Proceedings of the 38th Conference on Decision & Control
Phoenix, Arizona USA • December 1999

STABILITY ANALYSIS FOR A CLASS OF NONLINEAR SWITCHED SYSTEMS

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Abstract

In the present paper, we study several qualitative properties of a class of nonlinear switched systems under certain switching laws. First, we show that if all the subsystems are linear time-invariant and the system matrices are commutative componentwise and stable, then the entire switched system is globally exponentially stable under arbitrary switching laws. Next, we study the above linear switched systems with certain nonlinear perturbations, which can be either vanishing or non-vanishing. Under reasonable assumptions, global exponential stability is established for these systems. We further study the stability and instability properties, under certain switching laws, for switched systems with commutative subsystem matrices that may be unstable. Results for both continuous-time and discrete-time cases are presented.

I. Introduction

Switched systems are hybrid systems that consist of severals subsystems and are controlled by switching law. These switching laws may be either supervised or unsupervised, and may be time-driven or event-driven. Recently, there has been increasing interest in the stability analysis of such systems ([1]-[8]).

The methodologies used in studying switched systems are very diverse. For example, multiple Lyapunov functions were employed to establish certain general Lyapunov-like results for both linear time-invariant switched systems [1] and nonlinear switched systems [2], and Linear Matrix Inequality (LMI) techniques were formulated to study stability and robust stability problems [5]. In [6], a class of general nonlinear switched systems were treated as a special case of sampled-data control systems with multiple sampling periods and some stability criteria were obtained. A conic switching law was proposed in [7], [8] to study second-order linear time-invariant switched systems, which was shown to be very efficient. However, this conic switching law does not seem to be applicable to higher dimensional problems. In [4], recent developments in the study of these issues is summarized in detail.

In the present paper, we propose an approach to study the stability properties of a specific class of nonlinear switched systems which differs significantly from the existing works. The switched systems under investigation in the present paper consist of dominant linear parts, for which the matrices are commutative, and certain nonlinear perturbation terms. We demonstrate that under reasonable assumptions, the qualitative properties of the above linear time-invariant switched systems are preserved if the perturbation terms are sufficiently "small".

The paper is organized as follows. In Section 2, the stability analysis for continuous-time switched systems is conducted, while in Section 3, similar results are established for discrete-time switched systems.

II. Continuous-Time Switched System

We will consider three general cases for the class of switched systems considered.

Case 1: All the dominant subsystems are linear time-invariant and Hurwitz stable

Consider linear switched systems described by equations of the form

$$\dot{x}(t) = A_i x(t), \qquad i = 1, 2, \cdots, m, \tag{2.1}$$

where $m \geq 2$ is an integer and $x(t) \in R^{m_1}$, $A_i \in R^{m_1 \times m_1}$.

In the following, we will always assume that the solutions of (2.1) are right differentiable. We will use the notation $\{i_k\}_{k\geq 0} \subset \{1, 2, \cdots, m\}$ to denote the switching sequence and we let $[t_k, t_{k+1})$ denote the time period over which the i_k-th subsystem is activated. Assume that

(A1) $\lim_{k\to\infty} t_k = \infty$;

(A2) the A_i's are all Hurwitz stable, i.e., for each i, all the eigenvalues of matrix A_i lie in the left-half complex plane;

(A3) $A_i A_j = A_j A_i$ for all $i \neq j$.

We have the following result.

Theorem 2.1. Assume that hypotheses (A2)-(A3) are true. Then the equilibrium $x_e = 0$ of switched sys-

[1]Supported in part by a Center of Applied Mathematics Fellowship, University of Notre Dame.

[2]Supported in part by a Center of Applied Mathematics Fellowship, University of Notre Dame.

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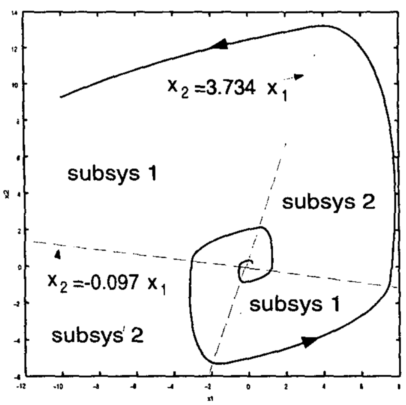

Figure 1: Phase plot of the switched system in Example 2.1 with initial value $x_1(0) = x_2(0) = 0.2$.

tem (2.1) is *globally exponentially stable* under arbitrary switching laws satisfying (A1).

Proof : Since the A_i's are Hurwitz stable, it is well-known that there exist constants $k_i, \alpha > 0$, such that for each $i = 1, 2, \cdots, m$, the following inequality holds:

$$\|e^{A_i t}\| \le k_i e^{-\alpha t}, \qquad i = 1, 2, \cdots, m. \qquad (2.2)$$

Now, for any initial condition (t_0, x_0) (Without loss of generality, we assume in the sequel that $t_0 \ge 0$) and $t \in [t_k, t_{k+1})$, we have

$$x(t) = e^{A_{i_k}(t-t_k)} e^{A_{i_{k-1}}(t_k - t_{k-1})} \cdots e^{A_{i_1}(t_2-t_1)} e^{A_{i_0}(t_1-t_0)} x_0.$$

If $T_i(t_0, t)$, $i = 1, 2, \cdots, m$, denotes the total time that the i-th subsystem is activated during the time t_0 to t, then we have that $\sum_{i=1}^m T_i(t_0, t) = t - t_0$. Since the A_i's are commutative pairwise, we rewrite the above expression as

$$x(t) = e^{A_1 T_1(t_0,t)} \cdots e^{A_m T_m(t_0,t)} x_0. \qquad (2.3)$$

By (2.2), we have
$\|x(t)\| \le (\prod_{i=1}^m k_i) e^{-\alpha(T_1(t_0,t)+\cdots+T_m(t_0,t))}\|x_0\|$
$= (\prod_{i=1}^m k_i) e^{-\alpha(t-t_0)}\|x_0\|$, which implies that switched system (2.1) is globally exponentially stable under arbitrary switching laws. □

It is natural to ask whether the above result remains true if the subsystem matrices of (2.1) do not commute pairwise. The answer is in general not true. Counterexamples can be found in [3] and [7]. Using the reversed **conic switching law** as proposed in [7], we know that even when every subsystem has stable foci, the entire system may still not be asymptotically stable under arbitrary switching laws. The following example shows that there exist certain switching laws which make a switched system unstable even though the subsystems have stable foci.

Example 2.1. Consider the switched system (2.1) with $m = 2$, and

$$A_1 = \begin{bmatrix} -1 & -13 \\ 2 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -2 \\ 10 & -3 \end{bmatrix}.$$

For initial point $(x_1(0), x_2(0))^T = (0.2, 0.2)^T$, we incorporate the following inverse conic switching law: whenever the trajectory is inside the region of subsystem i (which is partitioned by two straight lines: $x_2 = 0.0097x_1$ and $x_2 = 3.734x_1$, as depicted in Figure 1), subsystem A_i is employed until the trajectory intersects the above boundary lines. Then another subsystem will be activated. From the phase plot, we know that the two subsystems with stable foci are made unstable by the switching law specified above. □

We may now ask: what happens if all the subsystems are "almost commutative"? In the following, we study the qualitative properties of system (2.1) under certain perturbations, which may either be vanishing or non-vanishing. Thus, we consider switched nonlinear systems described by equations of the form

$$\dot{x}(t) = A_i x(t) + f_i(t, x(t)), \quad i = 1, 2, \cdots, m, \qquad (2.4)$$

where the perturbation term f_i are either vanishing in the sense that

$$\|f_i(t, x(t))\| \le \gamma\|x(t)\|, \quad i = 1, 2, \cdots, m, \qquad (2.5)$$

or are non-vanishing in the sense that

$$\|f_i(t, x(t))\| \le \gamma\|x(t)\| + \beta(t), \quad i = 1, 2, \cdots, m, \quad (2.6)$$

where γ is a constant and $\beta(\cdot)$ is a nonnegative Lebesgue integrable function such that $\int_0^\infty e^{\alpha\tau}\beta(\tau)d\tau < \infty$, where α is the constant given in (2.2) or will be specified later.

Theorem 2.2. Assume that there exist constants k_i, α, $\gamma > 0$ and a nonnegative Lebesgue integrable function $\beta(\cdot)$ such that conditions (2.2) and (2.6) hold. If hypotheses (A2)-(A3) are true, then for switched system (2.4) with initial condition (t_0, x_0), under arbitrary switching law satisfying (A1), it is true that

$$\|x(t)\| \le K_0\left(\int_0^\infty e^{\alpha\tau}\beta(\tau)d\tau + \|x_0\|\right)e^{-(\alpha - K_0\gamma)(t-t_0)} \quad (2.7)$$

where $K_0 = \prod_{i=1}^m k_i$. Therefore, under the condition that $\gamma < \frac{\alpha}{K_0}$, if $\beta(t) \equiv 0$, then $x_e = 0$ is an equilibrium which is globally exponentially stable. Otherwise, the entire system is uniformly bounded and and the solutions converge to the origin exponentially.

Proof : For $t \in [t_k, t_{k+1})$, we have

$$x(t_1) = e^{A_{i_0}(t_1-t_0)}x_0 + \int_{t_0}^{t_1} e^{A_{i_0}(t_1-\tau)} f_{i_0}(\tau, x(\tau))d\tau,$$

$$x(t_2) = e^{A_{i_1}(t_2-t_1)}x(t_1) + \int_{t_1}^{t_2} e^{A_{i_1}(t_2-\tau)} f_{i_1}(\tau, x(\tau))d\tau$$
$$= e^{A_{i_1}(t_2-t_1)+A_{i_0}(t_1-t_0)}x_0$$

$$+\int_{t_0}^{t_1} e^{A_{i_1}(t_2-t_1)+A_{i_0}(t_1-\tau)} f_{i_0}(\tau,x(\tau))d\tau$$
$$+\int_{t_1}^{t_2} e^{A_{i_1}(t_2-\tau)} f_{i_1}(\tau,x(\tau))d\tau,$$

and by induction, we obtain that

$$\begin{aligned}
x(t) &= e^{A_{i_k}(t-t_k)+A_{i_{k-1}}(t_k-t_{k-1})+\cdots+A_{i_1}(t_1-t_0)}x(t_0)\\
&+\int_{t_0}^{t_1} e^{A_{i_k}(t-t_k)+A_{i_{k-1}}(t_k-t_{k-1})+\cdots+A_{i_0}(t_1-\tau)} f_{i_0}(\tau,x(\tau))d\tau\\
&+\int_{t_1}^{t_2} e^{A_{i_k}(t-t_k)+A_{i_{k-1}}(t_k-t_{k-1})+\cdots+A_{i_1}(t_2-\tau)} f_{i_1}(\tau,x(\tau))d\tau\\
&+\cdots+\int_{t_{k-1}}^{t_k} e^{A_{i_k}(t-t_k)+A_{i_{k-1}}(t_k-\tau)} f_{i_{k-1}}(\tau,x(\tau))d\tau\\
&+\int_{t_k}^{t} e^{A_{i_k}(t-\tau)} f_{i_k}(\tau,x(\tau))d\tau. \qquad (2.8)
\end{aligned}$$

Grouping the elapsed time intervals for each subsystem as in the proof of Theorem 2.1, we have, since $k_i \geq 1$,

$$\begin{aligned}
\|x(t)\| &\leq K_0[e^{-\alpha(t-t_0)}\|x_0\| + \int_{t_0}^{t_1} e^{-\alpha(t-\tau)}(\gamma\|x(\tau)\|\\
&+\beta(\tau))d\tau + \cdots + \int_{t_{k-1}}^{t_k} e^{-\alpha(t-\tau)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau\\
&+\int_{t_k}^{t} e^{-\alpha(t-\tau)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x(t)\|e^{\alpha t} &\leq K_0e^{\alpha t_0}\|x_0\| + K_0\int_{t_0}^{t} e^{\alpha\tau}(\gamma\|x(\tau)\| + \beta(\tau))d\tau\\
&\leq K_0(\int_0^\infty e^{\alpha\tau}\beta(\tau))d\tau + e^{\alpha t_0}\|x(t_0)\|)\\
&+\gamma K_0\int_{t_0}^{t} e^{\alpha\tau}\|x(\tau)\|d\tau.
\end{aligned}$$

By the Gronwall inequality (see, e.g., [9]), we have

$$\begin{aligned}
\|x(t)\| &\leq K_0(\int_0^\infty e^{\alpha\tau}\beta(\tau)d\tau + e^{\alpha t_0}\|x_0\|)e^{\gamma K_0(t-t_0)}e^{-\alpha t}\\
&\leq K_0(\int_0^\infty e^{\alpha\tau}\beta(\tau)d\tau + \|x_0\|)e^{-(\alpha-\gamma K_0)(t-t_0)}.
\end{aligned}$$

This proves the theorem. □

Application to switched interval systems

Consider switched interval systems described by equations of the form

$$\dot{x}(t) = B_ix(t) \triangleq (A_i + \Delta A_i)x(t), \quad i = 1,2,\cdots,m, \quad (2.9)$$

where for each i, the interval matrix B_i is centered at A_i entrywise, with radii $\frac{d_ir_{kl}^{(i)}}{2}$ ($d_i > 0$, $r_{kl}^{(i)} \geq 0, k,l = 1,2,\cdots,m_1$), at the (k,l)-entry.

In order to guarantee the global exponential stability of interval switched systems (2.9), we need to add some constraints on the radii. Let $d = \max_{1\leq i\leq m} d_i$. Then by Theorem 2.2, under the condition that $\max_{1\leq i\leq m}\|\Delta A_i\| \leq d\max_{1\leq i\leq m}(\sum_{k,l=1}^n (r_{kl}^{(i)})^2)^{\frac{1}{2}} < \frac{\alpha}{K_0}$, i.e.,

$$d < \frac{\alpha}{K_0}\min_{1\leq i\leq m}(\sum_{k,l=1}^n (r_{kl}^{(i)})^2)^{-\frac{1}{2}},$$

the switched interval system (2.9) is globally exponentially stable.

Case 2: All subsystem matrices are unstable, but the corresponding negative subsystem matrices are Hurwitz stable

Rather than (A2), we now assume that the following assumption holds for switched system (2.1):

(A4) $-A_i$ is Hurwitz stable for $i = 1,2,\cdots,m$.

Suppose that there exist constants $k_i, \alpha > 0$ such that

$$\|e^{-A_it}\| \leq k_ie^{-\alpha t}, \qquad i = 1,2,\cdots m. \quad (2.10)$$

We have the following result.

Theorem 2.3. Assume that there exist constants $k_i, \alpha, \gamma > 0$ such that conditions (2.5) and (2.10) hold. If hypotheses (A3) and (A4) are true, then for switched system (2.4) with initial condition (t_0, x_0), and under arbitrary switching law satisfying (A1), it is true that

$$\|x(t)\| \geq \frac{1}{K_0}e^{(\alpha-\frac{\gamma}{K_0})(t-t_0)}\|x_0\|. \quad (2.11)$$

Therefore, if $\alpha > \frac{\gamma}{K_0}$, then the switched system is exponentially unbounded.

Proof : The proof can be given by reversing the process in the proof of Theorem 2.2. It is omitted due to space limitation. □

Note that in the above theorem, condition (2.5) instead of (2.6) is used.

Case 3: Mixed-type switched systems

When the switched systems consist of both stable and unstable subsystems, their qualitative analysis becomes more difficult. In such cases, the trajectory behavior depends greatly on the switching law.

Example 2.2. Consider the switched system described by

$$\dot{x}(t) = A_ix(t), \quad A_1 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is not difficult to show that under different switching laws, the properties of the solutions of this system vary significantly. Assume that $T > 0$ is a constant:

i) if subsystems A_1 and A_2 are activated alternatively with the same duration T, then the entire system is **exponentially stable**;

ii) if subsystems A_1 and A_2 are activated alternatively with durations T and $2T$, respectively, then the entire system is **uniformly stable** but not asymptotically stable;

iii) if subsystems A_1 and A_2 are activated alternatively with durations T and $3T$, respectively, then the entire system is **unstable**. □

From above we conclude that, in order to study the qualitative properties of the switched systems of the present case, the switching law has to be specified. Denote $\{A_i : i = 1,2,\cdots,m\} = \{A_i^- : i = 1,\cdots,r_1\} \cup \{A_i^+ : i = r_1+1,\cdots,m\}$, where $1 \leq r_1 \leq m-1$ is an integer, the A_i^-'s are Hurwitz stable, while the A_i^+'s are not. Assume that there exist $k_i, \alpha_1, \alpha_2 > 0$ such that

$$\|e^{A_i^-t}\| \leq k_ie^{-\alpha_1t}, \quad i = 1,2,\cdots r_1 \quad (2.12)$$
$$\|e^{A_i^+t}\| \leq k_ie^{\alpha_2t}, \quad i = r_1+1,\cdots m. \quad (2.13)$$

In $[\tau, t]$, let $T^-(\tau,t)$ (resp., $T^+(\tau,t)$) denote the total time period that stable subsystems from $\{A_i^- : i = 1,\cdots,r_1\}$ (resp., unstable subsystems from $\{A_i^+ : i = r_1+1,\cdots,m\}$) are activated. In the next result, we assume that

(A5) the switching law guarantees that for any given initial time t_0, $\inf_{t\geq t_0} \frac{T^-(t_0,t)}{T^+(t_0,t)} = q > \frac{\alpha_2}{\alpha_1}$, where α_1 and α_2 are specified in (2.12) and (2.13).

Theorem 2.4. Assume that there exist constants $k_i, \alpha_1, \alpha_2 > 0$ and such that conditions (2.12), (2.13) hold. If hypothesis (A3) is true, then for switched system (2.1) with initial condition (t_0, x_0), and under any switching law satisfying (A1) and (A5), then $\|x(t)\| \leq K_0 e^{-\alpha(t-t_0)}$ with $\alpha = \frac{\alpha_1 q - \alpha_2}{1+q}$.

Proof : Notice that under assumption (A5), it is true that

$$-\alpha_1 T^-(t_0,t) + \alpha_2 T^+(t_0,t) \leq -(\alpha_1 - \tfrac{\alpha_2}{q})T^-(t_0,t) \\ \leq -(\alpha_1 - \tfrac{\alpha_2}{q})\tfrac{q}{1+q}(t-t_0) = -\alpha(t-t_0).$$

Thus, the inequality in the above theorem can be derived directly from (2.8). □

In the presence of perturbations, we have similar results to Theorem 2.2, yet we need to add more restrictive condition (A6) on switchings.

(A6) Let $t_0^1 < t_0^2 < t_1^1 < t_1^2 < \ldots < t_i^1 < t_i^2 < \ldots$ ($\lim_{i\to\infty} t_i = +\infty$) be the time instants such that for switched system (2.1) only in intervals $[t_i^1, t_i^2)$ is one of the unstable subsystems A_i^+ activated. Assume that $\inf_{i\geq 0} \frac{t_{i+1}^1 - t_i^2}{t_i^2 - t_i^1} = q > \frac{\alpha_2}{\alpha_1}$, $\sup_i\{t_i^2 - t_i^1\} = T < \infty$ and $t_0^1 - t_0 \leq T_1$, where $T_1 > 0$ is a constant and α_1 and α_2 are specified in (2.12) and (2.13).

We have the following result.

Theorem 2.5. Assume that there exist constants $k_i, \alpha_1, \alpha_2 > 0$ and a nonnegative Lebesgue integrable function $\beta(\cdot)$ such that conditions (2.12), (2.13) and (2.6) (with $\alpha = \frac{\alpha_1 q - \alpha_2}{1+q}$) hold. If $\gamma < \frac{\alpha}{K_0}$ (where $K_0 = \prod_{i=1}^m k_i$) and hypothesis (A3) is true, then for switched system (2.4) with initial condition (t_0, x_0), there exist constants $c_1, c_2, c_3 > 0$ such that under any switching law satisfying (A1) and (A6), for $t \in [t_i^1, t_{i+1}^1)$, $i = 0, 1, \ldots$, the following estimate holds.

$$\|x(t)\| \leq (c_1\|x_0\| + c_2\int_0^\infty e^{\alpha\tau}\beta(\tau)d\tau)e^{-(\alpha-K_0\gamma)(t-t_0)} \\ + c_3 e^{-\alpha t}\int_{t_i^1}^{t_i^1+(q+1)T} e^{\alpha\tau}\beta(\tau)d\tau. \quad (2.14)$$

Therefore, if $\beta(t) \equiv 0$, then $x_e = 0$ is an equilibrium which is globally exponentially stable. Otherwise, the entire system is uniformly bounded and the solutions converge to the origin exponentially.

Proof : We calculate the following three numbered inequalities and then combine them together.

First, for $t \in [t_i^2 + q(t_i^2 - t_i^1), t_{i+1}^1]$, let $(t_0, x_0) = (t_0^1, x(t_0^1))$ in (2.8), we have the following inequality.

$$\|x(t)\| \leq K_0 e^{-\alpha_1 T^-(t_0^1,t)} e^{\alpha_2 T^+(t_0^1,t)}\|x(t_0^1)\| \\ + K_0\int_{t_0^1}^{t_1} e^{-\alpha_1 T^-(\tau,t)} e^{\alpha_2 T^+(\tau,t)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau + \cdots \\ + K_0\int_{t_{k-1}}^{t_k} e^{-\alpha_1 T^-(\tau,t)} e^{\alpha_2 T^+(\tau,t)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau \\ + K_0\int_{t_k}^{t} e^{-\alpha_1 T^-(\tau,t)} e^{\alpha_2 T^+(\tau,t)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau, \quad (2.15)$$

where $t_1, \ldots, t_k$ are the time instants (between t_0^1 and t) at which certain subsystem is enabled. Under assumption (A6), for $\tau \in [t_{j-1}, t_j]$ ($j = 0, 1, \ldots, k$), we must have

$$-\alpha_1 T^-(\tau,t) + \alpha_2 T^+(\tau,t) \leq -(\alpha_1 - \tfrac{\alpha_2}{q})T^-(\tau,t) \\ \leq -(\alpha_1 - \tfrac{\alpha_2}{q})\tfrac{q}{1+q}(t-\tau) = -\alpha(t-\tau).$$

Thus, (2.15) yields $\|x(t)\| \leq K_0 e^{-\alpha(t-t_0^1)}\|x(t_0^1)\| + K_0\int_{t_0^1}^{t} e^{-\alpha(t-\tau)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau$, which implies that

$$\|x(t)\| \leq K_0(\int_{t_0^1}^{\infty} e^{\alpha\tau}\beta(\tau)d\tau + \|x(t_0^1)\|)e^{-(\alpha-K_0\gamma)(t-t_0^1)}. \quad (2.16)$$

(If for some i, $t_{i+1}^1 - t_i^2 = q(t_i^2 - t_i^1)$, the above inequality still holds for $t = t_{i+1}^1$. We can simply derive this by allowing the last activated stable subsystem (before $t = t_{i+1}^1$) to be held until $t > t_{i+1}^1$.)

Next, for $t \in [t_0, t_0^1)$, only the stable subsystems are activated, similar to the above arguments, we obtain that

$$\|x(t)\| \leq K_0 e^{-\alpha_1(t-t_0)}\|x_0\| \\ + K_0\int_{t_0}^{t} e^{-\alpha_1(t-\tau)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau.$$

Using Granwall inequality, we have

$$\|x(t)\| \leq (K_0\|x_0\| + K_0\int_{t_0}^{t_0^1} e^{\alpha_1\tau}\beta(\tau)d\tau)e^{(-\alpha_1+\gamma K_0)(t-t_0)}.$$

Therefore,

$$\|x(t_0^1)\| \leq K_0\|x_0\| + K_0\int_0^{T_1} e^{\alpha_1\tau}\beta(\tau)d\tau. \quad (2.17)$$

For $t \in [t_i^1, t_i^2 + q(t_i^2 - t_i^1))$, let $t_0 = t_i^1$ in (2.8), we have the following estimate

$$\|x(t)\| \leq K_0 e^{\alpha_2(t-t_i^1)}\|x(t_i^1)\| \\ + K_0\int_{t_i^1}^{t} e^{\alpha_2(t-\tau)}(\gamma\|x(\tau)\| + \beta(\tau))d\tau,$$

which implies that

$$\|x(t)\| \leq \\ (K_0\|x(i^1)\| + K_0\int_{t_i^1}^{t_i^2+q(t_i^2-t_i^1)} \beta(\tau)d\tau)e^{(\alpha_2+K_0\gamma)(t-t_i^1)} \\ \leq \tilde{c}_1\|x(t_i^1)\| + \tilde{c}_2 e^{-\alpha t}\int_{t_i^1}^{t_i^1+(q+1)T} e^{\alpha\tau}\beta(\tau)d\tau, \quad (2.18)$$

where $\tilde{c}_1, \tilde{c}_2 > 0$ are two constants.

Combining inequalities (2.16), (2.17) and (2.18), we know that inequality (2.14) holds. □

Similar inequality results can be established as Theorem 2.3 to Theorem 2.2.

Remark 2.1. It is always easy to choose a switching law that satisfies either (A5) or (A6). For example, the switching law can be chosen as follows: beginning with a stable subsystem, we alternatively require that each stable subsystem be activated for a time period between $3\alpha_2 T$ to $4\alpha_2 T$, while each unstable subsystem be activated for a time period between $\alpha_1 T$ to $2\alpha_1 T$, where $T > 0$ is a given constant. In fact, we do not require that all the activating time periods be uniformly bounded as time elapses. □

Remarks 2.2. Some of the above results can be generalized to switched systems that do not possess commutative properties by using switching laws based on average dwell time (see, e.g., [10], [11]). □

III. Discrete-Time Switched System

For discrete-time switched systems, we can study similar problems as was done in Section 2.

Consider discrete-time switched systems described by difference equations of the form

$$z[n+1] \quad = \quad D_i z[n], \qquad i = 1, 2, \cdots, m \qquad (3.1)$$

where $m \geq 2$ is an integer, $z[n] \in R^{m_1}$, $D_i \in R^{m_1 \times m_1}, i = 1, 2, \cdots, m$. As in the continuous-time case, we use the notation $\{i_k\}_{k \geq 0} \subset \{1, 2, \cdots, m\}$ to denote the switching sequence and we let $[n_k, n_{k+1}) \cap N = \{n_k, n_k + 1, \cdots, n_{k+1} - 1\}$ denote the discrete-time instants when the i_k-th subsystem is activated. Without loss of generality, we assume that in the subsequent discussion $n_0 \geq 0$. Assume that

(B1) the D_i's are Schur stable, i.e., there exist constants $k_i \geq 1$, $0 < r < 1$ such that

$$\|D_i^n\| \quad \leq \quad k_i r^n, \qquad i = 1, 2, \cdots, m. \qquad (3.2)$$

(B2) $D_i D_j = D_j D_i$ for all $i \neq j$.

We have the following result.

Theorem 3.1. Assume that (B1) and (B2) are true. Then the equilibrium $z_e = 0$ of switched system (3.1) is ***globally exponentially stable*** under arbitrary switching law.

Proof : Similar to the proof of Theorem 2.1. Omitted. □

Next, we endow system (3.1) with perturbation terms. Thus, we consider perturbed switched systems described by equations of the form

$$z[n+1] = D_i z[n] + g_i(n, z[n]), \quad i = 1, 2, \cdots, m, \quad (3.3)$$

with either vanishing perturbations described by

$$\|g_i(n, z[n])\| \leq p\|z[n]\|, \quad i = 1, 2, \cdots, m \qquad (3.4)$$

or non-vanishing perturbations described by

$$\|g_i(n, z[n])\| \leq p\|z[n]\| + q[n], \quad i = 1, 2, \cdots, m, \quad (3.5)$$

where $p > 0$ is a constant and $\{q[n]\}_{n \geq 0}$ is a nonnegative sequence satisfying the condition $\sum_{j=0}^{\infty} r^{-j} q[j] < \infty$, where r is either the constant in (3.2) or will be specified later.

Theorem 3.2. Under hypotheses (B1) and (B2), assume that (3.5) holds. Then for switched system (3.3) with initial value (n_0, z_0), and under arbitrary switching law, it is true that

$$\|z[n]\| \leq K_0((1 + \frac{p}{r})\|z_0\| + \sum_{j=0}^{\infty} r^{-j-1} q[j])(pK_0 + r)^{n-n_0} \qquad (3.6)$$

where $K_0 = \prod_{i=1}^{m}$. Therefore, under the condition that $p < \frac{1-r}{K_0}$, if $q[n] = 0$ for $n \geq n_0$, $z_e = 0$ is an equilibrium of switched system (3.3) and is globally exponentially stable. Otherwise, the entire system is uniformly bounded and the solution satisfies the condition $\lim_{n \to \infty} z[n] = 0$.

Proof: By induction, for $n_k \leq n < n_{k+1}$, we have

$$\begin{aligned}
z[n_0+1] &= D_{i_0} z[n_0] + g_{i_0}(n_0, z[n_0]), \\
z[n_0+2] &= D_{i_1} z[n_1] + g_{i_1}(n_1, z[n_1]) \\
&= D_{i_0}^2 z[n_0] + D_{i_0} g_{i_0}(n_0, z[n_0]) + g_{i_1}(n_1, z[n_1]), \\
&\cdots \\
z[n_1] &= D_{i_1}^{n-n_0} z[n_0] + D_{i_0}^{n_1-n_0-1} g_{i_0}(n_0, z[n_0]) \\
&\quad + D_{i_0}^{n_1-n_0-2} g_{i_0}(n_0+1, z[n_0+1]) + \cdots \\
&\quad + g_{i_0}(n_1 - 1, z[n_1 - 1]), \\
&\cdots \\
z[n] &= D_{i_k}^{n-n_k} D_{i_{k-1}}^{n_k - n_{k-1}} \cdots D_{i_0}^{n_1 - n_0} z[n_0] \\
&\quad + D_{i_k}^{n-n_k} D_{i_{k-1}}^{n_k - n_{k-1}} \cdots D_{i_0}^{n_1 - n_0} g_{i_0}(n_0, z[n_0]) + \cdots \\
&\quad + D_{i_k}^{n-n_k} D_{i_{k-1}}^{n_k - n_{k-1}} \cdots D_{i_1}^{n_2 - n_1} g_{i_0}(n_1 - 1, z[n_1 - 1]) \\
&\quad + D_{i_k}^{n-n_k} D_{i_{k-1}}^{n_k - n_{k-1}} \cdots D_{i_1}^{n_2 - n_1 - 1} g_{i_0}(n_1, z[n_1]) \\
&\quad + \cdots + D_{i_k}^{n-n_k} \cdots D_{i_2}^{n_3 - n_2} g_{i_1}(n_2 - 1, z[n_2 - 1]) \\
&\quad + \cdots + D_{i_k}^{n-n_k} D_{i_{k-1}}^{n_k - n_{k-1} - 1} g_{i_{k-1}}(n_{k-1}, z[n_{k-1}]) \\
&\quad + \cdots + D_{i_k}^{n-n_k} g_{i_k - 1}(n_k - 1, z[n_k - 1]) \\
&\quad + D_{i_k}^{n-n_k-1} g_{i_k}(n_k, z[n_k]) \\
&\quad + \cdots + g_{i_k}(n-1, z[n-1]).
\end{aligned} \qquad (3.7)$$

By (3.2) and (3.7), we have that

$$\begin{aligned}
\|z[n]\| \leq K_0(&r^{n-n_0}\|z[n_0]\| + r^{n-n_0-1}(p\|z[n_0]\| + q[n_0]) \\
&+ r^{n-n_0-2}(p\|z[n_0+1]\| + q[n_0+1]) + \cdots \\
&+ (p\|z[n-1]\| + q[n-1])).
\end{aligned}$$

Therefore,

$$\begin{aligned}
r^{-n}\|z[n]\| &\leq K_0 r^{-n_0}(1 + \tfrac{p}{r})\|z[n_0]\| \\
&\quad + \tfrac{K_0}{r} \textstyle\sum_{j=n_0}^{n-1} r^{-j-1} q[j] \quad + \tfrac{K_0 p}{r}(r^{-(n-1)}\|z[n-1]\| \\
&\quad + r^{-(n-2)}\|z[n-2]\| + \cdots r^{-(n_0-1)}\|z[n_0+1]\|) \\
&\leq K_0 r^{-n_0}(1 + \tfrac{p}{r})\|z[n_0]\| + K_0 \textstyle\sum_{j=0}^{\infty} r^{-j-1} q[j] \\
&\quad + \tfrac{K_0 p}{r}(r^{-(n-1)}\|z[n-1]\| + r^{-(n-2)}\|z[n-2]\| \\
&\quad + \cdots r^{-(n_0-1)}\|z[n_0+1]\|).
\end{aligned} \qquad (3.8)$$

We require the following intermediate lemma to proceed.

Lemma 3.1. If for a nonnegative sequence $\{y[n]\}_{n\geq 0}$, there exist two constants $h_0, h_1 > 0$ such that for every $n \geq n_0$ (n_0 fixed) the inequality: $y[n] \leq h_0 + h_1(y[n-1] + y[n-2] + \cdots + y[n_0+1])$ holds, then $y[n] \leq h_0(1+h_1)^{n-n_0-1}$ for $n \geq n_0+1$; If there exist two constants $h_0, h_1 > 0$ such that for every $0 \leq n_0 < n$ (n fixed) the inequality: $y[n_0] \leq h_0 + h_1(y[n_0+1] + y[n_0+2] + \cdots + y[n-1])$ holds, then $y[n_0] \leq h_0(1+h_1)^{n-n_0-1}$ for $n_0 \geq n-1$.

Proof: By induction. Omitted due to space limitation. □

By the first part of the above lemma and the inequality (3.8), we obtain that

$$\begin{aligned} r^{-n}\|z[n]\| &\leq (K_0 r^{-n_0}(1+\frac{p}{r})\|z[n_0]\| \\ &\quad + K_0 \sum_{j=0}^{\infty} r^{-j-1} q[j])(1+\frac{K_0 p}{r})^{n-n_0}, \end{aligned}$$

which implies that

$$\begin{aligned} \|z[n]\| &\leq K_0((1+\frac{p}{r})\|z[n_0]\| \\ &\quad + \sum_{j=0}^{\infty} r^{-j-1} q[j])(r+K_0 p)^{n-n_0}. \end{aligned}$$

Thus, Theorem 3.2 follows. □

When all the eigenvalues of D_i lie outside the unit disc, we have the following result.

Theorem 3.3. Assume that there exist constants $k_i \geq 1$, $0 < r < 1$, $p > 0$ such that $\|D_i^{-n}\| \leq k_i r^n$ for $i = 1, 2, \cdots, m$ and (3.4) holds. If hypothesis (B2) is true and $K_0 pr < 1$, then for switched system (3.3) with initial value (n_0, z_0), the inequality

$$\|z[n]\| \geq \frac{1}{K_0}(\frac{1-K_0 pr}{r})^{n-n_0}\|z_0\|. \qquad (3.9)$$

holds. Therefore, if $p < \frac{1-r}{K_0 r}$, then the equilibrium $z_e = 0$ of switched system (3.3) is exponentially unstable.

Proof: By (3.7), we have that

$$\begin{aligned} &z[n_0] = (D_{i_k}^{-1})^{n-n_k}(D_{i_{k-1}}^{-1})^{n_k-n_{k-1}} \cdots (D_{i_0}^{-1})^{n_1-n_0} z[n] \\ &-(D_{i_k}^{-1})^{n-n_k}(D_{i_{k-1}}^{-1})^{n_k-n_{k-1}} \cdots (D_{i_0}^{-1})^{n_1-n_0} \\ &\times g_{i_k}(n-1, z[n-1]) - \cdots \\ &-(D_{i_k}^{-1})(D_{i_{k-1}}^{-1})^{n_k-n_{k-1}} \cdots (D_{i_0}^{-1})^{n_1-n_0} g_{i_k}(n_k, z[n_k]) \\ &-(D_{i_{k-1}}^{-1})^{n_k-n_{k-1}} \cdots (D_{i_0}^{-1})^{n_1-n_0} g_{i_k}(n_k-1, z[n_k-1]) \\ &- \cdots - (D_{i_{k-1}}^{-1})(D_{i_{k-2}}^{-1})^{n_{k-1}-n_{k-2}} \cdots (D_{i_0}^{-1})^{n_1-n_0} \\ &\times g_{i_{k-1}}(n_{k-1}, z[n_{k-1}]) - \cdots \\ &-(D_{i_1}^{-1})^{n_2-n_1}(D_{i_0}^{-1})^{n_1-n_0} g_{i_1}(n_2-1, z[n_2-1]) \\ &- \cdots - (D_{i_1}^{-1})(D_{i_0}^{-1})^{n_1-n_0} g_{i_1}(n_2-1, z[n_2-1]) \\ &-(D_{i_0}^{-1})^{n_1-n_0} g_{i_0}(n_1-1, z[n_1-1]) \\ &- \cdots - (D_{i_0}^{-1}) g_{i_0}(n_0, z[n_0]). \end{aligned}$$

Therefore, we obtain that $\|z[n_0]\| \geq K_0(r^{n-n_0}\|z[n]\| + pr^{n-n_0}\|z[n-1]\| + \cdots + pr\|z[n_0]\|)$, i.e.,

$r^{n_0}\|z[n_0]\| \geq (1-K_0 pr)^{-1} K_0 (r^n \|z[n]\|) + (1-K_0 pr)^{-1} \times K_0 pr (r^{n_0+1}\|z[n_0+1]\| + \cdots + r^{n-1}\|z[n-1]\|)$.

By the second part of Lemma 3.1 (fix n), we have that $r^{n_0}\|z[n_0]\| \geq (1-K_0 pr)^{-1} K_0 (r^n \|z[n]\|) \times (1+\frac{K_0 pr}{1-K_0 pr})^{n-n_0-1}$. Thus, $\|z[n_0]\| \geq \frac{1}{K_0}(\frac{1-K_0 pr}{r})^{n-n_0}\|z[n_0]\|$. □

For discret-time switched systems consisting of both stable and unstable subsystems, we can establish similar results as in Theorems 2.5 and 2.6.

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