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Abstract

For a class of second-order switched systems consisting of two linear time-invariant (LTI) subsystems, we show that the so-called *conic switching law* proposed previously by the present authors is robust, not only in the sense that the control law is *flexible* (to be explained further), but also in the sense that the Lyapunov stability (resp., Lagrange stability) properties of the switched system are preserved in the presence of certain kinds of vanishing perturbations (resp., nonvanishing perturbations). The analysis is possible since the conic switching laws always possess certain kinds of "quasi-periodic switching operations". We also propose for a class of nonlinear second-order switched systems with time-invariant subsystems a switching control law which locally exponentially stabilizes the entire nonlinear switched system, provided that the conic switching law exponentially stabilizes the linearized switched systems (consisting of the linearization of each nonlinear subsystem). This switched control law is robust in the sense mentioned above. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Second-order switched systems; Switching control law; Robustness; Perturbations

1. Introduction

Switched systems are hybrid systems that consist of two or more subsystems and are controlled by switching laws. These switching laws may be either supervised or unsupervised, time-driven or event-driven, and may be (logically) constrained or unconstrained. Many real-world processes and systems can be modeled as switched systems, including chemical processes, computer controlled systems, switched circuits, and so forth.

Recently, there has been increasing interest in the stability analysis of systems of this type (see, e.g., [1–5,7–9]). The methodologies used in studying the qualitative properties of switched systems are very diverse. In [1,2], multiple Lyapunov functions are introduced and a result for the stability of a switched system is established. In [9], linear matrix inequality (LMI) problems are formulated for the stability analysis of switched systems consisting of linear subsystems. The LMI approach (see, e.g. [4,9]) proves to be a very good way to determine sufficient conditions for the stability of switched systems with affine subsystems. Other related topics can be found in the survey paper [5] and the references therein.

Another important issue is the synthesis problem on how to derive stabilizing switching laws. Thus far,

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such results are quite rare, especially for high-order switched systems. In [9], a "region partition" procedure is mentioned, which is relevant in this regard. Actually, this problem was formulated in [9] as an LMI problem. The partitioning is possible if a solution to the LMI problem can be obtained. In many cases, however, the LMI problem turns out to be quite complicated and the existence of a solution can not be guaranteed. Note that another approach to the problem of robust stabilization via controller switching was presented in [10]. In [11], conic switching laws were proposed to study second-order linear time-invariant switched systems, and for switched systems whose subsystems have unstable foci, both necessary and sufficient conditions for stabilizability were established. This method can also be extended to study switched systems consisting of LTI subsystems not necessarily with foci (see [12]). We point out that by following the procedure in [9], even for a given second-order switched system consisting of two linear time-invariant subsystems, the system still may or may not be stabilizable if the LMI problem has no solution. This reinforces the fact that the approach involving LMI yields only sufficient conditions. Clearly, necessary and sufficient conditions for second-order LTI switched systems have advantages over the existing results in the literature.

In the present paper, we study the robustness properties of the conic switching control laws. For LTI switched systems, we know from Xu and Antsaklis [11] (refer also to Section 2) that the conic switching control laws rely heavily on the switching information at the boundaries of certain conic regions. It has not been shown rigorously whether conic control laws can still stabilize an entire switched system if the switching boundaries are not precisely reached when switching occurs. Also the question whether or not the stabilizing properties will be preserved in the presence of perturbations, either vanishing or nonvanishing is not answered. The answers to the above questions are affirmative and are given below. We show in Section 3 that for LTI switched systems the conic switching laws are endowed with a certain kind of robustness property, either in the sense that these event-driven control laws have certain flexibility on switching regions, or in the presence of vanishing/nonvanishing perturbations, or a combination of both.

In addition to the above, in a more interesting problem we ask whether or not we can determine conic switching laws for nonlinear switched systems and whether or not the conic switching laws are still robust. We will show that the answer to this question is also affirmative. For a class of *second-order time-invariant nonlinear switched systems* whose linearized subsystems have unstable foci, we propose a conic switching law in Section 4 and show that this switching law not only locally stabilizes the entire system, but also possesses robustness properties similar to those discussed in Section 3.

To demonstrate our results, we present some numerical examples along with simulations in Section 5.

For clarity of presentation, we primarily addressed in the present paper switched systems consisting of two subsystems with foci and of opposite directions. We point out, however, that similar results can also be established for more general second-order switched systems discussed in [12].

2. Conic switching laws for LTI switched systems

In the interests of completeness and clarity, we summarize in the present section the conic switching laws proposed in [11,12] for switched systems consisting of two subsystems with foci. As in [12], we say that a subsystem is of *clockwise* (*counterclockwise*) *direction* if starting from any nonzero initial condition in the phase plane its trajectory is a spiral around the origin in the clockwise (counterclockwise) direction.

Consider the switched systems described by $\dot{x}(t)=A_ix(t)$, $i=1,2$, whose subsystems are both assumed to have unstable foci. Let $x=(x_1,x_2)^{\mathrm{T}}$ be a nonzero point in the $\mathbb{R}^2$ plane, and denote $f_1(x)=A_1x=(a_1,a_2)^{\mathrm{T}}$, $f_2(x)=A_2x=(a_3,a_4)^{\mathrm{T}}$. We view x, f_1 and f_2 as vectors in $\mathbb{R}^2$ and define the angle θ_i, $i=1,2$ to be the angle between x and f_i measured counterclockwise with respect to x (θ_i is confined to $-\pi\leqslant\theta_i<\pi$). Thus, θ_i is positive (negative) if f_i, as a vector, is to the counterclockwise (clockwise) side of x (see Fig. 1(a)). Also as in [11], we define the regions

$$
\begin{aligned}
E_{is} &= \Big\{x \,|\, -\pi\leqslant\theta_i(f_i(x))\leqslant-\frac{\pi}{2} \\
&\qquad \text{or } \frac{\pi}{2}\leqslant\theta_i(f_i(x))<\pi\Big\} \\
&= \{x|x^{\mathrm{T}}f_i(x)=x^{\mathrm{T}}A_ix\leqslant 0\}, \quad i=1,2,
\end{aligned}
$$

$$
\begin{aligned}
E_{iu} &= \Big\{x \,|\, -\frac{\pi}{2}\leqslant\theta_i(f_i(x))\leqslant\frac{\pi}{2}\Big\} \\
&= \{x|x^{\mathrm{T}}f_i(x)=x^{\mathrm{T}}A_ix\geqslant 0\}, \quad i=1,2.
\end{aligned}
$$

Clearly, the interior of E_{is} (E_{iu}) is the set of all points in the $\mathbb{R}^2$ plane where the trajectory of the ith subsystem

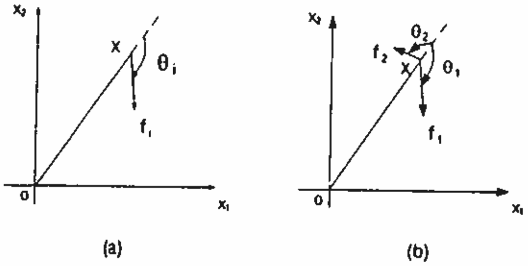

Fig. 1. (a). The angle θ_i. (b). Figure for Case 2.

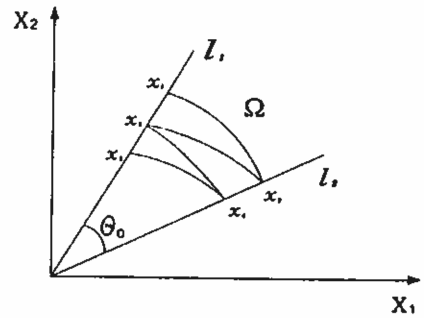

Fig. 2. Switching happens on the boundary of the safe region Ω: l_1 and l_2.

would be driven closer to (farther from) the origin if the subsystem evolves for sufficiently small amount of time starting from the point.

To design stabilizing switching control laws, we identify the following two different cases: (Case 1) two subsystems are of the same direction; (Case 2) two subsystems are of opposite directions. In accommodation with the subsequent discussion, without loss of generality, we only discuss the results for the latter case, assuming that subsystem 1 is of clockwise direction while subsystem 2 is of counterclockwise direction.

The basic idea for determining an asymptotically stabilizing switching law is motivated by the observation that in any conic region where $|\theta_1|+|\theta_2|\geqslant\pi$ (see Fig. 1(b)), the following trajectory will be bounded, where the trajectory starts from x_0 in the conic region and evolves following subsystem 1 and then switches to another subsystem upon intersecting the boundaries of the conic region. This basic idea is formalized below. Let

$$\Omega_1 = E_{1s}\cap E_{2s}, \qquad \Omega_2 = E_{1u}\cap E_{2u},$$

$$\Omega_3 = E_{1s}\cap E_{2u}\cap\{x|a_2a_3-a_1a_4\geqslant 0\},$$

$$\Omega_4 = E_{1s}\cap E_{2u}\cap\{x|a_2a_3-a_1a_4\leqslant 0\},$$

$$\Omega_5 = E_{1u}\cap E_{2s}\cap\{x|a_2a_3-a_1a_4\geqslant 0\},$$

$$\Omega_6 = E_{1u}\cap E_{2s}\cap\{x|a_2a_3-a_1a_4\leqslant 0\},$$

from Xu and Antsaklis [11] we have the following result concerning the stabilizability of the switched system.

Theorem 2.1. *The switched system* $\dot{x}(t)=A_ix(t)$, $i=1,2$, *consisting of two subsystems with foci and of opposite directions is asymptotically stabilizable if and only if* $\mathrm{Int}(\Omega_1)\cup\mathrm{Int}(\Omega_3)\cup\mathrm{Int}(\Omega_5)\neq\emptyset$, *where* $\mathrm{Int}(\Omega)$ *denotes the interior of set* Ω.

Conic switching law: First, by following subsystem 1, force the trajectory into the interior of one of the conic regions $\Omega_1,\Omega_3,\Omega_5$ *(there must be one available if the system is stabilizable according to Theorem 2.1), and then switch to another subsystem upon intersecting the boundary of the region so as to keep the trajectory inside the conic region.*

3. Robustness analysis of conic switching laws for LTI switched systems

In the present section, we investigate the robustness problem of the previous proposed switching control law for LTI switched systems.

We call the conic regions in Section 2, $\Omega_1,\Omega_3,\Omega_5$, *safe regions*, since the existence of nonempty interior of such regions guarantees the existence of a stabilizing switching control law.

The reason that the conic switching law applies lies in the fact that there exists a safe region Ω (see Fig. 2) such that for every point $x_1\in l_1\subset\partial\Omega$, by following an appropriate subsystem (for example, we assume subsystem A_1 in the subsequent discussion), the trajectory will intersect another boundary at $x_2\in l_2\subset\partial\Omega$; then switch to another subsystem A_2 until it intersects l_1 again at $x_3\in l_1$. Since each subsystem is a second-order LTI system, we can see that if there exists a switching control law which stabilizes the entire switched system then the following condition is satisfied: $x_3=qx_1$ for some constant $0<q<1$. From this, we know that if such a switching control law exists, it exponentially stabilizes the entire switched system.

In this section, we first study switched systems described by

$$\dot{x}(t)=A_ix(t), \quad i=1,2, \tag{3.1}$$

where A_1 and A_2 are with foci and of opposite directions. Without loss of generality, we assume the following conic switching law: for any $x_0 \in \mathbb{R}^2$, subsystem A_2 is first activated until the trajectory intersects l_1, and then proceeds following the procedure described above.

Before going further, we introduce the following lemmas.

Lemma 3.1. *Let $\dot{x}(t) = Ax(t)$ be a LTI system with focus, where*

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

The solution with $x(t_0) = x_0 \neq 0$ has the following properties: If $\alpha < 0$ and $\beta > 0$, then the solution $x(t) = e^{A(t-t_0)}x_0$ is a logarithmic spiral that converges to the origin clockwisely; If $\alpha < 0$ and $\beta < 0$, then the solution $x(t) = e^{A(t-t_0)}x_0$ is a logarithmic spiral that converges to the origin counterclockwisely; If $\alpha > 0$ and $\beta > 0$, then the solution $x(t) = e^{A(t-t_0)}x_0$ is a logarithmic spiral that diverges to ∞ clockwisely; If $\alpha > 0$ and $\beta < 0$, then the solution $x(t) = e^{A(t-t_0)}x_0$ is a logarithmic spiral that diverges to ∞ counterclockwisely.

Proof. See [12]. It follows also from the proof of Lemma 3.2. □

Remark 3.1. For an LTI system $\dot{x}(t) = Ax(t)$ with focus, if A is of general form, then by well known results, there exists a nonsingular matrix P (with $\det P > 0$, see [12]) such that

$$P^{-1}AP = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

Using the linear transformation $x = Py$, we can always map the system trajectory properties in the Y_2–Y_1 plane into the X_2–X_1 plane. By the property of nonsingular linear transformations, it is easy to see that the solutions are also spiral-like in the X_2–X_1 plane.

Lemma 3.2. *Consider an autonomous system*

$$\dot{x}(t) = Ax(t) + g(x(t)), \tag{3.2}$$

where $g \in C$, i.e., g is continuous, and $g(0) = 0$. For initial value $x(t_0) = x_0 \neq 0$, let $T > 0$ denote the time required for the solution of system $\dot{x}(t) = Ax(t)$ to move between two rays l_1 and l_2 once (see, e.g., Fig. 2) and let $T + \Delta T > 0$ denote the time period the solution of $\dot{x}(t) = Ax(t) + g(x(t))$ takes to move between two rays l_1 and l_2 once (if possible). We have the following properties:

(i) There exists a constant $v_0 > 0$ such that when $0 < v < v_0$, then for every $\varepsilon \geqslant 0$, whenever $\|g(x)\| \leqslant v\|x\| + \varepsilon$ is satisfied, there exists a constant $K > 0$ so that when the trajectory is outside the disc $O_{K\varepsilon}$, it proceeds along a spiral-like curve similarly to the solution of $\dot{x}(t) = Ax(t)$. Furthermore, if $\varepsilon \leqslant 1$, for a trajectory outside the disc $O_{K\sqrt{\varepsilon}}$ there exist two constants $C_1, C_2 > 0$ (independent of v) such that $|\Delta T| \leqslant C_1 v + C_2\sqrt{\varepsilon}$.

(ii) If $\lim_{x \to 0} \|g(x)\|/\|x\| = 0$, then there exists a constant $r_0 > 0$ such that when $0 < r < r_0$, each solution starting inside $O_r \triangleq \{x \in \mathbb{R}^2 \colon \|x\| < r\}$ goes towards the outside of O_r (or converges to 0) along a spiral-like curve similar to the solution of $\dot{x}(t) = Ax(t)$.

Proof. Without loss of generality, we only prove the case when

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

with $\alpha > 0$, $\beta > 0$. (If A is of general form, the proof follows similarly, by Remark 3.1, and the fact that the traveling time between two rays remains the same in both the original plane and the transformed (mapped) plane.) By Lemma 3.1, we know that all the solutions of $\dot{x}(t) = Ax(t)$ diverge to infinity clockwisely along a logarithmic spiral. Changing the coordinates to polar form, we have $x_1(t) = \rho(t)\cos\theta(t)$, $x_2(t) = \rho(t)\sin\theta(t)$. Let $g(x) = (g_1(x), g_2(x))^{\mathrm{T}}$ and express it in polar coordinates form: $g_1(x) = g_1(\rho,\theta)$, and $g_2(x) = g_2(\rho,\theta)$. Substitute the expression of $x_i(t)$, $i = 1, 2$, into $\dot{x}(t) = Ax(t) + g(x(t))$, we obtain that

$$\dot{\rho}\cos\theta - \rho\sin\theta \cdot \dot{\theta} = \alpha\rho\cos\theta + \beta\rho\sin\theta + g_1(\rho,\theta),$$

$$\dot{\rho}\sin\theta + \rho\cos\theta \cdot \dot{\theta} = -\beta\rho\cos\theta + \alpha\rho\sin\theta + g_2(\rho,\theta).$$

Then

$$\dot{\rho} = \alpha\rho + \cos\theta \cdot g_1(\rho,\theta) + \sin\theta \cdot g_2(\rho,\theta),$$

$$\dot{\theta} = -\beta + \tfrac{1}{\rho}(-\sin\theta \cdot g_1(\rho,\theta) + \cos\theta \cdot g_2(\rho,\theta)).$$

Since

$$\begin{aligned} &|\cos\theta \cdot g_1(\rho,\theta) + \sin\theta \cdot g_2(\rho,\theta)| \\ &\quad\leqslant \sqrt{\cos^2\theta + \sin^2\theta} \cdot \sqrt{g_1^2(\rho,\theta) + g_2^2(\rho,\theta)} \\ &\quad= \|g(x)\| \leqslant v\|x\| + \varepsilon = v\rho + \varepsilon, \end{aligned}$$

and similarly, since

$$|-\sin\theta \cdot g_1(\rho,\theta) + \cos\theta \cdot g_2(\rho,\theta)| \leqslant v\rho + \varepsilon,$$

we have

$$(\alpha - \nu)\rho - \varepsilon \leqslant \dot{\rho} \leqslant (\alpha + \nu)\rho + \varepsilon,$$
$$-\beta - \nu - \frac{\varepsilon}{\rho} \leqslant \dot{\theta} \leqslant -\beta + \nu + \frac{\varepsilon}{\rho}. \tag{3.3}$$

Now clearly, if we let $\nu_0 = \frac{1}{2}\min\{\alpha,\beta\}$ and $K = 4/\min\{\alpha,\beta\}$, the solution of (3.3) diverges to ∞ along a spiral-like curve as the solution of $\dot{x}(t) = Ax(t)$.

Furthermore, denote the angle from l_2 to l_1 by Θ_0 ($\Theta_0 > 0$, see Fig. 2). It is clear that it takes Θ_0/β time for the trajectory to move from l_2 to l_1. Now, by the second inequality of (3.3), we have $(-\beta-\nu-\varepsilon/\rho)(T+\Delta T) \leqslant -\Theta_0 \leqslant (-\beta+\nu+\varepsilon/\rho)(T+\Delta T)$. Therefore

$$|\Delta T| \leqslant \max\left\{\left|\frac{\Theta_0}{\beta-\nu-\varepsilon/\rho} - \frac{\Theta_0}{\beta}\right|, \left|\frac{\Theta_0}{\beta+\nu+\varepsilon/\rho} - \frac{\Theta_0}{\beta}\right|\right\} = \frac{\Theta_0(\nu+\varepsilon/\rho)}{\beta(\beta-\nu-\varepsilon/\rho)}.$$

For $\nu \leqslant \nu_0$ and the above K, if $\rho \geqslant K\sqrt{\varepsilon} \geqslant K\varepsilon$ (for $\varepsilon \leqslant 1$), we have that

$$\begin{aligned}|\Delta T| &\leqslant \frac{\Theta_0(\nu + (1/K)\sqrt{\varepsilon})}{\beta(\beta/2 - (1/K)\sqrt{\varepsilon})} \\ &\leqslant \frac{\Theta_0(\nu + (\beta/4)\sqrt{\varepsilon})}{\beta \times (\beta/4)} \\ &\leqslant \frac{\Theta_0}{\beta^2}(4\nu + \beta\sqrt{\varepsilon}) \\ &\triangleq C_1\nu + C_2\sqrt{\varepsilon}.\end{aligned}$$

Therefore, the results of (i) follow. For (ii), similar arguments can be applied by using "$\varepsilon - \delta$" arguments. Due to space limitations, we omit the details. □

We also need the following lemma for the analysis on nonlinear switched systems in Section 4.

Lemma 3.3. *For systems described by $\dot{x}(t) = Ax(t) + g(x(t))$ and initial condition (t_0, x_0), if $\|g(x)\| \leqslant \nu\|x\| + \varepsilon$, then it is true that $\|x(t)\| \leqslant (\|x_0\| + \varepsilon/\|A\| + \nu)e^{(\|A\|+\nu)(t-t_0)} - \varepsilon/\|A\| + \nu$.*

Proof. Write the equation in the form $x(t) = x(t_0) + \int_{t_0}^{t}(Ax(\tau) + g(x(\tau)))\,d\tau$, and then use the Gronwall inequality (see, e.g., [6]) to establish the above inequality. Due to space limitations, we omit the details. □

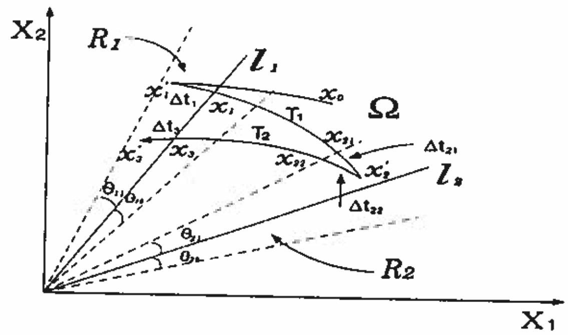

Fig. 3. Switchings occur within the conic regions R_1 and R_2.

3.1. Robustness for switchings only

Robustness Question 1: In view of the previous discussion, it is required that switchings occur exactly at times when a trajectory intersects l_1 or l_2. Can this requirement be slightly relaxed? This gives rise to the following question: are there any marginal conic regions R_1 and R_2 that include l_1 and l_2, respectively (see Fig. 3), so that any switchings that happen inside these two regions will lead to exponentially stable system?

It is clear from Fig. 3 that such marginal regions are characterized by angles $\theta_{ij} > 0$, $i,j = 1,2$, and in fact $R_i = \{x \in \mathbb{R}^2 \,|\, x = [r\cos\theta, r\sin\theta]^{\mathrm{T}},\ \theta_{li} - \theta_{i2} < \theta < \theta_{li} + \theta_{i1},\ 0 < r < \infty\}$, $i = 1,2$, where θ_{li} is the angle between l_i and X_1-axis. We need to show the existence of θ_{ij} that guarantees the robustness of the switching control law.

To answer the above questions, for solutions beginning from any initial condition (t_0, x_0), we assume that the trajectory follows subsystem A_2 for $t_1 - t_0$ time until it switches at $x_1' = e^{A_2(t_1-t_0)}x_0 \in R_1$. Then it follows subsystem A_1 for $t_2 - t_1$ time until it switches at $x_2' = e^{A_1(t_2-t_1)}x_1' \in R_2$. Next, it switches back to subsystem A_2 for $t_3 - t_2$ time until it arrives at $x_3' = e^{A_2(t_3-t_2)}x_2' \in R_1$, and so forth.

Assume that from any point $x_1 \in l_1$, it takes T_1 time to arrive at $x_2 = e^{A_1T_1}x_1 \in l_2$ while following subsystem A_1 and it takes T_2 time to return from x_2 to l_1 at $x_3 = e^{A_2T_2}x_2 = e^{A_2T_2}e^{A_1T_1}x_1 \in l_1$. Clearly, T_1 and T_2 are independent of the choice of x_1. As before, we assume that $x_3 = qx_1$ for some constant $0 < q < 1$. That is, $e^{A_2T_2}e^{A_1T_1}x_1 = qx_1$, which implies that q is an eigenvalue of matrix $e^{A_2T_2}e^{A_1T_1}$.

It is also clear that there exist quantities $\Delta t_1, \Delta t_{21}, \Delta t_{22}, \Delta t_3$, which might be negative, and points $x_{21}, x_{22} \in l_2$ and $x_1, x_3 \in l_1$ such that $x_1 = e^{A_1\Delta t_1}x_1' \in l_1$, $x_2' = e^{A_1\Delta t_{21}}x_{21}$, $x_{22} = e^{A_2\Delta t_{22}}x_2' \in l_2$, $x_3 =$

$e^{A_2(T_2-\Delta t_3)}x_{22} \in l_1$, $x_3' = e^{A_2\Delta t_3}x_3$, where $t_2 - t_1 = \Delta t_1 + T_1 + \Delta t_{21}$, $t_3 - t_2 = \Delta t_{22} + T_2 + \Delta t_3$.

Denoting $\theta \triangleq (\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22})$ (see Fig. 3), we have the following

Observation: There exists a nonnegative continuous function $c(\theta) \geqslant 0$ satisfying $\lim_{\theta \to 0} c(\theta) = c(0) = 0$ such that

$$\max\{|\Delta t_1|, |\Delta t_{21}|, |\Delta t_{22}|, |\Delta t_3|\} \leqslant c(\theta). \tag{3.4}$$

This observation follows intuitively from the idea given in the proof of Lemma 3.2.

Now due to the quasi-periodicity of the switching law (i.e., it switches back and forth for almost the same periods of time T_1 and T_2, respectively), it suffices to show that there exist switching regions R_1, R_2 (i.e., $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{21} \geqslant 0$) such that no matter when the switchings occur within regions R_1 and R_2, it is true that

$$\|x_3'\| \leqslant q_1 \|x_1'\| \quad \text{with a constant } 0 < q_1 < 1. \tag{3.5}$$

To see this, we compute

$$\begin{aligned}
x_3' &= e^{A_2(t_3-t_2)}e^{A_1(t_2-t_1)}x_1' \\
&= e^{A_2(\Delta t_{22}+T_2+\Delta t_3)}e^{A_1(\Delta t_1+T_1+\Delta t_{21})}x_1' \\
&= e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}e^{A_1(\Delta t_1+\Delta t_{21})}x_1' \\
&= e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}x_1' \\
&\quad + e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}(e^{A_1(\Delta t_1+\Delta t_{21})} - I)x_1' \\
&= e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}x_1 \\
&\quad + e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}(I - e^{A_1\Delta t_1})x_1' \\
&\quad + e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}(e^{A_1(\Delta t_1+\Delta t_{21})} - I)x_1' \\
&= qe^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_1\Delta t_1}x_1' \\
&\quad + e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}(I - e^{A_1\Delta t_1})x_1' \\
&\quad + e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}(e^{A_1(\Delta t_1+\Delta t_{21})} - I)x_1',
\end{aligned}$$

where I denotes the identity matrix. It is now not difficult to see that there exists $\varepsilon > 0$ such that when

$$\max\{|\Delta t_1|, |\Delta t_{21}|, |\Delta t_{22}|, |\Delta t_3|\} \leqslant \varepsilon,$$

we have

$$\|e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_1\Delta t_1}\| \leqslant 1 + (1-q)/8q,$$

$$\|e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}(I - e^{A_1\Delta t_1})\| \leqslant (1-q)/8,$$

$$\|e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}(e^{A_1(\Delta t_1-\Delta t_{21})} - I)\| \leqslant (1-q)/4.$$

Therefore, $\|x_3'\| \leqslant (q + (1-q)/8 + (1-q)/8 + (1-q)/4)\|x_1'\| = (1+q)/2\|x_1'\|$. If we let $q_1 = (1+q)/2$, then since $0 < q_1 < 1$, relation (3.5) holds.

Now by (3.4), we know that there exist constants $\theta_{11}^0, \theta_{12}^0, \theta_{21}^0, \theta_{22}^0 > 0$ such that whenever $0 \leqslant \theta_{ij} \leqslant \theta_{ij}^0$, $i,j = 1,2$, $c(\theta) \leqslant \varepsilon$. Therefore, (3.5) holds.

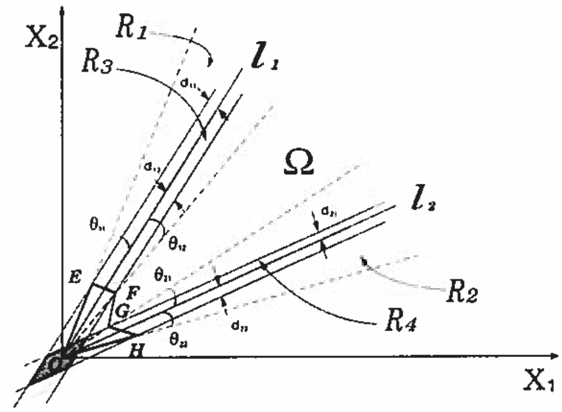

Fig. 4. Switching happens inside the strip regions R_3 and R_4.

By induction, whenever switchings occur within these conic regions, we always have that $\|x_{2k+1}'\| \leqslant (1+q)/2\|x_{2k-1}'\|$, for $k \in N \triangleq \{1, 2, \ldots\}$. Therefore, $x_{2k+1}' \to 0$ as $k \to \infty$. Since the trajectory between x_{2k-1}' and x_{2k+1}' for $k \geqslant 0$ is uniformly bounded by $\|x_{2k-1}'\|$ (denote $x_{-1}' = x_0$) and since the traveling time periods are uniformly bounded as well (for example, we can pick a bound like $\max\{T_0, 2(T_1+T_2)\}$ (where T_0 is a uniform upper bound of $t_1 - t_0$ for any initial condition (t_0, x_0)) for small θ_{ij}, $i,j = 1,2$), we conclude that $\|x(t)\| \leqslant c_0((1+q)/2)^{t-t_0}\|x_0\|$ for some constant $c_0 > 0$ (which depends on θ_{ij}, $i,j = 1,2$). Therefore the entire switched system is exponentially stable. This proves that the conic switching law is robust in the sense of Question 1. □

Robustness Question 2: If switchings happen inside the strip regions R_3 and R_4 (see Fig. 4), the best we can hope for is that the switching control law would drive the trajectory to the vicinity of the origin exponentially, but not to the origin. Since once the trajectory enters into the dark shaded region (see Fig. 4), either subsystems can be chosen, and clearly the arbitrary choice of switching may force the trajectory to go outwards.

The reason that we can force the trajectory to move to the vicinity of the origin is simple. From the answer to Question 1, we know that there exist $\theta_{ij} > 0$, $i,j = 1,2$, such that when switching happens inside conic regions R_1 and R_2, the trajectory converges to the origin exponentially. Now pick $d_{ij} > 0$, $i,j = 1,2$, sufficiently small (the choice depends on how close to the origin we require). From Fig. 4, we know that there exist intersecting points E, F, G, H. Clearly, the trajectory will finally move into the polygonal region $\overline{OEFGH}$. □

Once the trajectory enters the polygon $\overline{OEFGH}$, it may leave this region if we still employ the strip switching control law. For this reason, the robustness property of the first case is of much greater interest to us. In the following, we will mainly discuss the robust problem of the first kind.

3.2. *Robustness for perturbations only*

In this subsection, we will study the stability properties of the switched systems in the presence of perturbations, including both vanishing and nonvanishing perturbations.

Theorem 3.1. *For the switched system described by* (3.1) *where A_1, A_2 are with unstable foci and of opposite directions, suppose that there exists the aforementioned conic switching law that makes* (3.1) *exponentially stable. Then for the perturbed switched system described by*

$$\dot{x}(t) = A_i x(t) + g_i(x(t)), \quad i = 1,2, \tag{3.6}$$

and the switching law given below:

Switching Law (event-driven): *For any $x_0 \in \mathbb{R}^2$, follow A_2 until the trajectory intersects l_1 on x_1 at t_1, then alternatively switch on subsystems* 1 *and* 2 *when the trajectory crosses l_1 and l_2, respectively, where l_1 and l_2 are determined by the linear part of subsystems.*

We have the following conclusion:

(a) There exists a constant $v_0 > 0$ such that whenever $0 < v < v_0$, if $\|g_i(x)\| \leqslant v\|x\|$ (vanishing perturbations) is satisfied for $i = 1,2$, then the switching law stabilizes exponentially the entire switching system (3.6) with the following robust property: there exist two conic regions in which switchings are allowed as discussed in Section 3.1.

(b) There exists a constant $0 < \varepsilon_0 \leqslant 1$, such that whenever $\|g_i(x)\| \leqslant \varepsilon \leqslant \varepsilon_0$ (nonvanishing perturbations), the switching law will exponentially drive the trajectory to an open disc of radius $K_1\varepsilon$ for some constant $K_1 > 0$, i.e., $O_{K_1\varepsilon}$.

For $g_i(x)$, as in (a) and (b), the switching law is robust in the sense of Robustness Question 1 discussed in Section 3.1.

Proof of Theorem 3.1. We first prove (a). From Lemma 3.2(i), there exist small constants v_0 whenever $v \leqslant v_0$, such that the solution starting from (t_0, x_0) will intersect l_1 at $x_1 = x(t_1)$ by following subsystem 2. If the system switches to subsystem 1, the solution will intersect l_2 at $x_2 = x(t_2)$. If the system then switches back to subsystem 2, the solution will intersect l_1 at $x_3 = x(t_3)$. Continuing in this manner, we have $x_{2k-1} = x(t_{2k-1})$ on l_1 and $x_{2k} = x(t_{2k})$ on l_2.

For exponential stability, we first show that there exists $v_1 > 0$, such that whenever $0 < v < v_1$, then

$$\|x_3\| \leqslant q_1 \|x_1\| \quad \text{for a constant } 0 < q_1 < 1. \tag{3.7}$$

Let T_1 (resp., T_2) be the time period for the trajectory to move from l_1 to l_2 (resp., from l_2 to l_1) by following subsystem A_1 (resp., A_2). Since $t_2 - t_1 = T_1 + \Delta t_1$, then $t_3 - t_2 = T_2 + \Delta t_2$, and by Lemma 3.2(i) (with $\varepsilon = 0$), there exist constants $v_0 > 0$ and $C > 0$ (independent of the choice of v), such that whenever $0 \leqslant v \leqslant v_0$, it is true that $\Delta t_i \leqslant Cv$, $i = 1,2$. Write

$$x(t_2) = e^{A_1(t_2-t_1)}x(t_1) + \int_{t_1}^{t_2} e^{A_1(t_2-\tau)} g_1(x(\tau))\,d\tau,$$

$$\begin{aligned}
x(t_3) &= e^{A_2(t_3-t_2)}x(t_2) + \int_{t_2}^{t_3} e^{A_2(t_3-\tau)} g_2(x(\tau))\,d\tau \\
&= e^{A_2(t_3-t_2)}e^{A_1(t_2-t_1)}x(t_1) \\
&\quad + e^{A_2(t_3-t_2)} \int_{t_1}^{t_2} e^{A_1(t_2-\tau)} g_1(x(\tau))\,d\tau \\
&\quad + \int_{t_2}^{t_3} e^{A_2(t_3-\tau)} g_2(x(\tau))\,d\tau \\
&= q e^{A_2\Delta t_2}x(t_1) + e^{A_2\Delta t_2}e^{A_2T_2}e^{A_1T_1}(e^{A_1\Delta t_1} - I)x(t_1) \\
&\quad + e^{A_2(t_3-t_2)} \int_{t_1}^{t_2} e^{A_1(t_2-\tau)} g_1(x(\tau))\,d\tau \\
&\quad + \int_{t_2}^{t_3} e^{A_2(t_3-\tau)} g_2(x(\tau))\,d\tau,
\end{aligned} \tag{3.8}$$

since $e^{A_2T_2}e^{A_1T_1}x(t_1) = qx(t_1)$.

There exists $v_1 \leqslant v_0$ (which will be further specified later), such that whenever $0 < v \leqslant v_1$, it is true that

$$\max\{|\Delta t_1|, |\Delta t_2|\} \leqslant \max\{T_1, T_2\},$$

$$\|e^{A_2\Delta t_2}\| \leqslant 1 + \frac{1-q}{4q},$$

$$\|e^{A_2\Delta t_2}e^{A_2T_2}e^{A_1T_1}(e^{A_1\Delta t_1} - I)\| \leqslant \frac{1-q}{4}.$$

Let $T \triangleq 2\max\{T_1, T_2\}$ and $\lambda \triangleq \max\{\|A_1\|, \|A_2\|\}$. We know from Lemma 3.3 that for $t \in [t_1, t_2]$, $\|x(t)\| \leqslant e^{(\lambda+v)T}\|x(t_1)\|$, while for $t \in [t_2, t_3]$, $\|x(t)\| \leqslant e^{(\lambda+v)T}\|x(t_2)\| \leqslant e^{2(\lambda+v)T}\|x(t_1)\|$. Therefore, by (3.8), we have

$$\|x(t_3)\| \leqslant \frac{1+q}{2}\|x(t_1)\| + v\mathrm{e}^{\lambda T}\cdot T\mathrm{e}^{\lambda T}\mathrm{e}^{(v+\lambda)T}\|x(t_1)\| + vT\mathrm{e}^{\lambda T}\cdot \mathrm{e}^{2(v+\lambda)T}\|x(t_1)\|$$
$$= \left(\frac{1+q}{2} + vT\mathrm{e}^{\lambda T}\mathrm{e}^{(v+\lambda)T}(\mathrm{e}^{\lambda T} + \mathrm{e}^{(v+\lambda)T})\right)\|x(t_1)\|.$$

Pick v_1 sufficiently small such that whenever $0 < v < v_1$, it is true that $(1+q)/2 + vT\mathrm{e}^{\lambda T}\mathrm{e}^{(v+\lambda)T}(\mathrm{e}^{\lambda T} + \mathrm{e}^{(v+\lambda)T}) \leqslant 3(1+q)/4$. Then for $0 < v < v_1$, we have $\|x_3\| \leqslant q_1\|x_1\|$, with $0 < q_1 = 3(1+q)/4 < 1$.

By induction, we can show that $\|x_{2k+1}\| \leqslant q_1\|x_{2k-1}\|$ for $k \geqslant 1$. Therefore $x(t_k) \to 0$ as $k \to \infty$. Since there exists a constant $c > 0$ such that for $t \in [t_{2k-1}, t_{2k+1}]$ $(t_{-1} \triangleq t_0)$, it is always true that $\|x(t)\| \leqslant c\|x(t_{2k-1})\|$, and therefore, (a) is proved.

Now for case (b), by Lemma 3.2(i) (with $v=0$), we know that for any $\varepsilon > 0$, there exists a constant K such that outside $O_{K\varepsilon}$ the solutions of subsystems 1,2 behave like their corresponding linear subsystems. Suppose that for $t \in [t_0, t_3]$, the trajectory of the switched system does not go inside $O_{K\varepsilon}$. Then for

$t \in [t_0, t_1]$ (let $t_1 = t_0 + T_0(x_0, g_2)$),

$$\|x(t)\| \leqslant (\|x(t_0)\| + \varepsilon/\|A_2\|)\mathrm{e}^{\|A_2\|T_0(x_0,g_2)} - \varepsilon/\|A_2\|.$$

From (3.8) we have

$$x(t_3) = q\mathrm{e}^{A_2\Delta t_2}x(t_1) + \mathrm{e}^{A_2\Delta t_2}\mathrm{e}^{A_2T_2}\mathrm{e}^{A_1T_1}(\mathrm{e}^{A_1\Delta t_1} - I)x(t_1) + \mathrm{e}^{A_2(t_3-t_2)}\int_{t_1}^{t_2}\mathrm{e}^{A_1(t_2-\tau)}g_1(x(\tau))\,\mathrm{d}\tau + \int_{t_2}^{t_3}\mathrm{e}^{A_2(t_3-\tau)}g_2(x(\tau))\,\mathrm{d}\tau.$$

By Lemma 3.2(i), we know that there exists a sufficiently small constant ε_0 and a constant T such that whenever $0 < \varepsilon < \varepsilon_0$, it is true that

$$\max\{T_1, \Delta t_1, T_2, \Delta t_2, T_0(x_0, g_2)\} \leqslant T,$$
$$\|\mathrm{e}^{A_2\Delta t_2}\| \leqslant 1 + (1-q)/4q,$$
$$\|\mathrm{e}^{A_2\Delta t_2}\mathrm{e}^{A_2T_2}\mathrm{e}^{A_1T_1}(\mathrm{e}^{A_1\Delta t_1} - I)\| \leqslant (1-q)/4.$$

Therefore, we have that

$$\|x(t_3)\| \leqslant q(1 + (1-q)/4q)\|x(t_1)\| + (1-q)/4\|x(t_1)\| + 2T\mathrm{e}^{2\lambda T}\mathrm{e}^{2\lambda T}\varepsilon + 2T\mathrm{e}^{2\lambda T}\varepsilon \triangleq (1+q)/2\|x(t_1)\| + C_3\varepsilon.$$

Since for $t \in [t_0, t_1]$,

$$\|x(t)\| \leqslant (\|x(t_0)\| + \varepsilon/\|A_2\|)\mathrm{e}^{\|A_2\|T_0(x_0,g_2)} - \varepsilon/\|A_2\| \triangleq \mathrm{e}^{\lambda T}\|x(t_0)\| + C_4\varepsilon,$$

by induction we have that

$$\|x(t_{2k+1})\| \leqslant ((1+q)/2)^k(\|x(t_1)\| - 2C_3\varepsilon/(1-q)) + 2C_3\varepsilon/(1-q)$$
$$\leqslant ((1+q)/2)^k\|x(t_1)\| + 2C_3\varepsilon/(1-q)$$
$$\leqslant ((1+q)/2)^k\mathrm{e}^{\lambda T}\|x_0\| + (C_4 + 2C_3/(1-q))\varepsilon.$$

For $t \in [t_{2k-1}, t_{2k+1}]$, $\|x(t)\| \leqslant C_5\|x(t_{2k-1})\| + C_6\varepsilon$ always holds for some constants C_5, $C_6 > 0$. We conclude that the trajectory will finally enter into a disc $O_{K_1\varepsilon}$ of radius $K_1\varepsilon$ for some constant $K_1 > 0$.

The argument for robustness follows a similar approach as was done in answering Robustness Question 1 in Section 3.1. Due to space limitations, we omit the details. □

Remark 3.2. For more general perturbations satisfying $\|g_i(x)\| \leqslant v\|x\| + \varepsilon$, we can establish similar results for the switching law as was stated in Theorem 3.1.

Remark 3.3. Another switching law which is event-driven plus time-driven might also be worth mentioning here. This law is stated as follows: for any $x_0 \in \mathbb{R}^2$, follow A_2 until the trajectory intersects l_1, and then follow alternatively A_1 and A_2 for time periods T_1 and T_2, respectively (T_1 and T_2 are known a priori from the precise conic switching law). Unfortunately, this switching law does not stabilize the entire switched system because of the occurrence of accumulation of errors in switchings. Example 5.3 in Section 5 demonstrates this phenomenon, which also implies that a time-driven control law may eventually cause trouble to the entire switched system due to the accumulation of switching inaccuracy.

4. Stabilizing switching control law for nonlinear switched systems

In this section, we study the stabilization problem of nonlinear switched systems. To accomplish this, we will use linearization. The problem of interest is that if each subsystem is locally exponentially unstable, then is it still possible to determine switching laws to stabilize the entire switched system? If affirmative, are these laws robust in the sense discussed in Section 3?

In the present section, we study only local exponential stability.

As before, we will study only the following sample problem. For the remaining cases, similar approaches may be pursued. Consider the second-order nonlinear switched system described by

$$\dot{x}(t) = f_i(x(t)) = A_i x(t) + g_i(x(t)), \quad i = 1, 2, \quad (4.1)$$

where $f_i \in C^1$, i.e., f_i is continuously differentiable, $f_i(0) = 0$ and A_i is the Jacobian of f_i at the origin, i.e., $[\partial f_i(x)/\partial x]_{x=0}$. Clearly, $g_i \in C^1$, $g_i(0) = 0$ and $\lim_{x\to 0} \|g_i(x)\|/\|x\| = 0$.

Lemma 4.1. *For the system described by $\dot{x}(t) = Ax(t) + g(x(t))$, where $g \in C^1$ and $\lim_{x\to 0}\|g(x)\|/\|x\| = 0$, for every $\varepsilon > 0$ and any given constant $T > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any initial condition (t_0, x_0), whenever $\|x_0\| \leqslant \tilde{\delta} e^{-(\|A\|+\varepsilon)T}$ holds for some constant $\tilde{\delta} \leqslant \delta$, it is true for $t \in [t_0, t_0 + T]$ that*

$$\|x(t)\| \leqslant e^{(\|A\|+\varepsilon)T}\|x_0\| \leqslant \tilde{\delta}. \quad (4.2)$$

Proof. Since $\lim_{x\to 0}\|g(x)\|/\|x\| = 0$ and $g \in C^1$, then for $\varepsilon > 0$, there exists $\delta > 0$ such that $\|g(x)\| \leqslant \varepsilon\|x\|$ whenever $\|x\| \leqslant \delta$. Suppose that (4.2) is not true. Then by continuity, there exists a $t_1 \in [t_0, t_0 + T)$ such that $\|x(t)\| < e^{(\|A\|+\varepsilon)T}\|x_0\|$ for $t \in [t_0, t_1)$ and $\|x(t_1)\| = e^{(\|A\|+\varepsilon)T}\|x_0\|$. Hence, $\|g(x(t))\| \leqslant \varepsilon\|x(t)\|$ for $t \in [t_0, t_1]$. Also, we have for $t \in [t_0, t_1]$ that $x(t) = x_0 + \int_{t_0}^{t}(Ax(\tau) + g(x(\tau)))\,d\tau$, so that $\|x(t)\| \leqslant \|x_0\| + \int_{t_0}^{t}(\|A\| + \varepsilon)\|x(\tau)\|\,d\tau$. By the Gronwall inequality we obtain that $\|x(t)\| \leqslant e^{(\|A\|+\varepsilon)(t-t_0)}\|x_0\| < e^{(\|A\|+\varepsilon)T}\|x_0\|$, which implies that $\|x(t_1)\| < e^{(\|A\|+\varepsilon)T}\|x_0\|$, which is a contradiction. We conclude that Lemma 4.1 is true. □

Theorem 4.1. *For the switched system described by (4.1), where A_1 and A_2 have unstable foci with opposite directions, suppose that there exists a conic switching law that renders the linearized system $\dot{x}(t) = A_i x(t)$, $i = 1, 2$ exponentially stable. Then the switching law proposed in Theorem 3.1 will locally exponentially stabilize the nonlinear switched system (4.1). Furthermore, the robustness properties in the sense of Theorem 3.1 and Remark 3.2 are preserved.*

Proof. Due to space limitations, we only present a sketch of the proof of exponential stability of the switching law in case (a). Other results can be established similarly. The main idea follows the technique developed in [3] and the treatment indicated in Section 3.

First, by Lemma 3.2(ii) we know that there exists a disc O_{δ_1} (with radius $\delta_1 > 0$) inside which the solutions rotate outwards like the solutions of $\dot{x}(t) = A_2 x(t)$. From the proof of Lemma 3.1(ii), it is not hard to see that there exists another smaller disc $O_{\delta_2} \subset O_{\delta_1}$ ($\delta_2 \leqslant \delta_1$) and a constant $T_0 > 0$ such that the solutions of (4.1) with initial condition (t_0, x_0), $x_0 \in O_{\delta_2}$, intersect l_1 within finite time less than T_0 by following subsystem 2, and such that the entire trajectory stays inside O_{δ_1} after switching A_1 on (from l_1 to l_2) and A_2 on (from l_2 to l_1), consecutively and only once. Let T_1 and T_2 denote the activating time for the linearized switched system $\dot{x}(t) = A_i x(t)$, as before. As in the proof of Theorem 3.1, we have

$$x(t_1) = e^{A_2(t_1-t_0)}x(t_0) + \int_{t_0}^{t_1} e^{A_2(t_1-\tau)}g_2(x(\tau))\,d\tau \in l_1, \quad (4.3)$$

$$x(t_2) = e^{A_1(t_2-t_1)}x(t_1) + \int_{t_1}^{t_2} e^{A_1(t_2-\tau)}g_1(x(\tau))\,d\tau \in l_2, \quad (4.4)$$

$$\begin{aligned} x(t_3) &= e^{A_2(t_3-t_2)}x(t_2) + \int_{t_2}^{t_3} e^{A_2(t_3-\tau)}g_2(x(\tau))\,d\tau \in l_1 \\ &= qe^{A_2\Delta t_2}x(t_1) + e^{A_2\Delta t_2}e^{A_2T_2}e^{A_1T_1}(e^{A_1\Delta t_1} - I)x(t_1) \\ &\quad + e^{A_2(t_3-t_2)}\int_{t_1}^{t_2} e^{A_1(t_2-\tau)}g_1(x(\tau))\,d\tau \\ &\quad + \int_{t_2}^{t_3} e^{A_2(t_3-\tau)}g_2(x(\tau))\,d\tau, \end{aligned} \quad (4.5)$$

where $t_2 = t_1 + T_1 + \Delta t_1$, $t_3 = t_2 + T_2 + \Delta t_2$ and $e^{A_2T_2}e^{A_1T_1}x(t_1) = qx(t_1)$ with constant $0 < q < 1$. Let $\lambda = \max\{\|A_1\|, \|A_2\|\}$ and $T \triangleq 2\max\{T_0, T_1, T_2\}$.

By Lemma 3.2(i), there exists a constant $v_0 > 0$, such that whenever $0 \leqslant v < v_0$, if $\|g_i(x)\| \leqslant v\|x\|$, we have $\|\Delta t_i\| \leqslant Cv$. Now pick $v \leqslant v_0$ sufficiently small so that

$$\max\{|\Delta t_1|, |\Delta t_2|\} \leqslant \max\{T_0, T_1, T_2\},$$

$$\|e^{A_2\Delta t_2}\| \leqslant 1 + \frac{1-q}{4q},$$

$$\|e^{A_2\Delta t_2}e^{A_2T_2}e^{A_1T_1}(e^{A_1\Delta t_1} - I)\| \leqslant \frac{1-q}{4}$$

and $s = (1+q)/2 + vTe^{(3\lambda+v)T}(1 + e^{vT}) < 1$. For the above v, there exists a $\delta > 0$ such that $\|g_i(x)\| \leqslant v\|x\|$ whenever $\|x\| \leqslant \delta$. Pick $\delta_3 = \min\{\delta, \delta_2\}e^{-3(\lambda+v_0)T}$. Then for every $x_0 \in O_{\delta_3}$, we have by Lemma 4.1

that

$$\|x(t)\| \leqslant e^{(\lambda-v)T}\|x(t_i)\| \\ \leqslant \min\{\delta,\delta_1\} \quad \text{for } t_i \leqslant t < t_{i+1},\ i=0,1,2.$$

Therefore,

$$\|x(t_3)\| \leqslant (1+q)/2\|x(t_1)\| \\ +Tve^{(3\lambda+v)}\|x(t_1)\| + Tve^{\lambda T}\|x(t_2)\| \\ \leqslant ((1+q)/2 + vTe^{(3\lambda-v)T}(1+e^{vT}))\|x(t_1)\| \\ = s\|x(t_1)\|.$$

By induction, since $0 < s < 1$, it is readily shown that $\|x(t)\| \leqslant \min\{\delta,\delta_1\}$ holds for all $t \geqslant t_0$ and the solution approaches zero exponentially.

We have proved that the proposed switching law locally exponentially stabilizes the nonlinear switched system (4.1). For the robustness analysis, we can modify the arguments used in answering the Robustness Questions 1 and 2. □

5. Numerical examples and simulations

Example 5.1. Consider the switched system consisting of two unstable subsystems with foci and of opposite directions, given by

$$\dot{x}(t) = A_i x(t),$$

where

$$A_1 = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 5 & -8 \\ 4 & -3 \end{bmatrix}. \tag{5.1}$$

After some calculations, we can determine that the interior of Ω_5 is nonempty and Ω_5 is bounded by the two lines $l_1 : x_2 = 0.9087x_1$ with angle 42.261° and $l_2 : x_2 = -20.9087x_1$ with angle 92.738°. The trajectory of the system (starting from $x_0 = [2.0, 3.0]^T$) under the conic switching law is shown in Fig. 5(a).

We determine that $\|x_3\| = q\|x_1\|$, $q = 0.786$, $T_1 = 0.293$, and $T_2 = 0.2269$. Now if we let $\max\{|\Delta t_1|, |\Delta t_{21}|, |\Delta t_{22}|, |\Delta t_3|\} \leqslant 0.0022$, we find that

$$\|e^{A_2(\Delta t_{22}+\Delta t_3)}\| \leqslant 1.0244 \leqslant 1.0340 = 1 + (1-q)/8q,$$

$$\|e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}(I - e^{A_1\Delta t_1})\| \\ \leqslant 0.0263 \leqslant 0.0267 \leqslant (1-q)/8,$$

$$\|e^{A_2(\Delta t_{22}+\Delta t_3)}e^{A_2T_2}e^{A_1T_1}(e^{A_1(\Delta t_1+\Delta t_{21})} - I)\| \\ \leqslant 0.0526 \leqslant 0.0535 = (1-q)/4.$$

Corresponding to the above result, we find that if we choose conic regions R_1 and R_2 with $\theta_{11} = \theta_{12} = \theta_{21} = \theta_{22} = 0.2269^\circ$, then according to the discussion in Section 3.1, the switched system is robust with respect to variations in switchings. The trajectory of the system (starting from $x_0 = [2.0, 3.0]^T$) in this case is shown in Fig. 5(b).

Since the estimates in Section 3.1 are very conservative, we may try larger disturbances than the one given above. If we choose conic regions R_1 and R_2 with $\theta_{11} = \theta_{12} = \theta_{21} = \theta_{22} = 3.0^\circ$, then we will find that the system is still exponentially stable under the conic switching law. The trajectory of the system (starting from $x_0 = [2.0, 3.0]^T$) is shown in Fig. 5(c).

Example 5.2. Consider the nonlinear switched system whose linearizations are the switched systems in Example 5.1. Subsystems 1, 2 are described, respectively, by

$$\dot{x}_1 = x_1 + 3x_2 + x_1(x_1^2 + x_2^2), \\ \dot{x}_2 = -3x_1 + x_2 + x_2(x_1^2 + x_2^2)$$

and

$$\dot{x}_1 = 5x_1 - 8x_2 + x_2^2, \\ \dot{x}_2 = 4x_1 - 3x_2 + x_1x_2.$$

In the above two subsystems, the nonlinear terms can be viewed as vanishing perturbations to their corresponding linearized systems. Using the switching law proposed in Section 3.2, we find that the switched system is locally exponentially stable (Fig. 6(a) shows the trajectory starting from $x_0 = [0.05, 0.08]^T$).

Example 5.3. To show that the switching law stated in Remark 3.3 in Section 3 may not exponentially stabilize a switched system, we consider the same nonlinear switched system as in Example 5.2. Fig. 6(b) shows the trajectory starting from $x_0 = [0.05, 0.08]^T$. We find that the switched system is not locally exponentially stable.

Example 5.4. Consider the nonlinear switched system whose subsystems are subsystems in Example 5.1 with nonvanishing perturbations

$$\dot{x}_1 = x_1 + 3x_2 + 0.0071, \\ \dot{x}_2 = -3x_1 + x_2 + 0.0071$$

and

$$\dot{x}_1 = 5x_1 - 8x_2 + 0.0071, \\ \dot{x}_2 = 4x_1 - 3x_2 + 0.0071.$$

Here $\|g_i(x)\| = \varepsilon = 0.01$. Using the switching law proposed in Section 3.2, we can determine that the system trajectory can be driven exponentially into the open disc of radius 0.0198 (Fig. 6(c) shows the trajectory starting from $x_0 = [0.05, 0.08]^T$).

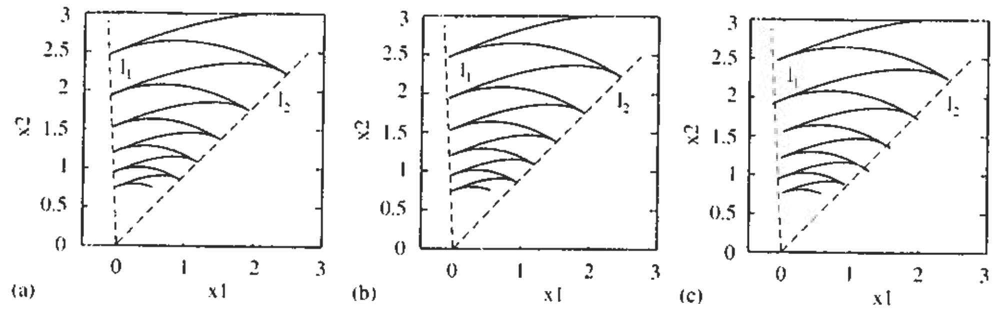

Fig. 5. Trajectories for Example 5.1.

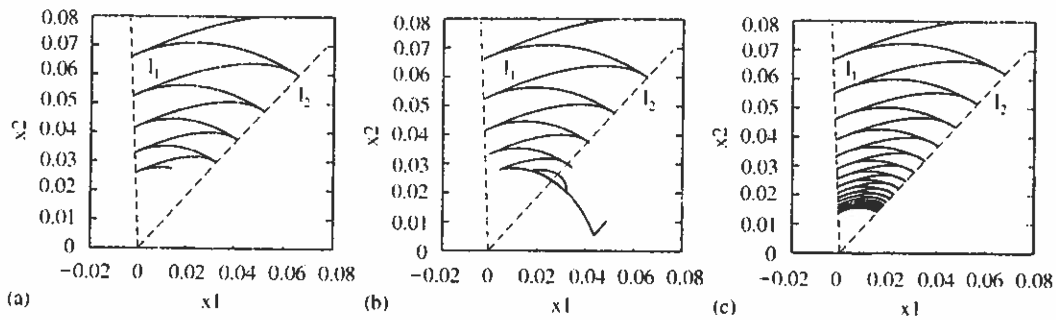

Fig. 6. Trajectories for Examples 5.2–5.4.

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