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Optimal Control of Switched Systems: New Results and Open Problems

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Abstract

In optimal control problems of switched systems, we may need to find both an optimal continuous input and an optimal switching sequence since the system dynamics vary before and after every switching instant. In this paper, optimal control problems for both continuous-time and discrete-time switched systems are formulated and investigated. In particular, we regard an optimal control problem as a two stage optimization problem and discuss its solution algorithm. The dynamic programming (DP) approach is also studied. Difficulties and open problems are discussed.

1 Introduction

A switched system is a system that consists of several subsystems and a switching law indicating the active subsystem at each time instant. Examples of switched systems can be found in power train systems, automotive systems, and electrical circuits, etc.

This paper gives an overview of optimal control problems of switched systems. Such problems have been studied before (see e.g., [2, 3, 6, 7, 9]). For an optimal control problem of a switched system, we need to find both an optimal continuous input and an optimal switching sequence since the system dynamics vary before and after every switching instant. Solving such problem is generally very difficult. The main contribution of the paper is as follows. We first formally define a switched system and formulate an optimal control problem for it. Then we prove that the optimal control problem can be formulated as a two stage optimization problem under additional assumptions. Difficulties in solving the problem are shown by examples. We propose ways to overcome these difficulties by taking into consideration minimum dwell time switching and costs for switchings. Hamilton-Jacobi-Bellman (HJB) equations for optimal control problems of both discrete-time and continuous-time switched systems are formulated and the complexity in solving them is discussed.

In the sequel, an optimal control problem for switched systems is formulated in Section 2. In Section 3, a two stage optimization method is proposed for the problem, also the difficulties of the problem and the related Zenoness problem (i.e., infinite switchings in finite time) are addressed. In Section 4, the DP approach is used and HJB equations are obtained. Section 5 mentions other approaches and related topics. Section 6 concludes the paper.

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2 Problem Formulation

2.1 Switched Systems

Switched systems: The following is the definition of a switched system.

Definition 1 (Switched System) *A switched system is a tuple $\mathcal{S} = (\mathcal{F}, \mathcal{D})$ where*
• $\mathcal{F} = \{f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ i \in I\}$ *with f_i describing the vector field for the ith subsystem $\dot{x} = f_i(x,u)$. $I = \{1, 2, \cdots, M\}$ is the set of indices of subsystems.*
• $\mathcal{D} = (I, E)$ *is a simple finite state machine which can also be viewed as a directed graph. I is the set of indices as defined above. Here I serves as the set of discrete states indexing the subsystems. $E \subseteq I \times I - \{(i,i) | i \in I\}$ is a collection of events. If an event $e = (i,j)$ takes place, the switched system will switch from subsystem i to j.*

In view of Definition 1, a switched system is a collection of subsystems which are related by a switching logic restricted by $\mathcal{D}$. The continuous state x and the continuous input u satisfy $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. If a particular switching law has been specified, then the switched system can be described as

$$\dot{x}(t) = f_{i(t)}(x(t), u(t)) \quad (1)$$

$$i(t) = \varphi(x(t), i(t^-), t), \quad (2)$$

where $\varphi : \mathbb{R}^n \times I \times \mathbb{R} \to I$ determines the active subsystem at time t. Note that (1)-(2) are used as the definition of switched systems in some of the literature (e.g., [1]). Here we adopt Definition 1 rather than (1)-(2) because in design problems, in general, φ is not defined a priori and it is a designer's task to find a switching law. A salient feature of a switched system is that its continuous state x does not exhibit jumps at switching instants.

Remark 1 If $f_i(x,u) = f_i(x), \forall i \in I$, then the switched system is said to be *autonomous.* We can also define discrete-time switched systems by letting all subsystems to be discrete-time systems.

Switching sequences: For a switched system $\mathcal{S}$, the inputs of the system consist of both a continuous input $u(t), t \in [t_0, t_f]$ and a switching sequence. We define a switching sequence as follows.

Definition 2 (Switching Sequence) *For a switched system $\mathcal{S}$, a switching sequence σ in $[t_0, t_f]$ is defined as*

$$\sigma = ((t_0, e_0), (t_1, e_1), (t_2, e_2), \cdots, (t_K, e_K)), \quad (3)$$

with $0 \le K < \infty$, $t_0 \le t_1 \le t_2 \le \cdots \le t_K \le t_f$, and

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$e_0 = i_0 \in I$, $e_k = (i_{k-1}, i_k) \in E$ *for* $k = 1, 2, \cdots, K$. *(If* $K = 0$, $\sigma = ((t_0, e_0))$.*)*

We also define $\Sigma_{[t_0,t_f]} = \{\sigma\text{'s in } [t_0, t_f]\}$.

A switching sequence σ as defined above indicates that subsystem i_k is active in $[t_k, t_{k+1})$ if $t_k < t_{k+1}$ ($[t_{K-1}, t_K]$ if $k = K-1$), and i_k is switched through at instant t_k if $t_k = t_{k+1}$ ('switched through' means that the system switches from subsystem i_{k-1} to i_k and then to i_{k+1} all at instant t_k). For a switched system to be well-behaved, we generally exclude the undesirable *Zeno* phenomenon, i.e., infinitely many switchings in finite amount of time. Hence in Definition 2, we only allow nonZeno sequences which switch finite number of times in $[t_0, t_f]$, though different sequences may have different numbers of switchings. We specify $\sigma \in \Sigma_{[t_0,t_f]}$ as a discrete input to a switched system.

Remark 2 In this paper, we mostly assume that a switching is *external* in the sense that it is forced by a designer or a controller. However, it is worth noting that for some switched systems, the boundaries for switchings are given in $\mathbb{R}^n$ so that switchings take place when crossing them, or some time instants or time constraints are given so that switchings take place at those instants; we call them *internal* switchings. We will mention internal switchings in Section 5.

2.2 An Optimal Control Problem

Problem 1 *For a switched system* $\mathcal{S} = (\mathcal{F}, \mathcal{D})$, *find a switching sequence* $\sigma \in \Sigma_{[t_0,t_f]}$ *and an input* $u \in \mathcal{U} = \{$*piecewise continuous function* u *on* $[t_0, t_f]$ *with* $u(t) \in U \subseteq \mathbb{R}^m$, $\forall t \in [t_0, t_f]\}$ *such that the cost functional*

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))dt \qquad (4)$$

is minimized, where t_0, t_f *and* $x(t_0) = x_0$ *are given,* $\psi : \mathbb{R}^n \to \mathbb{R}$, $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$.

Problem 1 is a fixed final time, free final state problem. We can similarly formulate other variations of optimal control problems (e.g., fixed final time, fixed final state, etc.). There are several results in the literature that discuss the optimal control problem in its various forms in either continuous time or discrete time [6, 9]. The involvement of σ makes the dynamics of the system vary in $[t_0, t_f]$, so the problem is very difficult to solve.

3 Two Stage Optimization

In general, Problem 1 requires the solution of an optimal control input (σ^*, u^*) such that

$$J(\sigma^*, u^*) = \min_{\sigma \in \Sigma_{[t_0,t_f]},\ u \in \mathcal{U}} J(\sigma, u). \qquad (5)$$

Notice that for any given switching sequence σ, Problem 1 reduces to a conventional optimal control problem for which we only need to find an optimal continuous input u so as to minimize $J_\sigma(u) = J(\sigma, u)$. The following lemma provides a way to formulate (5) into a two stage optimization problem. (The proof of Lemmas 1 to 4 may be found in [10].)

Lemma 1 *For Problem 1, if*

(1). an optimal solution (σ^*, u^*) *exists and*

(2). for any given switching sequence σ, *there exists a corresponding* $u^* = u^*(\sigma)$ *such that* $J_\sigma(u)$ *is minimized, then the following equation holds*

$$\min_{\sigma \in \Sigma_{[t_0,t_f]},\ u \in \mathcal{U}} J(\sigma, u) = \min_{\sigma \in \Sigma_{[t_0,t_f]}} \min_{u \in \mathcal{U}} J(\sigma, u). \qquad (6)$$

By (6), we adopt the following two stage optimization method for Problem 1 satisfying the conditions of Lemma 1.

Two stage optimization method

Stage 1: Fixing σ, solve the inner minimization problem.
Stage 2: Regarding the optimal cost for each σ as a function $J_1 = J_1(\sigma)$, minimize J_1 with respect to $\sigma \in \Sigma_{[t_0,t_f]}$.

In [9], a similar formulation for discrete-time switched systems is posed; however, the continuous-time case is not addressed.

The two stage optimization is still difficult to handle. We can implement the above method by the following algorithm.

Algorithm

1. Fix the total number of switchings to be K and the order of active subsystems, let the minimum value of J with respect to u be a function of the K switching instants, i.e., $J_1 = J_1(t_1, t_2, \cdots, t_K)$ for $K \geq 0$, and then find J_1.

2. (a) Minimize J_1 with respect to $t_1, t_2, \cdots, t_K$.

(b) Vary the order of active subsystems to find an optimal solution under K switchings.

(c) Vary the number of switchings K to find an optimal solution for Problem 1.

The above algorithm has high computational costs. In practice, we usually find suboptimal solutions with fixed number of switchings by using steps 1, 2(a), 2(b). Moreover, in many cases we only need to study problems with fixed number of switchings and fixed order of active subsystems (e.g., the speeding up of a power train).

3.1 Two Examples

The following two examples offer insights into optimal control problems and their difficulties.

Example 1 *Given a switched system* $\mathcal{S} = (\mathcal{F}, \mathcal{D})$, *where* $\mathcal{F} = \{f_1 = x + u,\ f_2 = -x + u\}$ *and* $\mathcal{D} = (I, E)$ *with* $I = \{1, 2\}$ *and* $E = \{(1, 2)\}$, *assume that* $t_0 = 0$, $t_f = 2$ *and the system switches once at* $t = t_1$ *(*$0 \leq t_1 \leq 2$*) from subsystem* 1 *to* 2. *Find an optimal switching instant* t_1 *and an optimal input* u *such that* $x(0) = 1$, $x(2) = 1$ *and the cost functional*

$$J = \int_0^2 u^2(t)dt \qquad (7)$$

is minimized.

We can define the Hamiltonian functions $H_1(x, u, \lambda_1)$ for $t \in [0, t_1)$ and $H_2(x, u, \lambda_2)$ for $t \in [t_1, 2]$. By using the state, costate and stationary conditions [4, Chapter 3] along with the Weierstrass-Erdmann corner condition $\lambda_1(t_1) = \lambda_2(t_1)$ at t_1 [12], we solve the minimum cost as a function of t_1 as $J_1(t_1) = \frac{2(e^{4t_1-4} - 2e^{2t_1-2} + 1)(e^{4t_1} - 2e^{2t_1} + e^4)}{(e^{4t_1-2} - 2e^{2t_1-2} + e^2)^2}$.

Figure 1 shows the plot of $J_1(t_1)$. $t_1 = 1$ is optimal with corresponding $J_1 = 0$. If we select subsystem 1 ($t_1 = 2$) or subsystem 2 ($t_1 = 0$), more control energy than 1.5 needs to be spent. □

Notice that even the above one switching problem has a complicated representation of $J_1(t_1)$. In general,

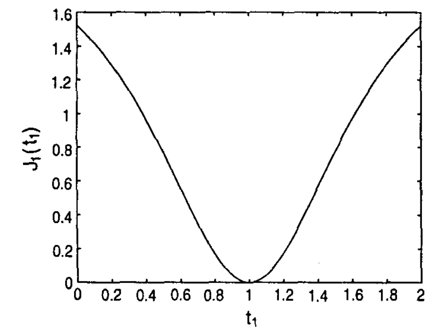

Figure 1: The plot of $J_1(t_1)$ in $[0,2]$.

for a switched system with $K > 1$ switchings, we can first fix the switching times $t_1, \cdots, t_K$ and use numerical methods to find $J_1(t_1, \cdots, t_K)$ and then vary $t_1, \cdots, t_K$ to optimize J_1 by using some nonlinear programming algorithms. In the above discussions about a solution algorithm based on the two stage optimization method for Problem 1, we assume that a solution exists. Yet this may not be true, even for simple cases — see Example 2.

Example 2 *Consider the same switched system as in Example 1 except that $E = \{(1,2),(2,1)\}$. We want to find an optimal control (σ, u) such that $x(0) = 1$, $x(2) = 1$ and the cost functional*

$$J = \int_0^2 [(x(t)-1)^2 + u^2(t)]dt \tag{8}$$

is minimized.

If we consider the switching sequence $\sigma_K = ((0,1),(1/K,(1,2)),(2/K,(2,1)),\cdots,((2K-1)/K,(1,2)))$ and $u(t) = 0$ for all $t \in [0,2]$, then as $K \to \infty$, $J(\sigma_K, 0) \to 0$. But $J = 0$ cannot be achieved because it requires infinite switchings in finite time. so the problem has no optimal solution in $\sigma \in \Sigma_{[0,2]}$ and $u \in \mathcal{U}$. □

3.2 Avoidance of Zenoness

As seen from Example 2, the undesirable Zeno phenomenon may prevent us from finding an optimum. We can impose more requirements to have nonZenoness.

Minimum dwell time switching: In practice, it may not be possible to switch from subsystem 1 to 2 and to 3 at the same instant (e.g., because of the nature of mechanical apparatus or hardware delay). Hence it is not restrictive to introduce a minimum dwell time $T > 0$ for a switched system (T may be different for different $e \in E$, however here we assume it is the same for all $e \in E$). That is, after the system switches from subsystem i to j, it must dwell at subsystem j at least for time T. With this additional assumption, valid switching sequences may further be restricted. We denote $\Sigma^T_{[t_0,t_f]}$ as the set of all valid switching sequences in this case.

Costs for switchings: Another way to avoid Zenoness is to introduce in J the costs for switchings. In practice, a switching usually consumes some energy. Hence it makes practical sense to introduce a cost $P : E \to \mathbb{R}^+$ for each switching. Then, for any switching sequence $\sigma = ((t_0,e_0),(t_1,e_1),\cdots,(t_K,e_K))$ we can extend our definition of P to be $P(\sigma) = \sum_{k=1}^{K} P(e_k)$. Now if we add to J in (4) the term $P(\sigma)$, we have

Problem 2 *For $\mathcal{S} = (\mathcal{F}, \mathcal{D})$, find $\sigma \in \Sigma_{[t_0.t_f]}$ and $u \in \mathcal{U}$ such that the expanded functional*

$$J_{exp}(\sigma, u) = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t),u(t))dt + P(\sigma)$$

is minimized, where t_0, t_f and $x(t_0) = x_0$ are given.

For Problem 2, we have the following Lemma.

Lemma 2 *If*

(1). $\psi \geq 0$ and $L \geq 0$ hold for all $x(t_f)$, $x(t)$, $u(t)$ and
(2). $P(e) > 0$ for any $e \in E$ and
(3). for any fixed number and fixed order of switchings, the corresponding optimal control problem with J as in Problem 1 has an optimal solution with $J_{min} < \infty$,

then Problem 2 has an optimal solution.

4 Dynamic Programming Approach

In this section, we formulate the Hamilton-Jacobian-Bellman (HJB) equation for Problem 1 using the DP approach. Let us start with the optimal control problem of discrete-time switched systems.

4.1 Discrete-time Switched Systems

Remark 1 mentions that, with slight modifications, we can define a discrete-time switched system similar to Definition 1 except that the ith subsystem is $x(k+1) = f_i(x(k),u(k))$. The switching sequence from step 0 to N is $\sigma = ((0,e_0),(t_1,e_1),\cdots,(t_K,e_K))$ with $0 \leq K < \infty$, $0 \leq t_1 \leq \cdots \leq t_K \leq N$ and t_k's are integers. Similarly we can define $\Sigma_{[0,N]}$. Problem 1 can be modified accordingly for the discrete-time case.

Problem 3 *For a discrete-time switched system $\mathcal{S} = (\mathcal{F}, \mathcal{D})$, find a switching sequence $\sigma \in \Sigma_{[0,N]}$ and an input $u \in \mathcal{U} = \{u(\cdot)|u(k) \in U \subseteq \mathbb{R}^m, \forall k = 0,1,\cdots,N-1\}$ such that the cost functional*

$$J = \psi(x(N)) + \sum_{k=0}^{N-1} L(x(k),u(k)) \tag{9}$$

is minimized, where $0, N, \psi, L$ and $x(0) = x_0$ are given.

Remark 3 We assume $U = \mathbb{R}^m$ in the followings.

For Problem 3, we can set up the two stage optimization problem as in Section 3 under similar assumptions as in Lemma 1

$$\min J = \min_{\sigma \in \Sigma_{[0,N]},\ u \in \mathcal{U}} J(\sigma,u) = \min_{\sigma \in \Sigma_{[0,N]}} \min_{u \in \mathcal{U}} J(\sigma,u).$$

Minimum dwell time switching: Now if at each time k, we can switch at most once, in other words, we have the minimum dwell time requirement with $T = 1$, we have the following result.

Lemma 3 *With the minimum dwell time $T = 1$, we have $|\Sigma_{[0,N]}| < \infty$.*

From Lemma 3, if a system has a minimum dwell time $T = 1$, then there are finitely many possible σ's. We can solve Problem 3 with minimum dwell time $T = 1$ by simply enumerating all possible $\sigma \in \Sigma_{[0,N]}$. Hence by finding $\min_{u \in \mathcal{U}} J_\sigma(u)$ for each possible σ and choosing the minimum among all of them, we can solve Problem 3. This is the so called *enumeration* method. Therefore, if each $\min_{u \in \mathcal{U}} J(\sigma,u)$ has a solution, Problem 3 must have a solution.

Although the enumeration method is straightforward, it does not show the relationship between different switching sequences. Another approach is to use the DP approach. The DP approach depends on the principle of optimality [4]. We denote $\Sigma^i_{[k,N]}$ to be a subset of $\Sigma_{[k,N]}$ which contains all $\sigma \in \Sigma_{[k,N]}$ that starts with subsystem i at instant k. We define the value functions $V^i(x(k),k)$, $i \in I$ to be the minimum value of J if the system starts at time k with state $x(k)$ and subsystem i, i.e.

$$V^i(x(k),k) = \min_{\sigma\in\Sigma^i_{[k,N]}} \min_{u\in\mathcal{U}} J(\sigma,u).$$

The principle of optimality gives us the HJB equation

$$V^i(x(k),k) = \begin{cases} \psi(x(N)), \text{ if } k = N, \\ \min_{u(k)}\{L(x(k),u(k)) \\ \quad + \min_{j\in\{i\}\cup\{i'|(i,i')\in E\}} V^j(x(k+1), \\ \quad k+1)\}, \text{ if } 0 \le k < N. \end{cases} \tag{10}$$

And at time $t = 0$, the optimal value of J is

$$V(x(0),0) = \min_{i\in I} V^i(x(0),0). \tag{11}$$

With (10), we can solve $V^i(x(k),k)$ backwards and finally find $V(x(0),0)$ and then construct an optimal solution (σ,u). It can be observed that because of the involvement of the term $V^j(x(k+1),k+1)$, the solution of (10) is nontrivial even for the following LQR problem.

Example 3 (An LQR Problem) *Consider a discrete-time switched system $\mathcal{S} = (\mathcal{F},\mathcal{D})$ where $\mathcal{F} = \{A_1x + B_1u, A_2x + B_2u\}$ and $\mathcal{D} = \{I,E\}$, $I = \{1,2\}$, $E = \{(1,2)\}$. Here N and $x(0) = x_0$ are given. Find an optimal solution (σ,u) such that*

$$\begin{aligned} J &= (1/2)x^T(N)Q_Nx(N) \\ &\quad +(1/2)\sum_{k=0}^{N-1}(x^T(k)Qx(k) + u^T(k)Ru(k)) \end{aligned}$$

is minimized, where $Q_N \ge 0$, $Q \ge 0$, $R > 0$.

We use the HJB equation to solve this problem (because the details are tedious and long, we do not include them here due to space limitation; see [10] for details). □

Using the HJB equation we may solve an optimal control problem, but the procedure is quite complicated and tedious. For Example 3, the enumeration method can also be used, and might not seem to be so complicated. However, the DP approach provides us with more insights into the relationship between subsystems that cannot be obtained by the enumeration method. Although it is still an largely open problem how to solve the HJB equation efficiently, it is expected that an efficient method based on (10) may be derived in the future, at least for some simple problems.

Costs for switchings: Another way to reduce the complexity of Problem 3 is to include a cost $P : E \to \mathbb{R}^+$ for switchings so that $J_{exp} = \psi(x(N)) + \sum_{k=0}^{N-1} L(x(k),u(k)) + P(\sigma)$ as in Section 3.2. In this case, we do not force a minimum dwell time at each instant k. Therefore at each instant k, we may have infinitely many switching patterns, so $|\Sigma_{[0,N]}|$ may not be finite. This seems to add more difficulties to our enumeration method. However, the next lemma justifies the use of the enumeration method.

Lemma 4 *If $P(e) > 0$, $\forall e \in E$, then the enumeration method can be used for optimizing J_{exp}. Moreover, in this case, it also applies to optimizing J in Problem 3.*

Under similar assumptions as in Lemma 1, we define a value function as

$$V^i(x(k),k) = \min_{\sigma\in\Sigma^i_{[0,N]}} \min_{u\in\mathcal{U}} J_{exp}(\sigma,u).$$

Then we can obtain the HJB equation

$$V^i(x(k),k) = \begin{cases} \psi(x(N)), \text{ if } k = N, \\ \min\{\min_{j\in\{i'\in I|(i,i')\in E\}}\{V^j(x(k),k) \\ \quad +P((i,j))\}, \min_{u(k)}\{L(x(k),u(k)) \\ \quad +V^i(x(k+1),k+1)\}\}, \\ \quad \text{if } 0 \le k < N. \end{cases}$$

The optimal value of J_{exp} is equal to $V(x(0),0) = \min_{i\in I} V^i(x(0),0)$. Notice that the above HJB equation is different from (10).

4.2 Continuous-time Switched Systems

Now we return to the continuous-time switched systems and use the DP approach to solve the problem with J_{exp}. The problem has been studied by Yong [11] for a general class of hybrid systems, here we restate the result in a way suitable for our purpose. Under similar assumptions as in Lemma 1, we define the value function

$$V^i(x,t) = \min_{\sigma\in\Sigma^i_{[t,t_f]}} \min_{u\in\mathcal{U}} J_{exp}(\sigma,u).$$

Note that the optimal cost at (x,t) is $V(x,t) = \min_{i\in I} V^i(x,t)$. By the principle of optimality we have

$$V^i(x,t) \le \min_{j\in\{i'|(i,i')\in E\}}\{V^j(x,t) + P((i,j))\}, \tag{12}$$

$$\begin{aligned} V^i(x,t) &\le \min_{u\in\mathcal{U}}\{\int_t^{t+\Delta t} L(x(t),u(t))dt \\ &\quad +V^i(x(t+\Delta t),t+\Delta t)\}. \end{aligned} \tag{13}$$

Note that in (13), we assume that subsystem i is active in $[t,t+\Delta t]$. Also note that at least one of (12),(13) becomes equality at each (x,t). If (12) satisfies strict inequality at (x,t) then $\exists \bar{x} > 0$ such that

$$\begin{aligned} V^i(x,t) &= \min_{u\in\mathcal{U}}\{\int_t^{t+\Delta t} L(x(t),u(t))dt \\ &\quad +V^i(x(t+\Delta t),t+\Delta t)\}, \\ &\quad \forall\Delta t \in [0,\bar{x}]. \end{aligned} \tag{14}$$

If $V^i(x,t) \in C^1[t_0,t_f]$, then we have the following HJB equation

$$\begin{aligned} 0 &= \min\{\min_{j\in\{i'|(i,i')\in E\}}\{V^j(x,t) + P((i,j))\} - V^i(x,t), \\ &\quad V^i_t(x,t) + \min_{u\in\mathcal{U}}\{L(x(t),u(t)) \\ &\quad +V^i_x(x,t)f_i(x,u)\}\} \end{aligned} \tag{15}$$

Remark 4 In the case when a solution does not exist, we may change min to inf and (15) still holds.

In [11], the HJB equation was derived for hybrid systems with continuous input, switchings and impulse effect. Yong proved the existence and uniqueness of the viscosity solutions for the HJB equation under some additional assumptions. However, (15) is very difficult to solve in general, even for LQR problems. In practice, we usually discretize a continuous switched system and then apply the method for discrete-time optimal control to solve the corresponding discrete-time version of the problem. In this way, we can obtain approximate optimal or suboptimal solutions to the continuous-time problem. Currently, how to solve the HJB equation efficiently is still a largely open problem.

5 Other Approaches and Related Topics

Now we briefly mention other optimal control approaches in the literature and some related topics.

Suboptimal control: A performance guided hybrid feedback control law was proposed in [8] for infinite horizon LQR problems. This control law can also be modified to find a suboptimal control for Problem 1 in finite horizon LQR forms with linear subsystems and quadratic cost functionals. The law provides us with a practical way to find a suboptimal control law for LQR problems. It is shown in [10] that, if the performance guided hybrid feedback law provides a valid switching sequence, then the resultant cost J at time t_0 is never worse than the best non-switching control.

Maximum principle and time optimal control: In [6], the authors regard a switching sequence σ as an input $\sigma(t) = (\sigma_1(t), \sigma_2(t), \cdots, \sigma_M(t))^T$, with $\sigma(t) \in D = \{\alpha \in \{0,1\}^M | \sum_{j=1}^M \alpha_j = 1\}$ for $\forall t \in [t_0, t_f]$ and then write a switched system equation as $\dot{x}(t) = \sum_{k=1}^M \sigma_k(t) f_k(x(t), u(t))$. The authors also formulated the cost functional in Problem 1 as $J = \sum_{k=1}^M \int_{t_0}^{t_f} \sigma_k(t) L_k(x(t), u(t))dt$, with L_k, $\frac{\partial L_k}{\partial x}$ defined and continuous on $\mathbb{R}^n \times U$. The above formulation allows the use of the Maximum Principle to obtain necessary conditions for optimal solutions. Moreover, the authors consider the time optimal control problems for switched systems with linear subsystems. Conditions for generality of position are given in [6] and the optimal control is a bang-bang control under this condition. However, it is still an important and largely open problem how to find the optimal trajectory from the solutions satisfying the necessary conditions.

Switched systems with internal switchings: As mentioned in Remark 2, internal switchings come along with a switched system with switching boundary (or active region for each subsystem) or time constraints being given. In this case, to each trajectory $\gamma(x, u, t)$ of the system that connects x_0 and x_f, there correspond points $t_0 < t_1 < \cdots < t_K = t_f$ and indices $i_0, \cdots, i_K$ such that on $[t_k, t_{k+1})$ the trajectory is in the active region for subsystem i_k and at $t_{i_{k+1}}$ it switches from subsystem i_k to i_{k+1}. This implies that we have a switching sequence generated by internal switchings $\sigma = ((t_0, i_0), (t_1, (i_0, i_1)), \cdots, (t_K, (t_{K-1}, t_K)))$. The optimal control problem is then to find an optimal input $u \in \mathcal{U}$ which minimizes the cost functional $J = \int_{t_0}^{t_f} L(x, u, t)dt$ with initial state x_0 and final state x_f. Numerical methods can be used to compute the solutions [12]. [5] addressed a discrete-time version of the problem and a numerical method was proposed. [7] proved the existence of an optimal solution for switched systems with two subsystems under some additional assumptions.

6 Conclusions

This paper formulates optimal control problems for switched systems and proposes some solution methods. Both continuous-time and discrete-time switched systems are considered. A two stage optimization method and a DP approach are studied in detail. It is shown that with the additional assumption of minimum dwelling time switching or costs for switchings, the problem complexity can be reduced. Comments on complexity and difficulties in finding solutions of the problem are made in the paper. Optimal control problems for switched systems are still in their early stage of investigation, even for linear switched systems, because the involvement of switchings complicates the behaviors of the system and makes the system nonlinear. Open problems including the existence of a solution, the exploration of necessary conditions, the solution of the HJB equation, the use of numerical methods, etc., still present considerable challenges.

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