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On the Reachability of a Class of Second-Order Switched Systems

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Abstract

In this paper, the reachability problem for a class of second-order LTI switched systems is solved. The reachability problem is first explored for switched systems consisting of two subsystems and switching control laws are proposed that can drive the system state from an initial point to a target point via finitely many switches. The method is then extended to the case of several subsystems. The relationship between stabilizability and reachability is also addressed.

1 Introduction

A switched system is a system that consists of several subsystems and a switching law that specifies which subsystem dynamics will be followed by the system trajectory at each instant of time. Recently, there has been increasing interest in the stability analysis and design of such systems (see, e.g., [1, 2, 3, 6, 7, 8, 9]). [5] provides a survey of recent development of stability and design of switched systems.

In this paper, we are interested in *switched systems consisting of second-order LTI subsystems with foci at the origin.* (A second-order system $\dot{x} = Ax$ is with focus at the origin if A has eigenvalues $\alpha_i \pm \beta_i j$, $\beta_i \neq 0$, see [4] Chapter 1.)

Definition 1.1 *A switched system consisting of second-order LTI subsystems with foci at the origin is a system described by system equations:*

$$\dot{x} = A_{i(t)}x \tag{1}$$
$$i(t) = \phi(x(t), i(t^-), t), \tag{2}$$

where $x \in \mathbb{R}^2$, $i(t) \in I = \{1, 2, \cdots, N\}$ and $A_i \in \mathbb{R}^{2\times 2}$ with foci at the origin. $\phi : \mathbb{R}^2 \times I \times \mathbb{R} \to I$ describes the switching law. It is assumed that the switching law can only generate a finite number of switches in any finite time period.

The switching law in Definition 1.1 is to be designed in control synthesis. Based on our earlier results on stabilization of switched systems [8, 9], we consider the system (1) and design the switching control law (2) for the reachability of the switched system. For the above class of switched systems, we define a switched system to be reachable as follows.

Definition 1.2 (Reachability) *The switched system (1) is said to be* **reachable** *if for any non-equilibrium initial point $x_i \neq 0$ and target point $x_t \neq 0$, there exists a switching control law that can generate a finite switching sequence to drive the system state from x_i to x_t in finite amount of time.*

The outline of this paper is as follows. In Section 2, we review some results from [8] and then, based on those results, design switching control laws for the reachability problem of switched systems with two unstable subsystems. Section 3 and 4 discuss the reachability problem for switched systems with two stable subsystems, and switched systems with one stable and one unstable subsystems, respectively. The switching control laws are extended to switched systems with several subsystems in Section 5. Section 6 addresses the relationship between stabilizability and reachability. Section 7 contains concluding remarks.

2 Reachability of Switched Systems with Two Unstable Subsystems

In this section, we consider the reachability problem for the switched system (1) consisting of two subsystems with unstable foci and derive switching control laws.

2.1 Unstable Subsystems

We first review some stability results from [8]. We shall say that a subsystem is of clockwise (counterclockwise) direction if starting from any nonzero initial condition in the phase plane its trajectory is a spiral around the origin in the clockwise (counterclockwise) direction.

Let $x = (x_1, x_2)^T$ be a nonzero point on $\mathbb{R}^2$ plane, and denote $f_1 = A_1x = (a_1, a_2)^T$, $f_2 = A_2x = (a_3, a_4)^T$. We view x, f_1 and f_2 as vectors in $\mathbb{R}^2$ and define $\theta_i, i = 1, 2$ to be the angle between x and f_i measured counterclockwise with respect to x (θ_i is confined to $-\pi \leq \theta_i < \pi$). So θ_i is positive (negative) if vector f_i is to the counterclockwise (clockwise) side of x. Also as in [8] we define the regions

$$\begin{aligned} E_{is} &= \{x| -\pi \leq \theta_i(f_i) \leq -\frac{\pi}{2} \text{ or } \frac{\pi}{2} \leq \theta_i(f_i) < \pi\} \\ &= \{x|x^T f_i(x) = x^T A_i x \leq 0\},\ i = 1, 2 \\ E_{iu} &= \{x| -\frac{\pi}{2} \leq \theta_i(f_i) \leq 0 \text{ or } 0 \leq \theta_i(f_i) \leq \frac{\pi}{2}\} \\ &= \{x|x^T f_i(x) = x^T A_i x \geq 0\},\ i = 1, 2. \end{aligned}$$

To design stabilizing switching control laws, we identify the following two distinct cases.

Case 1. Two Subsystems of the Same Direction

Without loss of generality, assume that both subsystems of (1) are of clockwise direction. We define the following conic regions.

$$\begin{aligned} \Omega_1 &= E_{1s} \cap E_{2u},\ \ \Omega_2 = E_{1u} \cap E_{2s}, \\ \Omega_3 &= E_{1s} \cap E_{2s} \cap \{x|a_2a_3 - a_1a_4 \leq 0\}, \\ \Omega_4 &= E_{1s} \cap E_{2s} \cap \{x|a_2a_3 - a_1a_4 \geq 0\}, \\ \Omega_5 &= E_{1u} \cap E_{2u} \cap \{x|a_2a_3 - a_1a_4 \leq 0\}, \\ \Omega_6 &= E_{1u} \cap E_{2u} \cap \{x|a_2a_3 - a_1a_4 \geq 0\}. \end{aligned}$$

The **conic switching law** proposed in [8] is as follows: switch the switched system to subsystem 1 whenever the system state enters Ω_1, Ω_3, Ω_5 and switch to subsystem 2 whenever the system state enters Ω_2, Ω_4, Ω_6.

The following theorem concerns the stabilizability of the switched system (see [8]). Note that this is a necessary and sufficient condition as opposed to other literature results.

Theorem 2.1 *Let l_1 be a ray that goes through the origin. Let $x_0 \neq 0$ be on l_1. Let x^* be the point on l_1 where the trajectory intersects l_1 for the first time after leaving x_0, when the switched system evolves according to the conic switching law. The switched system (1) with subsystems with unstable foci and of the same direction is asymptotically stabilizable if and only if $\|x^*\|_2 < \|x_0\|_2$ by the conic switching law.*

Example 2.1 illustrates how the conic switching law works.

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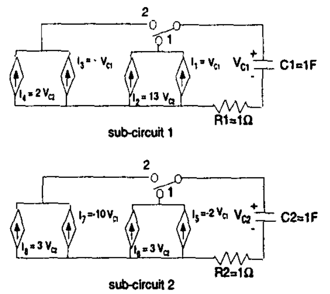


Figure 1: A simple circuit system consisting of two dependent sub-circuits for Example 2.1.

Example 2.1 *Figure 1 shows a simple circuit system consisting of two dependent sub-circuits. For every sub-circuit, there is a switch to connect the sub-circuit to one of the two voltage dependent current source circuits. So there are four possible switch combinations which would provide us with four subsystems. Here we assume that the switches for both sub-circuits can only be both at position 1 or both at position 2 simultaneously. This reduces the number of possible switch combinations to two. It is not difficult to derive the following differential equations for the circuit* ($V = [V_{c_1}, V_{c_2}]^T$).

Both switches are at position 1: $\dot{V} = \begin{bmatrix} 1 & 13 \\ -2 & 3 \end{bmatrix} V.$

Both switches are at position 2: $\dot{V} = \begin{bmatrix} -1 & 2 \\ -10 & 3 \end{bmatrix} V.$

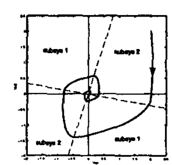


Figure 2: The trajectory of the switched circuit system by using the conic switching law $[(V_{c_1}(0), V_{c_2}(0)]^T = [2, 2]^T$).

The switched system consists of two unstable subsystems with foci at the origin and is asymptotically stabilizable by the conic switching law. Figure 2 shows the system trajectory from initial point $[(V_{c_1}(0), V_{c_2}(0)]^T = [2, 2]^T$. □

Case 2. Two Subsystems of Opposite Directions

Assume that subsystem 1 is of clockwise direction while subsystem 2 is of counterclockwise direction.

We introduce the following conic regions.

$$\begin{aligned}
\Omega_1 &= E_{1s} \cap E_{2s}, \quad \Omega_2 = E_{1u} \cap E_{2u}, \\
\Omega_3 &= E_{1s} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, \\
\Omega_4 &= E_{1s} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}, \\
\Omega_5 &= E_{1u} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, \\
\Omega_6 &= E_{1u} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}.
\end{aligned}$$

Theorem 2.2 concerns the stabilizability of the switched system (see [8]).

Theorem 2.2 *The switched system (1) with two subsystems with unstable foci and of opposite directions is asymptotically stabilizable if and only if* $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$.

If the switched system is asymptotically stabilizable, then the conic switching law can also be obtained as in [8] which makes the system asymptotically stable. The **conic switching law** is as follows: first, by following subsystem 1, force the trajectory into one of the conic regions $\Omega_1, \Omega_3, \Omega_5$, and then switch to another subsystem upon intersecting the boundary of the region so as to keep the trajectory inside the region.

Remark: The conic switching laws can also be extended to switched systems consisting of second-order LTI subsystems which are not necessarily with foci (see [8]). □

2.2 Reachability Results

In the following discussion, we assume that the switched system consists of unstable subsystems but is asymptotically stabilizable. Assume that $x_i \neq 0$ and $x_t \neq 0$ are given non-equilibrium initial and target points on $\mathbb{R}^2$ plane, respectively. We want to find a switching control law so as to drive the state of the system from x_i to x_t. We consider the following two cases.

Case 1. Two Subsystems of the Same Direction

Without loss of generality, assume both subsystems are of clockwise direction. It is known that the conic switching law will make the system asymptotically stable. Now consider the trajectory starting from x_t with time going backwards, i.e., consider the trajectory C^- for $-t, t \geq 0$ by following the conic switching law. It is clear that this trajectory would be away from the origin in a counterclockwise fashion. Let l_i be the ray that goes through the origin and x_i. Let l_t be the ray that goes through the origin and x_t. Let $x^* = x(-t^*)$ be the point on C^- that satisfies the following conditions.

1. The trajectory $E = \{x(-t) \in C^- | 0 < t \leq t^*\}$ intersects l_t at least once.
2. t^* is the minimum possible t such that condition 1 is satisfied and $x(-t^*)$ is on l_i and $\|x(-t^*)\|_2 \geq \|x_i\|_2$.

To obtain a switching control sequence, we let the system start from x_i at $t = 0$ following the trajectory of subsystem 1. Since subsystem 1 is unstable, it is clear that if the switched system stays at subsystem 1 for sufficiently long time, $x(t)$ will be outside the region formed by E and part of l_i(the region inside the bold curves in Figure 3(a)). By this we conclude that there exists a time instant t_1 such that the trajectory intersects E for the first time, i.e., $x_1 = x(t_1) \in E$.

So by the above discussion, we can adopt the following switching control law.

Switching control law:

Step 1. *Let the system trajectory start from* x_i *at* $t = 0$ *following the trajectory of subsystem 1 until it intersects* E *for the first time at* t_1 *on* $x_1 = x(t_1)$.

Step 2. *After it reaches* $x_1 = x(t_1)$, *let the system evolve following the conic switching law.*

The above switching control law can drive the system state from x_i to x_t via only a finite number of switches.

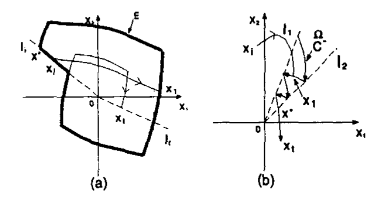


Figure 3: Switching control law for two unstable subsystems of (a). the same direction (b). opposite directions.

Example 2.2 *Consider the circuit system as in Example 2.1. The switched system is reachable since it is asymptotically stabilizable. Figure 4 shows the system trajectory from* $x_i = [-2, -2]^T$ *to* $x_t = [-0.5, 0.5]^T$ *and the corresponding switching sequences by using the switching control law.*

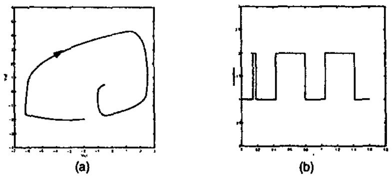


Figure 4: (a). The trajectory of the switched circuit system from $x_i = [-2,-2]^T$ to $x_t = [-0.5, 0.5]^T$. (b). The corresponding switching sequence.

Case 2. Two Subsystems of Opposite Directions

Assume subsystem 1 is of clockwise direction and subsystem 2 is of counterclockwise direction. Since the switched system is asymptotically stabilizable, $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$. We assume Ω is one conic region from the possible nonempty sets Ω_1, Ω_3 and Ω_5 which is between the two rays l_1 and l_2(Figure 3(b)). Consider the following trajectory C^-. Starting from x_t, let the system trajectory go backwards in time by following subsystem 1 until it intersects l_2 for the first time by x^* at $-t^*$, then let the trajectory going backwards in time following the conic switching law in Ω. In this way, we can get the trajectory C^- as depicted in Figure 3(b).

Now let the system start from x_i at $t = 0$ following the trajectory of subsystem 1. Since subsystem 1 is unstable, it is clear that if the switched system stays at subsystem 1 for sufficiently long time, $x(t)$ will intersect C^-. By this we conclude that there exists a time instant t_1 such that the trajectory intersects C^- for the first time at $x_1 = x(t_1)$.

By the above discussion, we can adopt the following switching control law.

Switching control law:

Step 1. *Let the system trajectory start from x_i at $t = 0$ following the trajectory of subsystem 1 until it intersects C^- for the first time at t_1 on $x_1 = x(t_1)$.*

Step 2. *After it reaches $x_1 = x(t_1)$, let the system evolve following the corresponding subsystems associated with the points on the trajectory C^-.*

The above switching control law can drive the system state from x_i to x_t via only a finite number of switches. □

3 Reachability of Switched Systems with Two Stable Subsystems

In the present section, we consider reachability problem for the switched systems (1) consisting of two subsystems with stable foci and derive switching control laws.

3.1 Stable Subsystems

If both subsystems are stable, then the stabilizability of the switched system can be easily established if we simply let the system stay at one subsystem and do not apply any switches. Yet we may ask a converse question: can we find a control law such that the switched system can be "destabilized", in other words, the trajectory of the switched system can be made unbounded. It is not quite difficult to see that such a control law can be found if the switched system with two unstable subsystems

$$\dot{x} = -A_1 x, \quad \dot{x} = -A_2 x, \tag{3}$$

is asymptotically stabilizable. Formally, we defined d-stabilizability as follows.

Definition 3.1 *An switched system (1) with two stable subsystems is said to be* **d-stabilizable** *if and only if the corresponding switched system (3) with two unstable subsystems is asymptotically stabilizable.*

A close look at the result for two unstable subsystems shows that the d-stabilizability problem is readily solved by some small modification of the aforementioned results. Similarly to Section 2.1, we will discuss the following two cases.

Case 1. Two Subsystems of the Same Direction

Assume both subsystems are of clockwise direction. Using the same notation as in Case 1 in the Section 2.1, we propose the following **conic switching law**: switch the switched system to subsystem 2 whenever the system state enters Ω_1, Ω_3, Ω_5 and switch to subsystem 1 whenever the system state enters Ω_2, Ω_4, Ω_6.

The following theorem concerns the d-stabilizability of the switched system.

Theorem 3.1 *Let l_1 be a ray that goes through the origin. Let $x_0 \neq 0$ be on l_1. Let x^* be the point on l_1 where the trajectory intersects l_1 for the first time after leaving x_0, when the switched system evolves according to the conic switching law. The switched system (1) with subsystems with stable foci and of the same direction is d-stabilizable if and only if $\|x^*\|_2 > \|x_0\|_2$ by the conic switching law.*

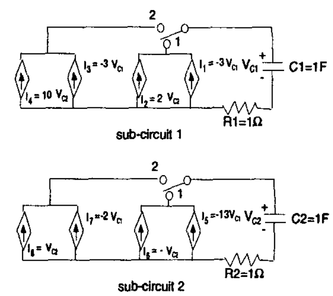


Figure 5: A simple circuit system consisting of two dependent sub-circuits for Example 3.1.

Example 3.1 shows how the conic switching law works.

Example 3.1 *Figure 5 shows a simple circuit system consisting of two dependent sub-circuits with coefficients different to Example 2.1. It can be readily obtained that:*

Both switches are at position 1: $\dot{V} = \begin{bmatrix} -3 & 2 \\ -13 & -1 \end{bmatrix} V.$

Both switches are at position 2: $\dot{V} = \begin{bmatrix} -3 & 10 \\ -2 & 1 \end{bmatrix} V.$

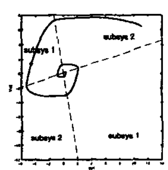


Figure 6: The trajectory of the switched circuit system by using the conic switching law $[(V_{c_1}(0), V_{c_2}(0)]^T = [0.2, 0.2]^T)$.

The switched system consists of two stable subsystems with foci at the origin and is d-stabilizable. Figure 2 shows the system trajectory from initial point $[(V_{c_1}(0), V_{c_2}(0)]^T = [0.2, 0.2]^T$. □

Case 2. Two Subsystems of Opposite Directions

Assume subsystem 1 is of clockwise direction and subsystem 2 is of counterclockwise direction. With the same notation as in Case 2 of Section 2.1, the following theorem concerns the d-stabilizability of the switched system.

Theorem 3.2 *The switched system (1) with two subsystems with stable foci of opposite directions is d-stabilizable if and only if* $Int(\Omega_2) \cup Int(\Omega_4) \cup Int(\Omega_6) \neq \emptyset$.

If the switched system is d-stabilizable, then a conic switching law can also be obtained. The **conic switching law** is: first, by following subsystem 1, force the trajectory into one of the conic regions Ω_2, Ω_4, Ω_6, and then switch to another subsystem upon intersecting the boundary of the region so as to keep the trajectory inside the region. □

3.2 Reachability Results

Consider the switched system (1) with two stable subsystems. In the following discussion, we assume that the switched system is d-stabilizable. The switching control law can be obtained by some modifications of the discussion in Section 2.2.

Case 1. Two Subsystems of the Same Direction

Assume that both subsystems are of the clockwise direction. Since the switched system is d-stabilizable, the conic switching law will "destabilize" the system asymptotically. Therefore, for this switched system, if we consider the trajectory starting from x_i with time going forward, i.e., consider the trajectory C^+ for $t, t \geq 0$ by following the conic switching law. It is clear that this trajectory would be farther and farther from the origin in a clockwise fashion. Let l_i be the ray that goes through the origin and x_i. Let l_t be the ray that goes through the origin and x_t. Let $x^* = x(t^*)$ be the point on C^+ that satisfies the following conditions.

1. The trajectory $E = \{x(t) \in C^+ | 0 < t \leq t^*\}$ intersects l_i at least once.
2. t^* is the minimum possible t such that condition 1 is satisfied and $x(t^*)$ is on l_t and $\|x(t^*)\|_2 \geq \|x_t\|_2$.

To obtain a switching control sequence, we let the system start from x_t and going backwards in time following subsystem 1, since subsystem 1 is stable, the backward trajectory will not be stable. So as the discussion in Case 1 in Section 2.2, the backward trajectory will intersects E for the first time at $x_1 = x(t_1)$.

So we propose the following switching control law.

Switching control law:

Step 1. *Let the system trajectory start from x_i at $t = 0$ following conic switching law until it reaches x_1.*

Step 2. *After it reaches x_1, let the system switch to subsystem 1 and evolve following subsystem 1.*

Such a switching control law drives the system state from x_i to x_t via only a finite number of switches (Figure 7(a)).

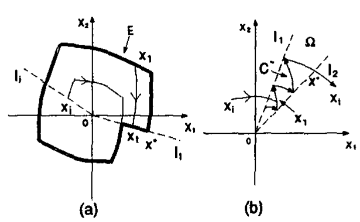


Figure 7: Switching control law for two stable subsystems of (a). the same direction (b). opposite directions.

Example 3.2 *Consider the circuit system in Example 3.1. The switched system is reachable since it is d-stabilizable. Figure 4 shows the system trajectory from $x_i = [0.2, 0.2]^T$ to $x_t = [4, -4]^T$ and the corresponding switching sequences by using the switching control law proposed above.* □

Case 2. Two Subsystems of Opposite Directions

Assume subsystem 1 is of clockwise direction and subsystem 2 is of counterclockwise direction. Since the switched system is d-stabilizable, $Int(\Omega_2) \cup Int(\Omega_4) \cup Int(\Omega_6) \neq \emptyset$. We assume Ω is one conic region from the possible nonempty sets Ω_2, Ω_4 and Ω_6 which is between the two rays l_1 and l_2(Figure 7(b)). Consider the following trajectory C^-. Starting from x_t, let the system trajectory go backwards in time by following subsystem 1 until it intersects l_2 for the first time by x^* at $-t^*$, then let the trajectory going backwards in time following the conic switching law in Ω. In this way, we can get the trajectory C^- as depicted in Figure 7(b).

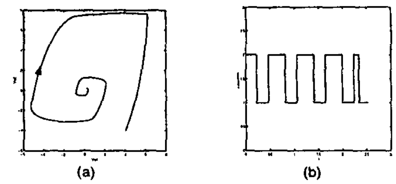


Figure 8: (a). The trajectory of the switched circuit system from $x_i = [0.2, 0.2]^T$ to $x_t = [4, -4]^T$. (b). The corresponding switching sequence.

Now let the system start from x_i at $t = 0$ following the trajectory of subsystem 1. Since subsystem 1 is stable, it is clear that if the switched system stay at subsystem 1 for sufficiently long time, $x(t)$ will intersect C^-. By this we conclude that there exists a time instant t_1 such that the trajectory intersects C^- for the first time at $x_1 = x(t_1)$.

So we propose the following switching control law.

Switching control law:

Step 1. *Let the system trajectory start from x_i at $t = 0$ following the trajectory of subsystem 1 until it intersects C^- for the first time at t_1 on $x_1 = x(t_1)$.*

Step 2. *After it reaches $x_1 = x(t_1)$, let the system evolve following the corresponding subsystems associated with the points on the trajectory C^-.*

The switching control law can drive the system state from x_i to x_t via only a finite number of switches. □

4 Reachability of Switched Systems with One Stable and One Unstable Subsystems

If the switched system (1) consists of stable subsystem 1 and unstable subsystem 2, we note that the switched system must be both asymptotically stabilizable (by just following subsystem 1) and d-stabilizable (by just following subsystem 2). For reachability, there are two cases to be discussed, where the first case is similar to Case 1 in Section 2.2 and the second case is slightly different from Case 2 in Section 2.2.

Case 1. Two Subsystems of the Same Direction

Let the trajectory C^- be the trajectory starting from x_t with time going backwards by following subsystem 1. Let $x^* = x(-t^*)$ be the point on C^- that satisfies:

1. The set $E = \{x(-t) \in C^- | 0 < t \leq t^*\}$ intersects l_t at least once.
2. t^* is the minimum possible t such that condition 1 is satisfied and $x(-t^*)$ is on l_i and $\|x(-t^*)\|_2 \geq \|x_i\|_2$.

Let the system start from x_i at $t = 0$ and follow subsystem 2. Since subsystem 2 is unstable, by the similar reason as in Case 1 in Section 2.2, the trajectory will intersect E at $x_1 = x(t_1)$ for the first time.

We propose the following switching control law.

Switching control law:

Step 1. *Let the system start from x_i at $t = 0$ and follow subsystem 2 until it reaches x_1.*

Step 2. *After it reaches x_1, let the system switch to subsystem 1 and evolve following subsystem 1.*

The switching control law requires only one switch to

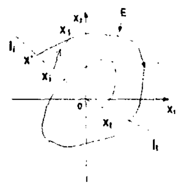


Figure 9: Switching control law for one stable subsystem and one unstable subsystem of the same direction.

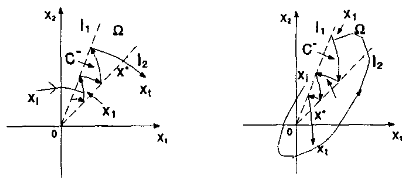
solve the reachability problem. □

Case 2. Two Subsystems of Opposite Directions

Assume in (1), subsystem 1 is of clockwise direction and subsystem 2 is of counterclockwise direction. Then we identify the following two sub-cases.

a. If $\exists c > 0$ such that $A_1 = -cA_2$, then $C^+_{A_1} = C^-_{A_2}$, $C^-_{A_1} = C^+_{A_2}$ for any initial point $x_0 \neq 0$ (See Section 2 of [8]). Therefore x_t is not reachable from x_i if x_t is neither on $C^+_{A_1}$ nor on $C^-_{A_1}$.

b. If there does not exist $c > 0$ such that $A_1 = -cA_2$, then there exists at most two lines on which $A_1x = kA_2x$ for some $k \neq 0$ (See Section 2 of [8]). In view of the θ_i introduced in Section 2.1, we can find a conic region in which either $|\theta_1| + |\theta_2| < \pi$ or $|\theta_1| + |\theta_2| > \pi$ holds for every point.



(a). $|\theta_1|+|\theta_2|<\pi$ (b). $|\theta_1|+|\theta_2|>\pi$

Figure 10: Switching control law for one stable subsystem and one unstable subsystem of opposite directions.

For $|\theta_1|+|\theta_2| < \pi$ case, we can obtain a switching control law similar to Case 2 in Section 3.2 (Figure 10(a)).

For $|\theta_1|+|\theta_2| > \pi$ case, we can obtain a switching control law similar to Case 2 in Section 2.2 except that we choose subsystem 2 between x_i and x_1 (Figure 10(b)). □

5 Several Subsystems

Consider the switched system (1) consisting of several second-order LTI subsystems with foci. The reachability results in Sections 2, 3 and 4 are readily extended to several subsystems as shown in the following.

Case 1. All Subsystems with Unstable Foci

We assume that the switched system is asymptotically stabilizable. If all subsystems are of the same direction, then we adopt the similar method as in Case 1 of Section 2.2 to drive x_i to x_t.

If $K(K > 0)$ subsystems $S^-_1, \cdots, S^-_K$ are of clockwise direction and $M(M > 0)$ subsystems $S^+_1, \cdots, S^+_M$ are of counterclockwise direction $(K + M = N)$, then we can use one of the following methods .

1. If $S^-_1, \cdots, S^-_K$ are asymptotically stabilizable, then adopt the similar method as in Case 1 of Section 2.2.
2. If $S^+_1, \cdots, S^+_M$ are asymptotically stabilizable, then adopt the similar method as in Case 1 of Section 2.2.
3. If there exists S^-_i and S^+_j such that the switched system consisting of S^-_i and S^+_j is asymptotically stabilizable, then adopt the similar method as in Case 2 of Section 2.2.

Case 2. All Subsystems with Stable Foci

The discussion is similar to the above case.

Case 3. K Subsystems with Stable Foci and M Subsystems with Unstable Foci

In this case, as long as there is one stable subsystem and one unstable subsystem satisfying the condition of Case 1 or the condition (b) in Case 2 of Section 4, we can always adopt the method therein.

6 Stabilizability and Reachability

Now we state the relationship between stabilizability and reachability. Without loss of generality, we just consider switched systems (1) with two subsystems.

If a switched system is reachable, then it must be asymptotically stabilizable and d-stabilizable. This is not difficult to show. For asymptotic stabilizability, we can start from x_i and drive the state to x_t which are on l_i and $\|x_t\|_2 \leq q\|x_i\|_2$, $q < 1$. Continuing this way, we can asymptotically stabilize the system. d-stabilizability can be similarly shown.

Combined with the results in Sections 2, 3 and 4, we can readily prove the following theorem which provides us with a necessary and sufficient condition for the reachability of the switched systems.

Theorem 6.1 *Consider the second-order switched systems (1) consisting of two LTI subsystems with foci. If there does not exist $c \neq 0$ such that $A_1 = cA_2$, then the switched system is asymptotically stabilizable and d-stabilizable if and only if it is reachable.*

7 Conclusions

This paper considers the reachability problem for switched systems consisting of second-order LTI subsystems with foci and it is concerned with switching control laws to drive the state from $x_i \neq 0$ to $x_t \neq 0$. Necessary and sufficient conditions for reachability are also obtained. The method to obtain a switching control law is constructive and it is based on the conic switching laws proposed in [8, 9]. Various cases are discussed according to the stability and directions of subsystems. Additional details can be found at: `http://www.nd.edu/~isis/tech.html`.

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