $(\hat{N}_2, \hat{A})$ are rlp, so there exist polynomial matrices $X_{\hat{A}}$ and $Y_{\hat{A}}$ such that $\hat{A} X_{\hat{A}} + \hat{N}_2 Y_{\hat{A}} = I$. Let $\hat{A}^{-1}\hat{N}_2 (= \tilde{N}_2 A^{-1}) = \overline{N}_2 \overline{A}^{-1}$ where $(\overline{N}_2, \overline{A})$ are rrp.

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

(25) now implies

$$B = Y_{\hat{A}}(-W\tilde{Q}_1) + \bar{A}V \tag{26a}$$

$$P_2QY_G = X_{\hat{A}}(-W\tilde{Q}_1) - \bar{N}_2V \tag{26b}$$

where V is an appropriate polynomial matrix. (26b) shows that if RPS1 has a solution, there exist polynomial matrices $\bar{a}_1$ and $\bar{a}_2$ so that

$$\bar{N}_2\bar{a}_1 + \bar{a}_2\tilde{Q}_1 = P_2QY_G \tag{27}$$

Note that $\tilde{N}_2 = \bar{N}_2G$, $A = \bar{A}G$ where $\sigma(G)\subset\mathbb{C}^-$ and that $\sigma(\tilde{Q}_1)\subset\mathbb{C}^+$. This, in view of [6, Lemma 7] implies that if (27) is satisfied, there exist polynomial matrices a_1 and a_2 so that

$$\tilde{N}_2a_1 + a_2\tilde{Q}_1 = P_2QY_G \tag{17}$$

To show sufficiency, assume that the conditions of the theorem are satisfied. Choose any polynomial matrix A of appropriate dimensions such that $|A|$ stable and determine $(\hat{N}_2,\hat{A})$ rlp from $\tilde{N}_2A^{-1} = \hat{A}^{-1}\hat{N}_2$. Premultiply (17) by $-\hat{A}$ to obtain

$$\hat{N}_2(-Aa_1) + (-\hat{A}a_2)\tilde{Q}_1 = -\hat{A}(P_2QY_G) \tag{28}$$

noticing that $\hat{A}\tilde{N}_2 = \tilde{N}_2A$. Comparing with (24), let $-Aa_1 = B$, $-\hat{A}a_2 = W$ and define M from (23b). Postmultiplying (23a) and (23b) by P_1 and Q_1 respectively and adding, (22) is obtained and finally in view of (16), $T_1 = \hat{A}^{-1}M$ which is stable.

Before showing that the general solution is given by (18), the following corollary is in order.

<u>Corollary 2</u>: RPS1 has a solution if and only if

a) $Q_2^{-1}Q_1 = Q$ a polynomial matrix (16)

and b) there exist polynomial matrices b_1 and b_2 so that

$$\hat{N}_2b_1 + b_2\tilde{Q}_1 = -\hat{A}(P_2QY_G) \tag{29}$$

where $(\hat{N}_2,\hat{A})$ are rlp determined from $\tilde{N}_2A^{-1} = \hat{A}^{-1}\hat{N}_2$ with A an arbitrary ($|A|$ stable) polynomial matrix. If a solution exists, this A and $B = b_1$, where (b_1,b_2) any solution of (29), is the solution to RPS1.

Proof: There exist b_1 and b_2 which satisfy (29) if and only if there exist a_1 and a_2 which satisfy (17) of Theorem 1. This was actually shown in the proof of the first part of the theorem (read b_1, b_2 instead of B, W in (24)). Therefore, since condition b) of the corollary is equivalent to condition b) of the theorem, RPS1 has a solution if and only if (16) and (29) are satisfied. To show the second part of the corollary, notice that if RPS1 has a solution B and A ($|A|$ stable) then (24) is satisfied or (29) with $b_1 = B$. Conversely, choose any A such that $|A|$ stable and any solution (b_1,b_2) of (29); consider b_1 and b_2 instead of B and W in (24). Backtracking (as in the sufficiency proof of the theorem) $T_1 = \hat{A}^{-1}M = \tilde{N}_2A^{-1}b_1P_1Q_1^{-1} + P_2Q_2^{-1}$ stable. That is A and b_1 are the solution A and B of RPS1. Q.E.D.

Proof of Theorem 1 (Cont.): The solution b_1 of (29) can be written as the sum of a particular solution and any solution of (19). In the sufficiency proof of the first part of the theorem it was shown that $-Aa_1$, where a_1 a solution of (17), is a particular solution of (24) or (29). Therefore, in view of Corollary 2, any solution to RPS1 is given by (18). This concludes the proof of the theorem.

Another set of necessary and sufficient conditions is given by the following corollary.

Corollary 3: RPS1 has a solution if and only if

a) $Q_2^{-1}Q_1 = Q$ a polynomial matrix (16)

and b) there exist polynomial matrices c_1 and c_2 so that

$$\tilde{N}_2c_1 + c_2Q_1 = P_2Q \tag{30}$$

If solutions exist then

$$\begin{aligned} A &= \text{any polynomial matrix } (|A| \text{ stable}) \\ B &= -Ac_1Y_G + m_1 \end{aligned} \tag{31}$$

where (c_1,c_2) a solution of (30) and m_1 any solution of (19).

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

Proof: Assume that (30) is satisfied. Postmultiply by Y_G to obtain

$$\tilde{N}_2(c_1Y_G) + (c_2\tilde{Y}_G)\tilde{Q}_1 = P_2QY_G$$

in view of (15) i.e., (17) is satisfied with $(a_1,a_2)=(c_1Y_G,\ c_2\tilde{Y}_G)$. Assume that (17) is satisfied and postmultiply by P_1 to obtain

$$\tilde{N}_2(a_1\tilde{P}_1) + a_2\tilde{Q}_1P_1 = P_2QY_GP_1$$

or

$$\tilde{N}_2(a_1\tilde{P}_1) + (a_2\tilde{P}_1)Q_1 = P_2Q(I-X_GQ_1)$$

or

$$\tilde{N}_2(a_1\tilde{P}_1) + (a_2\tilde{P}_1 + P_2QX_G)Q_1 = P_2Q$$

in view of (15) and (P1), i.e., (30) is satisfied. Therefore (30) has a solution if and only if (17) has a solution. Corollary 3 then directly follows from Theorem 1. Q.E.D.

Three equivalent sets of necessary and sufficient conditions for the solution of RPS1 were presented above. Note that the conditions in Theorem 1 and Corollary 3 do not depend on the choice of A(|A| stable is the characteristic polynomial of the closed loop). (17) and (30) involve, in general, matrices of different dimensions. Dimensional considerations therefore might play the decisive role in choosing between them. The conditions in Corollary 2 are dependent on A but the solution B is directly given by the solution of (29).

Stability and Regulation: As it will be shown in Theorem 4, A and B can always be chosen to give a proper compensator C while satisfying (18). It should be noted at this point that the formulation used in this paper to solve the regulator problem with stability, clearly reveals the distinction between the requirements imposed to the system structure and the compensator, due to the two inportant design objectives, internal stability and regulation. In particular, if internal stability is the only objective then all stabilizing compensators $C = \tilde{D}_c^{-1}\tilde{N}_c$ satisfy (9) or

$$\tilde{D}_cD_1 + \tilde{N}_cN_1 = A \tag{10a}$$

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

$$-\tilde{D}_c\tilde{Y}_1 + \tilde{N}_c\tilde{X}_1 = B \tag{10b}$$

where $|A|$ is the desired closed loop characteristic polynomial and B any polynomial matrix. (10b) therefore does not imply any restriction on $\tilde{D}_c$ and $\tilde{N}_c$ and this is why (10a) appears in the literature as the only equation to be satisfied when internal stability is the only issue. If in addition, regulation is to be achieved, B in (10b) is not arbitrary any longer but it must belong to an appropriate class of matrices (described here by (18)) which depends on the particular model representation used. Equation (10b) therefore corresponds to regulation (actually regulation with internal stability present), while (10a) to stabilization; it is (10b) which imposes, as it will be shown, certain structure on the output compensator and consequently on the closed loop system so that appropriate signals are generated in the loop to cancel the undesirable exogenous ones. Finally, note that if other design objectives are sought, in addition to internal stability, they will again be translated into requirements for B (and maybe A) to belong to a certain class, that is (10) are the design equations when internal stability is required.

Before showing that RPS has a solution if and only if RPS1 has a solution (Theorem 4) the form of the compensator $C = \tilde{D}_c^{-1}\tilde{N}_c$, given by (9) with $|A|$ stable and $B=-Aa_1+m_1$, will be studied. In particular, substituting (18) into (9) and (10):

$$\begin{aligned} \tilde{D}_c &= A\bar{X}_1 - m_1\tilde{N}_1 \\ \tilde{N}_c &= A\bar{Y}_1 + m_1\tilde{D}_1 \end{aligned} \qquad \text{or} \qquad [\tilde{D}_c, \tilde{N}_c] = [A, m_1]\bar{U} \tag{32}$$

$$\begin{aligned} \tilde{D}_c D_1 + \tilde{N}_c N_1 &= A \\ -\tilde{D}_c\tilde{\bar{Y}}_1 + \tilde{N}_c\tilde{\bar{X}}_1 &= m_1 \end{aligned} \qquad \text{or} \qquad [A, m_1] = [\tilde{D}_c, \tilde{N}_c]\bar{U}^{-1} \tag{33}$$

where $\bar{X}_1 \triangleq X_1 + a_1\tilde{N}_1$, $\bar{Y}_1 \triangleq Y_1 - a_1\tilde{D}_1$, $\tilde{\bar{X}}_1 \triangleq \tilde{X}_1 + N_1a_1$, $\tilde{\bar{Y}}_1 = \tilde{Y}_1 - D_1a_1$

and

$$\bar{U} = \begin{bmatrix} \bar{X}_1 & \bar{Y}_1 \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix} = \begin{bmatrix} 1 & -a_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 & Y_1 \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix} = \begin{bmatrix} 1 & -a_1 \\ 0 & 1 \end{bmatrix} U$$

$$\bar{U}^{-1} = U^{-1} \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} D_1 & -\tilde{Y}_1 \\ N_1 & \tilde{X}_1 \end{bmatrix} \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} D_1 & -\tilde{\tilde{Y}}_1 \\ N_1 & \tilde{\tilde{X}}_1 \end{bmatrix}$$

Clearly, $\bar{U}$ and $\bar{U}^{-1}$ are unimodular matrices. (32) (or (33)) give appropriate $\tilde{D}_c$, $\tilde{N}_c$ for any solution m_1 of

$$\hat{N}_2 m_1 + m_2 \tilde{Q}_1 = 0 \tag{19}$$

A rather large class of solutions is given by $(m_1, m_2) = (W\tilde{Q}_1, -\hat{N}_2 W)$ where W is any polynomial matrix of appropriate dimension. Substituting,

$$\begin{aligned} \tilde{D}_c &= A\bar{X}_1 - W\tilde{Q}_1\tilde{N}_1 \\ \tilde{N}_c &= A\bar{Y}_1 + W\tilde{Q}_1\tilde{D}_1 \end{aligned} \quad \text{or} \quad [\tilde{D}_c, \tilde{N}_c] = [A, W\tilde{Q}_1]\bar{U} \tag{32a}$$

and

$$\begin{aligned} \tilde{D}_c D_1 + \tilde{N}_c N_1 &= A \\ -\tilde{D}_c \tilde{\tilde{Y}}_1 + \tilde{N}_c \tilde{\tilde{X}}_1 &= W\tilde{Q}_1 \end{aligned} \quad \text{or} \quad [A, W\tilde{Q}_1] = [\tilde{D}_c, \tilde{N}_c]\bar{U}^{-1} \tag{33a}$$

For any W, (32a) gives appropriate $\tilde{D}_c, \tilde{N}_c$ which define C. It will now be shown that if RPS1 has a solution, then among the solutions A and B one can always choose a pair which produces a proper $C = \tilde{D}_c^{-1}\tilde{N}_c$; that is one can always choose W appropriately in (32a) to obtain a proper C.

<u>Theorem 4</u>: RPS has a solution if and only if RPS1 has a solution.

<u>Proof</u>: From the definition of RPS and RPS1 it is clear that if RPS has a solution ($C = \tilde{D}_c^{-1}\tilde{N}_c$ proper) so does RPS1 with A and B uniquely determined from $\tilde{D}_c, \tilde{N}_c$ via (10). Assume now that RPS1 has a solution given by (18). To each solution pair (A,B) there corresponds a unique pair $(\tilde{D}_c, \tilde{N}_c)$ determined from (9) (or (33)); if $|\tilde{D}_c| \neq 0$, $C = \tilde{D}_c^{-1}\tilde{N}_c$ will make T_{zd} stable and the closed loop internally stable. Since RPS requires C to be proper, to show that in this case RPS has a solution, it suffices to show that one can always choose a solution pair A,B (or an A and a W in (32a) or (33a)) so that $C = \tilde{D}_c^{-1}\tilde{N}_c$ exists

and it is proper†.

Proper C: Let $R \overset{\Delta}{=} C(I + H_1C)^{-1} = (I + CH_1)^{-1}C$. Then

$$C = R(I - H_1R)^{-1} \tag{34}$$

If $C=\tilde{D}_c^{-1}\tilde{N}_c$ with $\tilde{D}_c,\tilde{N}_c$ from (32a) ($|\tilde{D}_c|\neq 0$) then

$$R = D_1A^{-1}(A\bar{Y}_1 + W\tilde{Q}_1\tilde{D}_1) \tag{35}$$

Conversely, if (35) is used and C is determined from (34) then it is equal to $C=\tilde{D}_c^{-1}\tilde{N}_c$ with $(\tilde{D}_c,\tilde{N}_c)$ given by (32a). Note that if $\tilde{D}_c^{-1}$ exists, so does $(I-H_1R)^{-1}$ and vice-versa. Therefore, the relations (35) and (34) with $|I-H_1R|\neq 0$ can be used to determine C instead of the relations (32a) and $C=\tilde{D}_c^{-1}\tilde{N}_c$ with $|\tilde{D}_c|\neq 0$. It can be shown that if H_1 is proper (strictly proper) and R is strictly proper (proper) then C from (34) exists ($|I-H_1R|\neq 0$) and it is proper. In the following it will be shown how to choose A and W in (35) so that R is proper (strictly proper).

Let G_n be a greatest common right divisor (gcrd) of $\bar{Y}_1,\tilde{Q}_1\tilde{D}_1$. Note that $\sigma(G_n)\not\subset \mathbb{C}^+$ since they are some or all of the roots of $|\tilde{Q}_1|$ ($\bar{Y}_1,\tilde{D}_1$ are rlp). Let $\bar{Y}_1(\tilde{Q}_1\tilde{D}_1)^{-1}=N_3D_3^{-1}$ with (N_3,D_3) rrp and $A=A_3A_2A_1$ where $(A_2A_1N_3,D_3)$rrp, $\deg|A_1|>\deg|D_1|$, $\deg|A_2|>\deg|G_n|$ and A_3 to be determined. Then

$$R = (D_1A_1^{-1})A_2^{-1}[A_3^{-1}\ \bar{N}_3]G_n \tag{36}$$

where $\bar{N}_3\overset{\Delta}{=}A_3(A_2A_1N_3)+WD_3$. Since $(A_2A_1N_3,D_3)$ rrp, there exist polynomial matrices y_1,x_1 such that $y_1(A_2A_1N_3) + x_1D_3 = I$. The equation for $\bar{N}_3$ is then equivalent [14] to:

$$A_3 = \bar{N}_3y_1 + \bar{W}\hat{D}_3 \tag{37a}$$

$$W = \bar{N}_3x_1 - \bar{W}\hat{N}_3 \tag{37b}$$

where $(A_2A_1N_3)D_3^{-1}=\hat{D}_3^{-1}\hat{N}_3$ with $(\hat{N}_3,\hat{D}_3)$ rlp. Note that in (37a) y_1 and $\hat{D}_3$ are rrp. So A_3, with $|A_3|$ stable, and $\bar{N}_3,\bar{W}$ can always be found so that (37a) is satisfied with $A_3^{-1}\bar{N}_3$ proper (see [16]). Using (37b), W is also determined. Notice that $A_2^{-1}[A_3^{-1}\bar{N}_3]G_n$ in (36) is proper since $A_3^{-1}\bar{N}_3$ proper, $\deg|A_2|>\deg|G_n|$ and

†(32), for $|\tilde{D}_c|\neq 0$, generates a more general class of solutions C which includes nonproper compensators as well.

$\sigma(G_n)\mathbb{C}^+$ while $\sigma(A_3A_2)\mathbb{C}^-$. Since D_1A^{-1} was also chosen proper, (36) implies that R is proper (or strictly proper if $\deg|A_1|>\deg|D_1|$, $\deg|A_2|>\deg|G_n|$). Using this R in (34) C exists and it is proper. Q.E.D.

The above constructive proof of the existence of proper C formally shows a result which had been intuitively known for rather a long time. This is the fact that properness can always be achieved by choosing the compensator C of "high enough" order in which case the number of coefficients to be determined is large enough to satisfy the design objectives and keep C proper. Clearly this corresponds to choosing $\deg|A|$ (the number of closed loop poles) large enough which is exactly the result obtained above. Note that the above construction implies that $\deg|A|=\deg|A_3A_2A_1|$ will be greater than $\deg|\hat{D}_3|+\deg|G_n|+\deg|D_1|=\deg|\tilde{Q}_1|+\deg|\tilde{D}_1|+\deg|D_1|=2n+\deg|\tilde{Q}_1|$ where n is the order of H_1 which, in view of (33a), implies that the order of C will be greater than $n+\deg|\tilde{Q}_1|$. Proper compensators C can be determined from (33a) using an alternative, and computationally more efficient method, which involves substitution of real values in the indeterminate of the polynomials. This method will be presented in a future paper. Finally, it is of interest to notice that the procedure described in Theorem 4 to construct a proper C which solves RPS, gives a proper C which just internally stabilizes the system if $\bar{Y}_1$ and $\tilde{Q}_1$ are taken to be Y_1 and I respectively i.e., B in (9) and (10) is taken to be the polynomial matrix W determined in the theorem.

It was shown that the regulator problem with internal stability (RPS) has a solution if and only if the conditions of Theorem 1 (or Corollaries 2,3) are satisfied. The class of proper and nonproper compensators C which regulate and internally stabilize is characterized by (9) or (10) where A,B are determined in Theorem 1 (or Corollaries 2,3). Relations (32), (32a) or (34) define such appropriate compensators. Proper compensators C can be determined via Theorem 4. In the following, the structure of the compensators C will be studied.

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

VI. THE STRUCTURE OF THE COMPENSATOR

(32a) implies that $\tilde{D}_c$, $\tilde{N}_c$ must have certain structure which is independent of the particular choice of A and W. In particular, let G_d and G_n be greatest common right divisors (gcrd) of $(\bar{X}_1, \tilde{Q}_1\tilde{N}_1)$ and $(\bar{Y}_1, \tilde{Q}_1\tilde{D}_1)$ respectively. Then (32a) directly implies that

$$\tilde{D}_c = \hat{\tilde{D}}_c G_d, \quad \tilde{N}_c = \hat{\tilde{N}}_c G_n \tag{38}$$

and

$$C = G_d^{-1}\, \hat{C}\, G_n \tag{39}$$

where $\hat{C} \overset{\Delta}{=} \hat{\tilde{D}}_c^{-1}\hat{\tilde{N}}_c$. G_n and G_d are independent of the choice of A and they are introduced because of the requirement for regulation. Note that if internal stability were the only objective $\tilde{Q}_1=I$, $\bar{X}_1=X_1$, $\bar{Y}_1=Y_1$ and $G_n=I$, $G_d=I$. $\hat{\tilde{D}}_c$, $\hat{\tilde{N}}_c$ or $\hat{C}$ clearly depend on A or the stable but arbitrarily chosen closed loop characteristic polynomial $|A|$. Notice that $(\bar{X}_1, \tilde{N}_1)$ and $(\bar{Y}_1, \tilde{D}_1)$ are rrp; this implies that the zeros of $|G_d|$ and $|G_n|$ are some or all of the (unstable) zeros of $|\tilde{Q}_1|^{\dagger}$, which, in view of (13), are those unstable poles of G_1 that do not cancel with unstable poles of H_1. That is, for regulation, C must introduce into the loop those elements of the exogenous signal which are not present in the system. It should be pointed out that there are cases where due to certain structural similarities between $\hat{N}_2$ and $\tilde{Q}_1$ $(m_1, m_2)=(W\tilde{Q}_1, -\hat{N}_2W)$ is not the general solution of (19). (38) is still valid in this case however, if G_d, G_n are taken to be gcrds of $X_1, \tilde{\tilde{Q}}_1\tilde{N}_1$ and Y_1, $\tilde{\tilde{Q}}_1\tilde{D}_1$ respectively where $\tilde{Q}_1=\hat{Q}_1\tilde{\tilde{Q}}_1$, that is $\tilde{\tilde{Q}}_1$ some appropriate right divisor of $\tilde{Q}_1$. This means that in these cases C must introduce into the loop fewer elements of the exogenous signals than before because of the structure of H_2, H_1 and G_1.

In general, the role the compensator C plays in regulation is to introduce into the loop enough characteristics of the exogenous signal so that the closed loop system contains a duplicate model, an internal model, of the exogenous signal (see also [4]); this internal model will create appropriate signals which

---

†If $y=-z$, then $G_n=I$ and $|G_d|=|\tilde{Q}_1|$.

will effectively cancel the exogenous ones thus achieving regulation. The idea of an internal model was first introduced explicitly in [19] in a slightly different context, during discussion of robust regulators of a state-space model.

Note that if (38) is satisfied, this does not necessarily mean that regulation is automatically guaranteed, that is, (38) is only a necessary condition for regulation and it is not, in general, a sufficient condition. If it were sufficient then for $\tilde{D}_c=\hat{\tilde{D}}_c G_d$, $\tilde{N}_c=\hat{\tilde{N}}_c G_n$ the second equation in (33a) (which corresponds to regulation) should not impose any restrictions on $\hat{\tilde{D}}_c$, $\hat{\tilde{N}}_c$ and the only equation to be satisfied (for internal stability) should have been $\hat{\tilde{D}}_c(G_d D_1)+\hat{\tilde{N}}_c(G_n N_1)=A$. This is the case i.e. (38) is a necessary and sufficient condition for regulation, only when

$$|G_d||G_n| = \alpha|\tilde{Q}_1| \tag{40}$$

as it is now shown.

In view of (38), (32a) can be written as

$$[\hat{\tilde{D}}_c,\hat{\tilde{N}}_c] = [A,W]\hat{\bar{U}}$$

where

$$\hat{\bar{U}} = \begin{bmatrix} I & 0 \\ 0 & \tilde{Q}_1 \end{bmatrix} \bar{U} \begin{bmatrix} G_d^{-1} & 0 \\ 0 & G_n^{-1} \end{bmatrix} = \begin{bmatrix} \bar{X}_1 G_d^{-1} & ,\bar{Y}_1 G_n^{-1} \\ -\tilde{Q}_1 \tilde{N}_1 G_d^{-1}, & \tilde{Q}_1 \tilde{D}_1 G_n^{-1} \end{bmatrix}$$

which in view of the definition of G_n and G_d is a polynomial matrix; it is unimodular if and only if $|G_d||G_n|=\alpha|\tilde{Q}_1|$. If $\hat{\bar{U}}$ is unimodular

$$[A,W] = [\hat{\tilde{D}}_c,\hat{\tilde{N}}_c]\hat{\bar{U}}^{-1}$$

where $\hat{\bar{U}}^{-1}$ is also unimodular. Since W is arbitrary, the only equation $\hat{\tilde{D}}_c$, $\hat{\tilde{N}}_c$ must satisfy is

$$\hat{\tilde{D}}_c(G_d D_1) + \hat{\tilde{N}}_c(G_n N_1) = A$$

which is the internal stability equation. Therefore if (40) is satisfied, (38) guarantees regulation. Note that this is the case in the following example, as well as when y=-z (the measured are the regulated outputs).

Example: To illustrate the above, consider the case of a plant 1/(s-1) whose output z is to be regulated. The measured output y is contaminated by disturbance

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

$w_2=(1/p_2)d_2$ and the output of the compensator C by disturbance $w_1=(1/p_1)d_1$ where p_1, p_2 polynomials with roots in $\mathbb{C}^+$. The system (1) is in this case

$$y = -[-1/(s-1)]u + [1/p_1(s-1),\ 1/p_2]d$$

$$z = -[1/(s-1)]u + [1/p_1(s-1),\ 0]d$$

$$u = Cy$$

where $d=[d_1,d_2]^T$. In view of (3), (8) and (11), $\tilde{N}_1=N_1=-1$, $\tilde{D}_1=D_1=s-1$, $\tilde{X}_1=X_1=0$ $\tilde{Y}_1=Y_1=-1$ while (4) implies that $\tilde{N}_2=\tilde{D}_2=1$. It can be easily shown that if s-1 is a factor of p_2 or if p_1,p_2 have a common factor, solution does not exist since in these cases (16) is not satisfied i.e., $Q_2^{-1}Q_1$ is not a polynomial matrix. In all other cases solution does exist. In particular (13) implies: $\tilde{D}_1(G_1)_+=[1/p_1,(s-1)/p_2]=[1,s-1][\mathrm{diag}(p_1,p_2)]^{-1}=P_1Q_1^{-1}=[p_1p_2]^{-1}[p_2,(s-1)p_1]=\tilde{Q}_1^{-1}P_1$ and $(\tilde{N}_2Y_1G_1+\tilde{D}_2G_2)_+=(G_2-G_1)_+=[0,1/p_2]=[0,1][\mathrm{diag}(1,p_2)]^{-1}=P_2Q_2^{-1}$. Therefore, $Q_2^{-1}Q_1=\mathrm{diag}(p_1,1)=Q$ a polynomial matrix i.e. (16) is satisfied. If X_G, Y_G in (15) are $X_G=[x_{ij}]$, $Y_G=[y_{ij}]$ then (17) becomes: $a_1+a_2(p_1p_2)=-y_{21}=x_{21}p_1$ which always has a solution. To determine the solutions, choose $(a_1,a_2)=(x_{21}p_1,0)$ and note that the general solution of (19), $m_1+m_2p_1p_2=0$, is $(m_1,m_2)=(Wp_1p_2,-W)$ which implies that $B=-Ax_{21}p_1+Wp_1p_2$ (which is exactly the same as the general solution b_1 of (29)). Here $\bar{X}_1=-x_{21}p_1=\bar{\bar{X}}_1$, $\bar{Y}_1=-1-x_{21}p_1(s-1)=-x_{22}p_2=\bar{\bar{Y}}_1$. The solution is therefore given by(see (32a), (33a))

$$\tilde{D}_c=-Ax_{21}p_1+Wp_1p_2 \text{ and } \tilde{N}_c=-Ax_{22}p_2+W(s-1)p_1p_2$$

or by
$$\tilde{D}_c(s-1)-\tilde{N}_c=A \text{ and } \tilde{D}_cx_{22}p_2-\tilde{N}_cx_{21}p_1=Wp_1p_2.$$

Note that $G_d=p_1$, $G_n=p_2$ and observe that $|G_n||G_d|=|\tilde{Q}_1|=p_1p_2$. In view of (40), if $\tilde{D}_c=\hat{\tilde{D}}_cp_1$ and $\tilde{N}_c=\hat{\tilde{N}}_cp_2$ the regulation requirement is satisfied and the compensator $C=\dfrac{p_2}{p_1}\hat{C}$ where $\hat{C}=\hat{\tilde{D}}_c^{-1}\hat{\tilde{N}}_c$ satisfy $\hat{\tilde{D}}_c[p_1(s-1)]-\hat{\tilde{N}}_c[p_2]=A$. For $p_1=s-2$, $p_2=s^2$ a minimal order proper compensator which internally stabilizes and regulates is $C=\dfrac{s^2}{s-2}\hat{C}$ $=\dfrac{s^2(s-7)}{(s-2)(s^2+20s+6)}$ where A stable was arbitrarily chosen (of "high enough" degree) to be $A=s^4+7s^3+18s^2+22s+12$,[15], i.e., the closed loop poles are at $-1\pm j$, -2 and -3.

VII. SPECIAL CASE $y=-z$

An important special case is when the measured output coincides with the output to be regulated [4],[5],[10],[11],[12]. This case includes tracking where the input of the compensator C is the error to be regulated. Here $y=-z$ implies $H_1=H_2$ and $G_1=-G_2$; (2) becomes $z(=-y)=[H_1C[I+H_1C]^{-1}-I]G_1d=-(I+H_1C)^{-1}G_1d=T_{zd}d$. One can use in this case the new T_{zd} to solve the problem directly. Here the conditions will be derived from the general conditions.

Note that $H_2D_1=H_1D_1=N_1$, i.e., $\tilde{N}_2=N_1$, $\tilde{D}_2=I$ ($|\tilde{D}_2|=1$ a stable polynomial) and $(\tilde{N}_2Y_1G_1+\tilde{D}_2G_2)_+=(N_1Y_1-I)(G_1)_+=(-\tilde{X}_1\tilde{D}_1)(G_1)_+=-\tilde{X}_1P_1Q_1^{-1}=P_2Q_2^{-1}$ where (P1) and (13) were used. Therefore, $P_2Q_2^{-1}Q_1=-\tilde{X}_1P_1$ which implies that $Q_2^{-1}Q_1=Q$ a polynomial matrix ((P_2,Q_2)rrp), that is (16) is always satisfied. Condition (19) now becomes $N_1a_1+a_2\tilde{Q}_1=-\tilde{X}_1P_1Y_G$ which in view of [14, Theorem 3] has a solution if and only if

$$\tilde{N}_1e_1 + e_2\tilde{Q}_1 = I \tag{41}$$

has a solution. Furthermore, in this case $B=Aa_1+m_1=A(-Xe_1)+WQ_1$ where (e_1,e_2) a solution of (41) and W any polynomial matrix. It has been shown [14, Theorem 4] [20] that (41) has a solution if and only if there exist polynomial matrices D, N such that $\tilde{N}_1D^{-1}=\tilde{Q}_1^{-1}N$ with (N_1,D)rrp and (Q_1,N)rlp. Furthermore if a solution exists, there exist polynomial matrices $\hat{e}_1$, $\hat{e}_2$ such that $e_1\tilde{N}_1+\hat{e}_1D=I$, $\tilde{Q}_1e_2+N\hat{e}_2=I$ and (P1) are satisfied. In view of this and the value of B, $\bar{X}_1=X_1(I-e_1\tilde{N}_1)=X_1\hat{e}_1D$ and $\tilde{Q}_1\tilde{N}_1=ND$ in (32a). This implies that $G_d=D$ and $G_n=I$ since $(DN_1)[X_1\hat{e}_1]+(\hat{e}_2+D\tilde{Y}_1e_2)[N]=I$ and $(N_1)[\bar{Y}_1]+(\tilde{X}_1e_2)[\tilde{Q}_1\tilde{D}_1]=I$. Clearly in this case $|G_d||G_n|=|D|=\alpha|\tilde{Q}_1|$, that is (40) is satisfied. Therefore if $\tilde{D}_c=\hat{\tilde{D}}_cD$, regulation is guaranteed and the second equation in (33a) does not imply any other restrictions on the structure of C. This can be verified by noticing that $-\hat{\tilde{D}}_cD\tilde{\tilde{Y}}_1+\tilde{N}_c\tilde{\tilde{X}}_1= -\hat{\tilde{D}}_cD(\tilde{Y}_1+D_1X_1e_1)+\tilde{N}_c(\tilde{X}_1-N_1X_1e_1)=[-\hat{\tilde{D}}_c(\hat{e}_2+D\tilde{Y}_1e_2)+\tilde{N}_c\tilde{X}_1e_2]\tilde{Q}_1$ which can be written as $W\tilde{Q}_1$ where W some appropriate polynomial matrix.

It is therefore clear, in view of the above that

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

Corollary 5: When the measured outputs are the outputs to be regulated, RPS1 (and therefore, RPS) has a solution if and only if there exist polynomial matrices e_1, e_2 such that

$$\tilde{N}_1 e_1 + e_2 \tilde{Q}_1 = I \tag{41}$$

If a solution exists then the compensator is $C=D^{-1}\hat{C}=D^{-1}(\hat{\tilde{D}}_c^{-1}\tilde{N}_c)$ where $\hat{\tilde{D}}_c$, $\tilde{N}_c$ satisfy†

$$\hat{\tilde{D}}_c(DD_1) + \tilde{N}_c(N_1) = A \tag{42}$$

or equivalently

$$\hat{\tilde{D}}_c = AX_1\hat{e}_1 - WN \tag{43}$$

$$\tilde{N}_c = A(Y_1 + X_1 e_1 \tilde{D}_1) + W\tilde{Q}_1\tilde{D}_1$$

where $|A|$ stable and W an arbitrary polynomial matrix.

Remark: Note that here $(-T_{zd})_+=[I+H_1C]^{-1}(G_1)_+=[\tilde{D}_1+\tilde{N}_1D^{-1}\hat{C}]^{-1}\tilde{D}_1(G_1)_+=[\tilde{D}_1+\tilde{Q}_1^{-1}N\hat{C}]^{-1}\tilde{Q}_1^{-1}\tilde{P}_1=[\tilde{Q}_1\tilde{D}_1+N\hat{C}]^{-1}\tilde{P}_1=\hat{D}_c[(\tilde{Q}_1\tilde{D}_1)\hat{D}_c+N\hat{N}_c]^{-1}\tilde{P}_1$ which reveals the mechanism of regulation (the role D plays) and implies that $\hat{C}=\hat{N}_c\hat{D}_c^{-1}$ can also be found from the equivalent to (42) equation:

$$(\tilde{Q}_1\tilde{D}_1)\hat{D}_c + N\hat{N}_c = A_1 \tag{44}$$

where $|A_1|=|A|$ thus eliminating the need to calculate $N_1D_1^{-1}$.

As an illustration, consider the example in [4] where

$H_1 = \begin{bmatrix} s-1 & 1 \\ -s & s \end{bmatrix}^{-1}\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \tilde{D}_1^{-1}\tilde{N}_1$, and $G_1 = \dfrac{1}{s^2(s+1)}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. (41) in this case is $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} e_1 + e_2 \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix} = I$. $D = \begin{bmatrix} s & 0 \\ -1 & 2s \end{bmatrix}$, $N = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ are appropriate matrices and (44) becomes

$\begin{bmatrix} s^2 & 0 \\ -s^2 & s^2 \end{bmatrix}\hat{D}_c + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\hat{N}_c = A_1$. If A_1 is chosen to be $\begin{bmatrix} s^2+2s+2 & 0 \\ -s^2 & s^2+2s+2 \end{bmatrix}$ ($|A_1|$ has zeros at $-1\pm j$) then $C = D^{-1}\hat{C} = 1/2s^2 \begin{bmatrix} 2s & 0 \\ 1 & s \end{bmatrix}\begin{bmatrix} 0 & 2(s+1) \\ 2(s+1) & -2(s+1) \end{bmatrix} = \begin{bmatrix} 0 & 2(s+1)/s \\ (s+1)/s & -(s^2-1)/s^2 \end{bmatrix}$

is an appropriate compensator of second (minimal) order.

---

†Note that (N_1, DD_1)rrp since $N_1(DD_1)^{-1}=\tilde{D}_1^{-1}\tilde{N}_1D^{-1}$ and $(\tilde{N}_1,\tilde{D}_1)$rlp, $(\tilde{N}_1,D)$rrp, i.e., no cancellations take place.

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

VIII. DISCUSSION OF CONDITIONS

The regulator problem with internal stability has a solution if and only if (16) and (17) are satisfied. It can be shown that a necessary condition for the existence of solution is $Q_{G2}^{-1}Q_{G1}=Q_G$ a polynomial matrix where $(G_1)_+=P_{G1}Q_{G1}^{-1}$, $(G_2)_+=P_{G2}Q_{G2}^{-1}$, i.e., all unstable poles of G_2 must appear as poles of G_1 together of course with some similarity in the structure of the denominators Q_{G1}, Q_{G2}. The condition (16) used here, i.e., $Q_2^{-1}Q_1=Q$, implies more than that (but it is equivalent to the above when, for example, H_1 and therefore H_2 are stable). (16) (together with the assumption H_2D_1 stable) seems to guarantee that all the unstable modes which affect the trajectory of z can be observed through y. It is intuitively clear that if this is not the case, regulation of z by measuring y will be impossible. Note that when the measured outputs are the ones to be regulated, condition (16) is always satisfied for all types of exogenous signals (see above, the case of y=-z). In general, this condition, which involves exact cancellations, will be difficult to satisfy especially when small plant perturbations are possible. If however, H_1, H_2, G_1 and G_2 are "properly" related because of an appropriate choice of the signal to be measured, then (16) is always satisfied and actually it does not appear as a condition for the solution. This is the case in the state-space model when (A_1,B_1,D_1) is a controllable and observable system (see [3] for notation).

Condition (17), $\tilde{N}_2a_1+a_2\tilde{Q}_1=P_2QY_G$ (or equivalently (29) or (30)), explores the structural relations between the exogenous signal and the loop. If it can be satisfied then the compensator C will be able to introduce in the closed loop certain characteristics of the exogenous signal so that appropriate signals will be generated in the loop which will cancel the exogenous ones while internal stability is maintained. Equations of this form, (17), were first studied in [21] and more recently in [6] and [22]. Simpler forms of this general equation were also studied in [14] and [20]. Note that $\tilde{Q}_1$(pxp) or Q_1(qxq) contains (the zeros of $|\tilde{Q}_1|$ or $|Q_1|$ are) those unstable poles of G_1 which do not appear as poles of H_1 while $\tilde{N}_2(rxm)$ contains (the zeros of the gcd of all highest order minors are) the zeros of H_2 and those poles of H_1 which do not appear in H_2. If those values associated with $\tilde{Q}_1$ and $\tilde{N}_2$ are disjoint (or equivalently the zeros of the invariant factors of their Smith form are disjoint) and we concentrate on the usual case in practice where $r<m$ then in view of [6, Lemma 6], (17) has always a solution. Since the "zeros" of $\tilde{N}_2$ and $\tilde{Q}_1$ are almost always disjoint, if $r<m$, solution to (17) always exists generically. Note that in the case, for the state space model, $\tilde{N}_2$ contains the zeros of the system (A_1,B_1,D_1) (see [3] for notation); in view of the above, if $r<m$ and the system is of minimum phase, solution always exists. Furthermore, in view of the discussion on $Q_2^{-1}Q_1=Q$, if (A_1,B_1,D_1) is a controllable and observable system and $r<m$ then solution to the regulation problem with internal stability always exists generically (see [2]). Note also that $r<m$ has been shown elsewhere to be a condition for regulation with robusteness [13].

IX. CONCLUDING REMARKS

A general version of the regulator problem with internal stability was considered in this paper and a parametric characterization of all appropriate compensators C was derived. The solution was obtained using a characterization of the class of all internally stabilizing compensators and then restricting this class to achieve regulation as well. The dual role of the compensator C when regulation in addition to internal stability is to be achieved was fully explained and it was shown that regulation implies that C must be of the form $C=G_d^{-1}\hat{C}G_n$, that is it must introduce in the loop characteristics of the exogenous signals. It was also shown that properness of C can always be achieved by choosing the order of C to be "high enough". The necessary and sufficient conditions for the solution of the problem were discussed and simpler conditions were derived for the important special case when the measured and regulated outputs coincide.

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.

ACKNOWLEDGMENTS

The author is grateful to Prof. J. B. Pearson for his comments and support during the early stages of this work.

References

[1] W.M. Wonham and J.B. Pearson, "Regulation and internal stabilization in linear multivariable systems", SIAM J. Contr., Vol. 12, no. 1, pp. 5-18, 1974.

[2] W.M. Wonham, Linear Multivariable Control.A Geometric Approach, Springer-Verlag, 1974.

[3] P.J. Antsaklis and J.B. Pearson, "Stabilization and regulation in linear multivariable systems", IEEE Trans. Autom. Contr., Vol. AC-23, pp. 928-930, Oct. 1978.

[4] G. Bengtsson, "Output regulation and internal models. A frequency domain approach", Automatica, Vol. 13, pp. 333-345, 1977.

[5] W.A. Wolovich, "Multivariable system synthesis with step disturbance rejection", IEEE Trans. Autom. Contr., Vol. AC-19, pp. 127-130, 1974.

[6] L. Cheng and J.B. Pearson, "Frequency domain synthesis of multivariable linear regulators", IEEE Trans. Autom. Contr., Vol. AC-23, pp. 3-15, Feb. 1978.

[7] W.A. Wolovich and P. Ferreira, "Output regulation and tracking in linear multivariable systems", IEEE Trans. Autom. Contr., Vol. AC-24, pp. 460-465, June 1979.

[8] L. Pernebo, Algebraic control theory for linear multivariable systems, Ph.D. thesis, Department of Automatic Control, Lund Institute of Technology, Sweden, 1978.

[9] L. Cheng and J.B. Pearson, "Synthesis of linear multivariable regulators", Technical Report No. 8002, Department of Electrical Engineering, Rice University, Febr. 1980.

[10] B.A. Francis, "The multivariable servomechanism problem from the input-output viewpoint", IEEE Trans. Autom. Contr., Vol. AC-22, pp. 322-328, June 1977.

[11] C.A. Desoer and Y.T. Wang, "Linear time-invariant robust servomechanism problem: A self-contained exposition", Memorandum No. UCB/ERL M77/50, Electronics Research Laboratory, University of California, Berkeley, Aug. 1977.

[12] B.A. Francis and M. Vidyasagar, "Algebraic and topological aspects of the servo problem for lumped linear systems", Yale University, S & IS report No. 8003, Nov. 1980.

[13] E.J. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems", IEEE Trans. Autom. Contr., Vol. AC-21, pp. 25-34, Febr. 1976.

[14] P.J. Antsaklis, "Some relations satisfied by prime polynomial matrices and their role in linear multivariable system theory", Technical Report No. 7806, Department of Electrical Engineering, Rice University, June 1978 and IEEE Trans. Autom. Contr., Vol. AC-24, pp. 611-616, Aug. 1979.

[15] P.J. Antsaklis, "On multivariable regulators. An explicit description", Proceedings of the 19th IEEE Conference on Decision and Control, Albuquerque, New Mexico, Dec. 1980.

[16] W.A. Wolovich, Linear Multivariable Systems, Springer-Verlag, 1974.

[17] D.C. Youla, H.A. Jabr, and J.J. Bongiorno, Jr., "Modern Wiener-Hopf design of optimal controllers - Part II: The multivariable case", IEEE Trans. Automat. Contr., Vol. AC-21, pp. 319-338 , June 1976.

[18] C.A. Desoer, R.W. Liu, J. Murray and R. Saeks, "Feedback system design: the fractional representation approach to analysis and synthesis", IEEE Trans. Autom. Contr., Vol. AC-25, pp. 399-412 , June 1980.

[19] B.A. Francis and W.M. Wonham, "The internal model principle for linear multivariable regulators", J. Appl. Maths. Optimization 2(2), pp. 170-194, 1975.

[20] W.A. Wolovich, "Skew prime polynomial matrices", IEEE Trans. Automat. Contr., Vol. AC-23, pp. 880-887, Oct. 1978.

[21] W.E. Roth, "The equations $AX - YB = C$ and $AX - XB = C$ in matrices", Proc. Amer. Math. Soc., Vol. 3, pp. 392-396, June 1952.

[22] E. Emre and L.M. Silverman, "The polynomial equation $XR + QY = \Phi$:A characterization of solutions", Proceedings of the 19th Conference on Decision and Control, Albuquerque, New Mexico, Dec. 1980.

P. J. Antsaklis, "On Output Regulation with Stability in Multivariable Systems," Technical Report No. 810, Dept. of Electrical Engineering, University of Notre Dame, April 1981.