W.A. Wolovich, P. J. Antsaklis and H. Elliott, "On the Stability of the Minimal Design Problem," Paper No. WP5.6, P roc. o f t he 1 975 I EEE C onf. o n D ecision a nd C ontro l , Houston, TX, Dec. 10-12, 1975.

ON THE STABILITY OF THE MINIMAL DESIGN PROBLEM

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Abstract

A necessary condition for obtaining stable solutions to the minimal design problem is presented. The condition is shown to be sufficient for insuring the stability of solutions which need not be minimal. The results are based on the recently developed notion of minimal bases of rational vector spaces, and an example is employed to illustrate and clarify the procedure.

The minimal design problem (MDP) can be stated as follows: Given a pxm rational transfer matrix, $T_1(s)$ of rank $p(< m)$† and a pxq rational transfer matrix, $T_2(s)$, find a (mxq) proper rational transfer matrix, T(s), of minimal dynamic order‡ (if such a transfer matrix exists) such that

$$T_1(s)\ T(s) = T_2(s) \qquad (1)$$

It might be noted that the MDP represents an extension (to include "minimal order" input dynamic compensation) of the well known exact model matching problem, which has been the subject of numerous investigations. To resolve the MDP, we require some preliminary definitions. In particular, suppose that K(s) is a qxr polynomial matrix with q>r. K(s) is called column proper[1] if and only if the qxr constant matrix, $\Gamma_c[K(s)]$, consisting of the coefficients of the highest degree polynomials in each column of K(s) has full rank r. If K(s) is column proper and the degree of each (j-th) column is no greater than the degree of all subsequent columns; i.e. if $\partial_{cj}[K(s)] \leq \partial_{c,j+1}[K(s)]$ for $j \in \underline{r-1}$, K(s) will be called degree ordered as well. Forney has recently shown[3] that every rational vector space, defined by a basis of rational column vectors, has a minimal basis which, in view of [1], corresponds to the columns of a column proper polynomial matrix whose rows are r.r.p.

We now note that K(s) can be partitioned as $\begin{bmatrix} K_r(s) \\ K_{q-r}(s) \end{bmatrix}$ where $K_r(s)$ denotes the first r rows of K(s) and $K_{q-r}(s)$ denotes the final q-r rows. $\Gamma_c[K(s)]$ will now be written as $\begin{bmatrix} K_{r\gamma} \\ K_{q-r,\gamma} \end{bmatrix}$, noting that $K_{r\gamma}$ (or $K_{q-r,\gamma}$) does not necessarily equal $\Gamma_c[K_r(s)]$ (or $\Gamma_c[K_{q-r}(s)]$). With this notation in mind, we can now resolve the MDP.

Theorem I: Let $K(s) = \begin{bmatrix} K_m(s) \\ K_q(s) \end{bmatrix}$ be any (m+q)x(m+q-p) degree ordered, minimal basis for $\ker[T_1(s) -T_2(s)]$. The MDP has a solution, T(s), if and only if

$$\rho[K_{q\gamma}] = q \qquad (2)$$

Furthermore, if (1) holds, the minimal dynamic order of an appropriate T(s) is equal to the sum of the column degrees of the first (ordered from left to right) q columns of K(s) for which the corresponding (numbered) columns of $K_{q\gamma}$ are linearly independent. These q columns of K(s), $\begin{bmatrix} R(s) \\ P(s) \end{bmatrix}$, represent a proper, minimal order solution, $T(s) = R(s)P^{-1}(s)$, to (1).

Proof: Since Theorem I is not original, except for the conciseness of its statement, a formal proof will not be given here and the interested reader is referred to [3]. It should be noted that Wang and Davison[4][5] were first to resolve the MDP using a rather innovative, but belabored, approach. Forney[3] later employed the notion of minimal bases of rational vector spaces in order to facilitate the proof. More recently, Sain[6] has presented a more direct procedure for obtaining minimal bases which facilitates certain of the computational steps outlined by Forney.

Although we will not formally establish Theorem I here, we will illustrate its employment by example. In particular, if

$$T_1(s) = \begin{bmatrix} 1 & 0 & \frac{s^2+2s+2}{s^2+3s+2} \\ 1 & \frac{s-1}{s+2} & 0 \end{bmatrix} \text{ and } T_2(s) = I_2; \text{ i.e.}$$

if we wish to find a proper "right inverse" of $T_1(s)$, then we can first determine a degree ordered, minimal basis of $\ker[T_1(s) \ \ -T_2(s)]$. Using an algorithm outlined in [7], we find that

†If $p \geq m$, the MDP either has no solution or a unique solution (which can easily be found).

‡If $R(s)P^{-1}(s)$ is a relatively right prime (r.r.p.)[1][2] factorization of T(s), the degree of the determinant of P(s), $\partial[|P(s)|]$, is equal to the dynamic order of (a minimal state-space realization of T(s)[1].

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$$K(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s+2 & 0 \\ 0 & 0 & s^2+3s+2 \\ \hline 1 & 0 & s^2+2s+2 \\ 1 & s-1 & 0 \end{bmatrix} = \begin{bmatrix} K_m(s) \\ K_q(s) \end{bmatrix}$$

is such a basis. Since $\Gamma_c[K(s)] =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} K_{m\gamma} \\ K_{q\gamma} \end{bmatrix},\ \rho[K_{q\gamma}] = q = 2,$$ which establishes the existence of a proper right inverse. Since columns 1 and 2 of K(s) are the first two (=q) for which the corresponding columns of $K_{q\gamma}$ are linearly independent, the minimal dynamic order of a proper right inverse is $l = \partial_{c1}[K(s)] + \partial_{c2}[K(s)$; i.e. $T(s) = R(s)P^{-1}(s) =$

$$\begin{bmatrix} 1 & 0 \\ 0 & s+2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{s+2}{s-1} & \frac{s+2}{s-1} \\ 0 & 0 \end{bmatrix}$$

is a minimal (first) order proper solution to (1), the MDP. We finally note that this solution, while of minimal order, is also unstable and furthermore, that no stable minimal (first) order solutions exist, an observation which motivates the remaining results.

In particular, we now note that little has been said in the control literature regarding the ability or inability to achieve a stable solution to (1) when (2) holds. The purpose of this paper will be to partially resolve this question by presenting a necessary and sufficient condition for obtaining stable solutions to (1). The result given only partially resolves the MDP stability question, however, since minimality of the dynamic order of stable solutions cannot always be assumed. It might be noted at this point that the question of obtaining both a stable and a minimal dynamic order solution to (1) is analogous to the difficult and still unsolved question of stabilizing a linear system via constant gain output feedback.

Before we consider the stability question, some preliminary observations and definitions are required. In particular, if $T_1(s)$ and $T_2(s)$ are factored as the <u>relatively left prime (r.l.p)</u>[1][2] products $P_{1Q}^{-1}(s)Q_1(s)$ and $P_{2Q}^{-1}(s)Q_2(s)$ respectively, the zeros of the determinant, $\Delta_P(s)$, of any <u>greatest common right divisor (g.c.r.d.)</u>[1][2] $G_{RP}(s)$, of $P_{1Q}(s)$ and $P_{2Q}(s)$ will be called the common poles of $T_1(s)$ and $T_2(s)$. It now follows that

$$P_{2Q}(s)P_{1Q}^{-1}(s) = \hat{P}_{2Q}(s)G_{RP}(s)G_{RP}^{-1}(s)\hat{P}_{1Q}^{-1}(s) = \tilde{P}_1^{-1}(s)\tilde{P}_2(s) \qquad (3)$$

for some r.l.p. pair $\{\tilde{P}_1(s), \tilde{P}_2(s)\}$. The zeros of the determinant, $\Delta_Q(s)$, of any <u>greatest common left divisor (g.c.l.d.)</u>[1][2], $G_{LQ}(s)$, of $P_2(s)Q_1(s)$ and $P_1(s)Q_2(s)$ will be called the common zeros of $T_1(s)$ and $T_2(s)$. In light of these definitions, we can now state and constructively establish the main result of this paper.

<u>Theorem II</u>: Let $\Delta_T(s)$ represent the determinant of my g.c.l.d., $G_2(s)$, of (the columns of) $G_{LQ}^{-1}(s)\tilde{P}_2(s)Q_1(s)$. The poles of any solution, T(s), to (1) will equal the zeros of $\Delta_T(s)\Delta_D(s)$. Furthermore, if (2) holds, a proper solution can be found which arbitrarily assigns the zeros of $\Delta_D(s)$.

<u>Proof</u>: For notational convenience, let $M(s) = [M_m(s)\ \ -M_q(s)] \triangleq G_{LQ}^{-1}(s)[\tilde{P}_2(s)Q_1(s)\ \ -\tilde{P}_1(s)Q_2(s)]$. As formally established in [7], we now simply observe that when a proper T(s) does exist, one can append to $\underset{\sim}{M}(s) = [M_m(s)\ \ -M_q(s)] = G_{LQ}^{-1}(s)[\tilde{P}_2(s)Q_1(s)\ \ -\tilde{P}_1(s)Q_2(s)]$ (m-p) additional rows, $[\alpha(s)\ \ 0]$, such that $\left|\begin{bmatrix} M_m(s) \\ \alpha(s) \end{bmatrix}\right| = \beta\Delta_T(s)\Delta_D(s)$ with Δ_D arbitrary, except for degree, and

$$\begin{bmatrix} M_m(s) \\ \alpha(s) \end{bmatrix}^{-1} \begin{bmatrix} M_q(s) \\ 0 \end{bmatrix} = T(s), \qquad (4)$$

a proper solution to (1).

It is of interest to note that neither the common zeros nor the common poles of $T_1(s)$ and $T_2(s)$ affect T(s) since they appear, and can therefore be "cancelled", on both sides of (1). The fixed poles of T(s), which are now defined as the zeros of $\Delta_T(s)$, do correspond to all of the zeros of $T_1(s)$ which are not common to $T_2(s)$, as well as the zeros of $|\tilde{P}_2(s)|$, which represent the poles of $T_2(s)$ which are not common to $T_1(s)$. We thus note that in order to insure stable solutions to the MDP, $T_2(s)$ should be chosen to be stable and to have in common with $T_1(s)$ any and all unstable zeros of $T_1(s)$. This observation, which is intuitively obvious in the scalar case, therefore has an analogous interpretation in the more general, multivariable case.

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