ON THE COMMON (LEFT,RIGHT) DIVISORS OF TWO POLYNOMIAL MATRICES

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Summary

The notions of common right and common left divisors of two polynomial matrices are well established. They can be used to characterize the unobservable and the uncontrollable modes of a system and to study polynomial matrix equations of the form \( a_1 V_1 + a_2 V_2 = V_3 \). The new notion of common (right, left) divisors of two polynomial matrices is introduced here and it is used to characterize the modes of the system which are both uncontrollable and unobservable; it is also used to study polynomial matrix equations of the form

\[
N X_1 + X_2 D = V
\]

where \( N(qxp), D(qxp) \) and \( V(qxm) \) are given polynomial matrices; (1) has attracted recently considerable attention ([1] to [7]) mainly because of its importance in the output regulation problem with internal stability.

To introduce the common (right, left) divisors, consider the equation

\[
N X_1 + X_2 D = 0
\]

which is important in solving (1), since the complete solution of (1) can be written as the sum of a particular solution of (1) and the general solution of (2). Assume that \( D \) is square and nonsingular which is the case of interest in the problem of regulation.

Theorem. The general solution of (2) is given by

\[
(x_1,x_2) = ((W + M_2 G_2^{-1}) D, -N(W + G_2^{-1} M_1))
\]

where \( W \) is any polynomial matrix, \( M_1 \) and \( M_2 \) polynomial matrices such that \( M_2 G_2^{-1} = G_1^{-1} M_1 \) strictly proper, \( (M_1, G_1) \) left prime and \( G_2 \) right, \( G_2 \) right, left divisors of \( N, D \) respectively.

Proof. Let \((x_1,x_2)\) be a solution of (2). Then

\[
N X_1 D^{-1} = -X_2.
\]

Write \( X_1 D^{-1} = W + M_2 G_2^{-1} W + G_1^{-1} M_1 \) (using division algorithm [8]). where \( W \) some polynomial matrix, \( (M_2, G_2^{-1}) \) left prime, \( (M_1, G_1) \) left prime with \( N G_2^{-1} = G_1^{-1} M_1 \) strictly proper; note that \( G_2^{-1} D \) is a polynomial matrix i.e. \( G_2 \) is a left divisor of \( D \). Then \( N(W + G_2^{-1} M_1) = -X_2 \) which implies that \( N G_2^{-1} \) is a polynomial matrix i.e. \( G_2 \) is a right divisor of \( N \). Therefore the solution \((x_1,x_2)\) is of the form (3). Conversely \( x_1 \) and \( x_2 \) from (3) satisfy (2).

Definition. \((G_1, G_2)\) is a pair of common (right, left) divisors of \((N,D)\) (or \((G_1, G_2)\) is \(c(r,l)d\) of \((N,D)\) if

\[
N = N G_1, \quad D = G_2 D
\]

and there exist \((pxm)\) polynomial matrices \( M_1, M_2 \) such that

\[
G_1^{-1} M_1 = M_2 G_2^{-1}
\]

with \((M_1, G_1)\) left prime and \((M_2, G_2)\) right prime polynomial matrices. \((5)\) implies that \(|G_1| = 3|G_2|\). \(G_1\) and \(G_2\) not only have the same invariant polynomials but they also contain certain structure common to \( N \) and \( D \). \( G_1 \) is related to \( G_2 \) in the same way the denominators \( P_1, P_2 \) of two prime factorizations of a transfer matrix \( T = P_1 P_2^{-1} \) are related [9][10]. A greatest common (right, left) divisor \((gc(r,l)d)\) of \((G_1, G_2)\) can also be defined; consider it to be a \(c(r,l)d\) \((G_1^*, G_2^*)\) such that: degree \(|G_1^*| = \) degree \(|G_2^*|\) is maximum.

Note that the notion of \(c(r,l)d\) is unique to matrices, since if polynomials are considered then a \(c(r,l)d\) of \((G_1, G_2)\) is a common divisor (i.e. a common factor) of the polynomials.

In view of the definition it is clear that the general solution (3) is written in terms of \(c(r,l)d\) of \((N,D)\). If all such divisors are unimodular i.e. the \(gc(r,l)d\) of \((N,D)\) unimodular, the general solution to (2) is: \((x_1,x_2) = (0D, -NW)\). An example of such case is when \( N \) is also square and \(|N|, |D|\) are prime polynomials. (See [5, Theorem 4]).

The \(c(r,l)d\) plays a role in characterizing the modes of the system.
which are both uncontrollable and unobservable. In particular, let \( G_L \) and \( G_R \) be greatest common left (gcl) and right (gcr) divisors of \( P, Q \) and \( P, R \) respectively. It is known that the roots of \( |G_L| \) are all the uncontrollable (c) modes [8] (the input decoupling zeros [11]) of the system. Similarly, the roots of \( |G_R| \) are all the unobservable (co) modes (the output decoupling zeros) of the system. Furthermore, \( G_L \) and \( G_R \) each contain certain structural information about the system which can be translated into information about the corresponding uncontrollable and unobservable subspaces. Write \( P = G_L P_L = G_R P_R \) and let \( G \) be a gcr divisor of \( P_L, R \). Then \( G_R \) will also be a gcr divisor of \( G_L \) and \( P, R \). That is:

\[
G_R = G_L G_R, G_L = G_L G_R
\]  

with \( G_2, P_2 \) right prime. \( G_R \) contains those \( c \) modes which are \( c \), while \( G_L \) contains those \( c \) modes which are also \( c \). (Note that similar approach is described in [11]) Similarly, let \( G_L \) be a gcl divisor of \( P_R, R \). Then:

\[
G_L = G_L G_1, G_L = G_L G_1
\]  

with \( G_1, P_1 \) left prime. \( G_L \) contains those \( c \) modes which are \( c \), while \( G_L \) contains those \( c \) modes which are also \( c \). Note that:

\[
G_L^{-1} P_2 G_1^{-1} = P_2 G_1^{-1} = G_2^{-1} P_2 G_1^{-1} = G_1^{-1} P_2 G_1^{-1} = G_1^{-1} P_2 G_1^{-1}
\]  

The zeros of \( G_2 \) are exactly the \( c \) modes. Furthermore \( G_2 \) and \( G_L \) contain structural information which can be translated into information about the uncontrollable part of the unobservable subspace \( G_2 \) or about the unobservable part of the uncontrollable subspace \( G_L \).

\[
P = G_L P_L = G_L G_1 P_2 G_2 G_1 = G_R P_R = \begin{pmatrix} c & c \\ c & c \end{pmatrix}
\]

which corresponds to the (Kalman) state-space decomposition into \( c \), \( c \), \( c \), \( c \) subspaces.

Note that, from (9), \( P_2 G_2^{-1} = G_1^{-1} P_1 \) where \( (P_2, G_2) \) right prime, \( (P_1, G_1) \) left prime with \( G_1 \) and \( G_2 \) right and left divisors of \( G_L \) and \( G_R \) respectively. i.e. \( G_L G_R \) is a \( c(r,l) \)d of \( G_L, G_R \); it is also a \( c(l,r) \)d of \( G_L, G_R \).

The new notion of \( c(r,l) \)d appears to be a natural extension of the notions of \( c \) and \( c \) divisors. Further investigation is needed to study the full potential of this new idea and its implications.

References