MATRIX INTERPOLATION IN CONTROL DESIGN

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ABSTRACT
In this paper, recent developments of the matrix interpolation method and its applications in control design are reported and references are given. Certain fundamental results from the theory of polynomial and rational matrix interpolation are reviewed.

Introduction
Many system and control problems can be formulated in terms of matrix equations where polynomial or rational solutions with specific properties are of interest. It is known that equations involving just polynomials can be solved by either equating coefficients of equal power of the indeterminate s or equivalently by using the values obtained when appropriate values for s are substituted in the given polynomials; in the latter case one uses results from the theory of polynomial interpolation. Similarly one may solve polynomial matrix equations using the theory of polynomial matrix interpolation discussed here; polynomial matrix interpolation of the type \( Q(s) a_j = b_j \), where \( Q(s) \) is a matrix and \( a_j, b_j \) vectors, has been introduced recently as a generalization of the scalar polynomial interpolation of the form \( a_j = b_j \) for degree \( \ell - 1 \) for which

\[
q(s_j) = b_j \quad j = 1, \ldots, \ell
\]

That is, an nth degree polynomial \( q(s) \) can be uniquely represented by the \( \ell = n + 1 \) interpolation (points or doublets or pairs \( (s_j, b_j) \) \( j = 1, \ldots, \ell \)).

The polynomial matrix interpolation theory deals with this interpolation problem in the matrix case. Let \( S(s) \triangleq \text{blk diag } \{ [1, \ldots, s^m]^T \} \) where \( d_i = 1, \ldots, m \) are non-negative integers; let \( a_j \neq 0 \) and \( b_j \) denote \((m \times 1)\) and \((p \times 1)\) complex vectors respectively and \( s_j \) complex scalars.

**Theorem 1**: Given interpolation (points) triplets \( (s_j, a_j, b_j) \) \( j = 1, \ldots, \ell \) and nonnegative integers \( d_i \) with \( \ell = \sum d_i + m \) such that the \((\sum d_i + m) \times \ell \) matrix

\[
S_j \triangleq \begin{bmatrix} S(s_j) a_1, & \cdots, & S(s_j) a_{\ell} \end{bmatrix}
\]

has full rank, there exists a unique \((p \times m)\) polynomial matrix \( Q(s) \), with \( i \)th column degree equal to \( d_i \), \( i = 1, \ldots, m \) for which

\[
Q(s_j) a_j = b_j \quad j = 1, \ldots, \ell
\]
Proof: Since the column degrees of \( Q(s) \) are 
\( d_i, Q(s) \) can be written as
\[
Q(s) = QS(s)
\]
(4)
where \( Q(p \times (\Sigma d_i + m)) \) contains the coefficients of the polynomial entries. Substituting in (3), \( Q \) must satisfy
\[
QS_i = B_i
\]
(5)
where \( B_i \triangleq [b_{1i}, \ldots, b_{li}] \). Since \( S_i \) is nonsingular, \( Q \) and therefore \( Q(s) \) are uniquely determined. \( \square \)

It should be noted that when \( p = m = 1 \) and \( d_i = l - 1 = n \) this theorem reduces to the polynomial interpolation theorem; if \( a_j = 1 \), then \( S_i \) is a Vandermonde matrix.

Rational Matrix Interpolation

Similarly to the polynomial matrix case, the problem here is to represent a \((p \times m)\) rational matrix \( H(s) \) by interpolation tripods or points \((s_j, a_j, b_j)\), \( j = 1, \ldots, l \) which satisfy
\[
H(s_j) a_j = b_j, \quad 1, \ldots, l
\]
(6)
where \( s_j \) are complex scalars and \( a_j \neq 0, b_j \) complex \((m \times 1), (p \times 1)\) vectors respectively.

It can be shown that the rational matrix interpolation problem reduces to a special case of polynomial matrix interpolation. To see this:

Write \( H(s) = \tilde{D}^{-1}(s) \tilde{N}(s) \) where \( \tilde{D}(s) \) and \( \tilde{N}(s) \) are \((p \times p)\) and \((p \times m)\) polynomial matrices respectively. Then (6) can be written as
\[
N(s_j) a_j = \tilde{D}(s_j) b_j \text{ or as}
\]
\[
[\tilde{N}(s_j), -\tilde{D}(s_j)] \begin{bmatrix} a_j \\ b_j \end{bmatrix} = Q(s_j) c_j = 0, \quad j = 1, \ldots, l
\]
(7)
That is the rational matrix interpolation problem for a \( p \times m \) rational matrix \( H(s) \) can be seen as a polynomial interpolation problem for a \( p \times (p + m) \) polynomial matrix \( Q(s) = [\tilde{N}(s), -\tilde{D}(s)] \) with interpolation points \((s_j, c_j, 0) = (s_j, [a_j^T, b_j^T]^T, 0), j = 1, \ldots, l\). There is also the additional constraint that \( \tilde{D}^{-1}(s) \) exists.

Solution of Matrix Equations

In this section polynomial matrix equations of the form \( M(s)L(s) = Q(s) \) are studied. The main result is Theorem 2 where it is shown that all solutions \( M(s) \) of degree \( r \) can be derived by solving equation (16). In this way, all solutions of degree \( r \) of the polynomial equation, if they exist, are parameterized. It is also shown that Theorem 2 can be applied to solve rational matrix equations of the form \( M(s)L(s) = Q(s) \).

Consider the equation
\[
M(s)L(s) = Q(s)
\]
(8)
where \( L(s)(l \times m) \) and \( Q(s)(k \times m) \) are given polynomial matrices. Determine the polynomial matrix solutions \( M(s)(k \times l) \) when they exist.

First consider the left hand side of equation (8). Let
\[
M(s) \triangleq M_0 + \cdots + M_r s^r
\]
(9)
and \( d_i \triangleq \deg[L(s)] \), \( i = 1, \ldots, m \) that is the column degrees of \( L(s) \). If
\[
\tilde{Q}(s) \triangleq M(s) L(s)
\]
(10)
then \( \deg[L(s)] = d_i + r \) for \( i = 1, \ldots, m \). According to the polynomial matrix interpolation Theorem 1, the matrix \( \tilde{Q}(s) \) can be uniquely specified using \( \Sigma_{i=1}^{n} (d_i + r) + m - \Sigma_{i=1}^{n} (d_i + m)(r + 1) \) interpolation points. Therefore consider \( l \) interpolation points \((s_j, a_j, b_j)\), \( j = 1, \ldots, l \) where
\[
i = \Sigma d_i + m(r + 1)
\]
(11)
Let \( S_r(s) \triangleq \text{blk diag} \{[1, s, \ldots, s^{k+r}]^T\} \) and assume that the \((\Sigma d_i + m(r + 1)) \times l \) matrix
\[
S_r \triangleq [S_r(s_1) a_1, \cdots, S_r(s_l) a_l]
\]
(12)
is full rank; that is the assumptions in Theorem 1 are satisfied. Note that for distinct \( s_j, S_r \) will have full column rank for almost any set of nonzero \( a_j \).

Now in view of Theorem 1 the matrix \( Q(s) \) which satisfies
\[
\tilde{Q}(s_j) a_j = b_j, \quad j = 1, \ldots, l
\]
(13)
is uniquely specified given these \( l \) interpolation points \((s_j, a_j, b_j)\). To solve (8) these interpolation points must be appropriately chosen so that the equation \( \tilde{Q}(s)(= M(s) L(s)) = Q(s) \) is satisfied:

Write (8) as
\[
M L_r(s) = Q(s)
\]
(14)
where
\[
M \triangleq \begin{bmatrix} M_0 & \cdots & M_r \end{bmatrix} (k \times \ell(r + 1))
\]
\[
L_r(s) \triangleq \begin{bmatrix} L(s)^T, \ldots, s^l L(s)^T \end{bmatrix}^T (l(r + 1) \times m).
\]
Let \( s = s_j \) and postmultiply by \( a_j, j = 1, \ldots, l \); note that \( s_j \) and \( a_j \), \( j = 1, \ldots, l \) must be so that \( S_r \) above has full rank. Define
\[
b_j \triangleq Q(s_j) a_j, \quad j = 1, \ldots, l
\]
(15)
and combine the equations to obtain
\[
M L_r = B_l
\]
(16)
where 
\[ L_r = \begin{bmatrix} L_r(s_1) a_1, \cdots, L_r(s_l) a_l \end{bmatrix} (t(r + 1) \times l) \]
and 
\[ B_i = \begin{bmatrix} b_1, \cdots, b_l \end{bmatrix} (k \times l). \]

**Theorem 2** Given \( L(s), Q(s) \) in (8), let \( d_i = \text{deg}_a[L(s)] \) \( i = 1, \cdots, m \) and select \( r \) to satisfy
\[
\text{deg}_a[Q(s)] \leq d_i + r \quad i = 1, \cdots, m \tag{17}
\]
Then a solution \( M(s) \) of degree \( r \) exists if and only if a solution \( M \) of (4.16) does exist;
\[
M(s) = M[l, s_1, \cdots, s_r] T.
\]
It is not difficult to show that solving (16) is equivalent to solving
\[
M(s_j) c_j = b_j \quad j = 1, \cdots, l \tag{18}
\]
where
\[
c_j = L(s_j) a_j, \quad b_j = Q(s_j) a_j \quad j = 1, \cdots, l \tag{19}
\]
\( M(s) \) that satisfy (18) are obtained by solving
\[
M S_i = B_i \tag{20}
\]
where \( S_i = \begin{bmatrix} S_i(s_1) c_1, \cdots, S_i(s_l) c_l \end{bmatrix} (t(r + 1) \times l) \), with 
\[
S_i(s) = \begin{bmatrix} I, s, s I, \cdots, s^r I \end{bmatrix}^{T} (t(r + 1) \times t) \] and 
\[
B_i = \begin{bmatrix} b_1, \cdots, b_l \end{bmatrix} (k \times l). \]
Solving (20) is an alternative to solving (16).

**Constraints on Solutions:** When there are more unknowns \((t(r + 1) + l)\) than equations \((l = Ed_i + m(r + 1))\) in (16), the additional freedom can be exploited so that \( M(s) \) satisfies additional constraints. In particular, \( k \geq l \) additional linear constraints, expressed in terms of the coefficients of \( M(s) \) \( M \), can be satisfied in general. The equations describing the constraints can be used to augment the equations in (4.16). In this case the equations to be solved become
\[
M[L_{ci}, C] = [B_i, D] \tag{21}
\]
where \( M = D \) represents the \( k \) linear constraints imposed on the coefficients \( M \); \( C \) and \( D \) are matrices (real or complex) with \( k \) columns each.

The **Diophantine Equation**

An important case of (8) is the **Diophantine equation**
\[
X(s)D(s) + Y(s)N(s) = Q(s) \tag{22}
\]
where the polynomial matrices \( D(s), N(s) \) and \( Q(s) \) are given and \( X(s), Y(s) \) are to be found. Note that if
\[
M(s) = [X(s), Y(s)], \quad L(s) = \begin{bmatrix} D(s) \n N(s) \end{bmatrix} \tag{23}
\]
it is immediately clear that the Diophantine equation is a polynomial equation of the form (8) and all the previous results do apply. That is, **Theorem 2** guarantees that all solutions of (22) of degree \( r \) are found by solving (16). In the theory of Systems and Control the Diophantine equation used involves a matrix \( L(s) = [U^T(s), N^T(s)]^T \) which has rather specific properties. These are exploited to solve the Diophantine equation and to derive conditions for existence of solutions of (22) of degree \( r \). It can be shown that:

**Theorem 3:** Let \( r \) satisfy
\[
\text{deg}_a[Q(s)] \leq d_i + r \quad i = 1, \cdots, m \quad \text{and} \quad r \geq \nu - 1. \tag{24}
\]
where \( \nu \) is the observability index of the system \( \{D, J, N, 0\} \). Then the Diophantine equation (22) has solutions of degree \( r \) which can be found by solving (16) or (20).

**Solving Rational Matrix Equations**

Now let us consider the rational matrix equation:
\[
M(s)L(s) = Q(s) \tag{25}
\]
where \( L(s)(t \times m) \) and \( Q(s)(k \times m) \) are given rational matrices. The polynomial matrix interpolation theory developed above can be used to solve this equation and determine the rational matrix solutions \( M(s)(k \times t) \). Let \( \tilde{M}(s) = \tilde{D}^{-1}(s) \tilde{N}(s) \), a polynomial fraction form of \( M(s) \) to be determined. Then (24) can be written as:
\[
[\tilde{N}(s), -\tilde{D}(s)] \begin{bmatrix} L(s) \\ Q(s) \end{bmatrix} = 0 \tag{26}
\]
Note that one could equivalently solve
\[
[\tilde{N}(s), -\tilde{D}(s)] \begin{bmatrix} L_p(s) \\ Q_p(s) \end{bmatrix} = 0 \tag{27}
\]
where \( [L_p(s)^T, Q_p(s)^T]^T = [L(s)^T, Q(s)^T]^T \phi(s) \) a polynomial matrix with \( \phi(s) \) the least common denominator of all entries of \( L(s) \) and \( Q(s) \); in general, \( \phi(s) \) could be any matrix denominator in a right fractional representation of \( [L(s)^T, Q(s)^T]^T \). The problem to be solved is now (8), a polynomial matrix equation, where \( L(s) = [L_p(s)^T, Q_p(s)^T]^T \) and \( Q(s) = 0 \). Therefore, all solutions \( [\tilde{N}(s), -\tilde{D}(s)] \) of degree \( r \) can be determined by solving (16) or (20). Let \( s = s_j \) and postmultiply (27) by \( a_j \) \( j = 1, \cdots, l \) with \( a_j \) and \( l \) chosen properly. Define
\[
c_j = \begin{bmatrix} L_p(s) \\ Q_p(s) \end{bmatrix} a_j \quad j = 1, \cdots, l \tag{28}
\]
Note that one could equivalently solve
The problem now is to find a polynomial matrix \( [\tilde{N}(s), -\tilde{D}(s)] \) which satisfies

\[
[\tilde{N}(s_j), -\tilde{D}(s_j)] c_j = 0 \quad j = 1, \cdots, l
\]  

(29)

Note that restrictions on the solutions can be easily imposed to guarantee that \( \tilde{D}^{-1}(s) \) exists and/or that \( M(s) = \tilde{D}^{-1}(s)N(s) \) is proper. Additional constraints can be added so the solution satisfies additional specifications; see (21).

Pole Placement
Output Feedback. All proper output controllers of degree \( r \) (of order \( mr \)) that assign all the closed loop eigenvalues to arbitrary locations are characterized in a convenient way using interpolation results.

Given \( N(s)D^{-1}(s) \) proper, we are interested in solutions \( [X(s), Y(s)](m \times (p + m)) \) of the Diophantine equation where only the roots of \( Q(s) \) are specified; furthermore \( X^{-1}(s)Y(s) \) should exist and be proper. The equation to be solved is

\[
(X(s_j)D(s_j) + Y(s_j)N(s_j))c_j = 0 \quad j = 1, \cdots, l
\]

or \( ML_{r1} = 0 \) (i.e., \( \Sigma d_i + mr \)); that is the \( \Sigma d_i + mr \) roots of \( [X(s)D(s) + Y(s)N(s)] \) are to be assigned the values \( s_j, \quad j = 1, \cdots, l \). Note the difference between the problem studied above where \( Q(s) \) is known, and the problem studied here where only the roots of \( |Q(s)| \) or \( |Q(s)| \) within multiplication by some nonzero real scalar are given. Here the vectors \( c_j \) can be seen as design parameters and they can be selected almost arbitrarily to satisfy requirements in addition to pole assignment.

Consider the solution of the Diophantine equation

\[
[X(s), Y(s)](m \times (p + m)) = M(s) \quad \text{subject to} \quad X^{-1}(s)Y(s) \text{ exists and is proper,}
\]

(30)

Then \( (X(s_j), Y(s_j)) \) exists such that all the \( n + mr \) zeros of \( [X(s)D(s) + Y(s)N(s)] \) are arbitrarily assigned and \( X^{-1}(s)Y(s) \) is proper.

Example: Let \( D(s) = \begin{bmatrix} s - 2 & 1 \\ 0 & s + 1 \end{bmatrix}, \quad N(s) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) with \( n = \deg [D(s)] = 2 \). Here there are \( n + mr = 2 + 2r \) closed-loop poles to be assigned. Note that \( r \geq \nu - 1 = 1 - 1 = 0 \).

i) For \( r = 0 \) and \( \{(s_j, a_j), j = 1, 2\} = \{(1, 0)^T, (-2, [0 \; 1]^T)\} \), a solution of \( ML_{r1} = 0 \) is

\[
M = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 2 & 1 \end{bmatrix}
\]

For this case, \( M = M(s) = [X(s), Y(s)] \).

ii) For \( r = 1 \), and \( \{(s_j, a_j), j = 1, \cdots, 4\} = \{-1, [1 \; 0]^T, -2, [0 \; 1]^T, -3, [1 \; 0]^T, -4, [0 \; -1]^T\} \), a solution of \( ML_{r1} = 0 \) is \( [X(s), Y(s)] \)

\[
\begin{bmatrix} s - 7 & -1 & 12 & s + 1 \\ 5 & s + 4 & -6 & s + 4 \end{bmatrix}
\]

Note that here \( X(s)^{-1}Y(s) \) exists and it is proper.

State Feedback: Let \( A, B, F \) be \( n \times n, m \times m, m \times n \) real matrices respectively. Note that \( |sI - (A + BF)| = |sI - A| \cdot |I_m - (sI - A)^{-1}BF| = |sI - A| \cdot |I_m - F(sI - A)^{-1}B| \). If now the desired closed-loop eigenvalues \( s_j \) are different from the eigenvalues of \( A \), then \( F \) will assign all \( n \) desired closed loop eigenvalues \( s_j \) if and only if

\[
F([s_jI - A]^{-1}B)a_j = a_j \quad j = 1, \cdots, n
\]

(31)

The \( m \times 1 \) vectors \( a_j \) are selected so that \( (s_jI - A)^{-1}B \) are linearly independent vectors. Alternatively one could approach the problem as follows: let \( M(s)(n \times m), D(s)(m \times m) \) be right coprime polynomial matrices such that \( (sI - A)^{-1}B = M(s)D^{-1}(s) \). An internal representation equivalent to \( \delta = A\xi + Bu \) in polynomial matrix form is \( D\xi = u \) with \( z = M\xi \). The eigenvalue assignment problem is then to assign all the roots of \( |D(s) - FM(s)| \); or to determine \( F \) so that

\[
FM(s)a_j = D(s)a_j \quad j = 1, \cdots, n
\]

(32)

The \( m \times 1 \) vectors \( a_j \) are selected so that \( M(s)a_j \) are linearly independent. Note that when \( s_j \) are distinct, \( a_j \) can almost be arbitrarily selected. For all the appropriate choices of \( a_j \), \( M(s)a_j \) \( j = 1, \cdots, n \) linearly independent, the \( a_j \) are different from \( F \) that produce, in general, different closed loop behavior.

The exact relation of the eigenvectors to the \( a_j \) can be found as follows: \( [s_jI - (A + BF)]M(s)a_j = (s_jI - A)M(s)a_j - BF(M(s)a_j) = BD(s)a_j - BD(s)a_j = 0 \). Therefore, \( M(s)a_j = u_j \) are the closed-loop eigenvectors corresponding to \( s_j \).

Conclusions
Interpolation is a very general and flexible way to deal with problems involving polynomial and rational matrices. Results presented provide an appropriate theoretical setting and algorithms to deal with such problems. Matrix interpolation has been used successfully in a number of control problems to determine polynomial and rational matrix solutions. It has also been shown to have excellent numerical properties in deriving solutions of polynomial matrix equations. There is need to explore the full potential of the approach, to identify strengths and weaknesses and to develop a full range of efficient computer algorithms to meet the many needs of control design problems.

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