ABSTRACT

The exact relation between the interactor and the Hermite normal form of a system $P$ is established and their relation to state feedback compensation is shown. The Smith-McMillan form at infinity of $P$ is then derived from these canonical forms.

INTERACTOR AND HERMITE NORMAL FORM

The interactor $\xi_P$ of a proper plant $P(p\times m)$ and its extension, the Hermite normal form $H_P$, were introduced in [1], [2] respectively as appropriate canonical forms of $P$ under dynamic compensation. It was shown in [3] that $H_P = \xi_P^{-1}$ when $P$ nonsingular. The main difficulty in establishing the relation between $\xi_P$ and $H_P$ in the general case lies in the fact that $\xi_P$ in [1] is defined only when $P$ has full rank. A generalized version of the interactor is introduced here to overcome this difficulty.

If rank $P = r = p(m)$, the interactor is defined in [1] as the unique polynomial matrix $\xi_P$ ($p \times p$) which satisfies:

$$\lim_{s \to \infty} \xi_P P = K_P, \quad \text{rank } K_P = p$$

with

$$\xi_P = \begin{bmatrix} 1 & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 1 \\ \end{bmatrix} \text{ diag } [s^i]$$

where $u_{ij}$ is divisible by $s$ (or is 0).

The generalized interactor of a proper $P$, where rank $P = r < \min(p,m)$, is defined as follows: Consider the top first $r$ lin. indep. rows of $P$ and let $P_r (r \times p)$ denote these rows; let $P_{p-r}$ denote the remaining $p-r$ rows of $P$. This interchange of rows can be expressed as

$$C P = \begin{bmatrix} P_r \\ P_{p-r} \end{bmatrix}$$

where $C$ is nonsingular with entries 0 and 1. Define the interactor $\xi_P$ of $P$ by:

$$\xi_P = \begin{bmatrix} \xi_{Pr} \\ \xi_{P_{p-r}} \end{bmatrix}$$

where $\xi_{Pr}$ is the interactor of $P_r$ defined in (1), (2) and

$$[y_r, y_{p-r}] C P = \begin{bmatrix} [y_r, y_{p-r}] P_r \\ [y_r, y_{p-r}] P_{p-r} \end{bmatrix} = 0$$

where $[y_r, y_{p-r}] = \gamma$ a minimal basis of the left kernel of $C P$ with $y_{p-r}$ row proper and in (lower left) Hermite normal form; note that such basis is uniquely specified by $CP$ [4]. The unique $\xi_P$ satisfies:

$$\lim_{s \to \infty} \xi_P P = \lim_{s \to \infty} \begin{bmatrix} \xi_{Pr} P_r \\ \xi_{P_{p-r}} P_{p-r} \end{bmatrix} = K_P, \quad \text{rank } K_P = r$$

When rank $P = r = \min(p,m)$ and the top $r$ rows of $P$ are lin. indep. then $C = I$ and the above definition reduces to the definition of the interactor in [1].

The Hermite normal form $H_P$ of $P$, where rank $P = r < \min(p,m)$ satisfies [2]:

$$P K_P = \begin{bmatrix} H_P \end{bmatrix}$$

where $P$ biproper (i.e. $P, P^{-1}$ proper) and the top first $r$ lin. indep. rows of $H$ ($p \times p$) are:

$$H^* = \begin{bmatrix} 1/s & 0 \\ \cdots & \cdots \\ 0 & m_r \end{bmatrix}$$

where $q_{ij} = 0$ when $m_j = 0$ or $q_{ij} = a/s$ proper when $m_j > 0$. Here $H_P$ is the Hermite normal form of $P$ over the principal ideal domain of proper transfer functions ($s = \text{all monic polyn. in } R[s]$).

Proposition 1: $\xi_P H_P = \begin{bmatrix} I_r \\ 0 \\ 0 \\ \end{bmatrix}$

Proof: $\xi_P$ is defined in (4), $[y_r, y_{p-r}] C H_P = \gamma C \begin{bmatrix} P \\ 0 \end{bmatrix} = 0$ in view of (5), (7). The first $r$ rows of $C H_P$ are $[H^*, 0]$, that is $[\xi_{Pr} C H_P] = [\xi_{Pr} H^*, 0] = [I_r 0]$ since it has been shown in [3], that $\xi_{Pr} H^* = I$. ΔΔΔ

Linear state feedback (lsf) compensation. It is now shown that $H_P$ can be obtained as the closed loop transfer matrix when appropriate lsf is applied on $P$. To define lsf, consider the factorization $P = ND^{-1}$ which corresponds to the controllable realization $Dz = u$, $y = Nz$ [5]. Let $D$ be column proper with column degrees $s_{Di} = d_i$, and define the lsf control law $(F,G)$ by:

$$u = r z, G = I \quad \text{real} \quad |G| = 0 \quad \text{the closed loop transfer matrix is } N(D-F)^{-1} C = (ND^{-1}) \quad \Delta \Delta \Delta$$

Lemma 2. Let rank $P = r = p(< m)$ and let 0 be a ($p \times p$) polynomial matrix such that

$$\lim_{s \to \infty} 0P = P_K, \quad \text{rank } K_P = p$$

Then there exists lsf $(F,G)$ such that

$$0 P F_F, G = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$$

If $p = m$, $(F,G)$ is unique.

Proof: Find $F$ and $G$ so that ON = $K_P(D-F)$ and $K_P G = \begin{bmatrix} Ip, 0 \end{bmatrix}$. It can be shown that such $(F,G)$ always exists; it is unique when $p = m$.

Proposition 3 Let rank $P = r < \min(p,m)$. There exists lsf $(F,G)$ such that

$$P F_F, G = \begin{bmatrix} I_r \\ 0 \\ 0 \end{bmatrix}$$

If $r = m$, $(F,G)$ is unique.

The interactor $\xi_P$ and the Hermite normal form $H_P$ of $P$ are established and shown in [3] that $H_P = \xi_P^{-1}$. The Smith-McMillan form at infinity of $P$ is then derived from these canonical forms.
Proof: Use Lemma 2 to find \( \bar{P}_{F,G} \) such that \( \xi_F P_T \bar{P}_{F,G} \Delta \xi_F \).

A special case of this result (mp) has been shown in [6]; note also that \( P_{F,G} = P_T = 1 \) in [2] and used in [3] and elsewhere. Here \( (P,G) \) is easily derived and it is shown to be unique when \( P = m \).

**ZERO STRUCTURE AT INFINITY**

\( P(s) \) has a finite zero of order \( k \) if \( P(1/\lambda) \) has a finite zero of order \( k \) at \( \lambda = 0 \) [4,7,8]. The infinite zeros \( \zeta \) of \( P \) are directly available if the Smith-McMillan factorization at infinity is known, namely

\[
P = B_1 \begin{bmatrix} A_r & 0 \\ 0 & \hat{A}_2 \end{bmatrix} B_2 \tag{13}
\]

where \( B_1, B_2 \) biproper and \( \hat{A}_2 \) is diagonal. Note that a version of Silverman’s structure algorithm was used in [15] to derive \( B_1, B_2 \) of (13). Lemma 6 then the row degrees of \( c_p \) are the zero orders of \( P \). Note that a version of Silverman’s structure algorithm was used in [15] to derive \( B_1, B_2 \) of (13).

**Proposition 4.** Assume rank \( P = p \) and let \( P \) satisfy (10). If \( P \) is row proper, its row degrees are the infinite zero orders of \( P \). Proof: Interchange rows so that \( K_0(s) \) has row degrees \( \hat{\zeta} \) and write \( K_0 = \text{diag} s^{-\hat{\zeta}} \hat{\zeta} \). \( \hat{\zeta} \) is biproper.

In view of (11), (13) is derived with \( B_1 = \hat{0}^{-1}, B_2 = [K \ 0] \). Smith form is \( S(1/s) \) is the Smith McMillan form at infinity of \( F \).

**Lemma 6.** Let rank \( P = p \). There exists real nonsingular matrix \( C \) so that \( \xi_F C_p \) is row proper.

Let rank \( P = r < \min (p,m) \) and choose \( C \) in (4) as follows: Find row proper minimal basis \( y \) of left kernel of \( P \) and collect \( r-\gamma \) columns of \( y \) to obtain \( y_{\gamma} \). Let \( y_{\gamma} \) row proper with row degrees those of \( y \); this specifies \( P_{F,G} \). Note that \( y_{\gamma}^{-1} y_{\gamma} \) proper. \( P_{F,G}^{-1} y_{\gamma} \) specifies the zero orders at infinity (Prop. 5). If the remaining \( r-\gamma \) rows of \( P \) are rearranged to satisfy Lemma 6 then the row degrees of \( \xi_F P_T \) are the zero orders at infinity of \( P \). Having established the relation between the zeros at infinity and \( H_{P}(\xi_F) \), it is straightforward to study the effect of feedback and cascade compensation on these zeros.

Example \( P = \begin{bmatrix} 1/s+1 & 1/s+2 \\ 1/s+3 & 1/s+4 \end{bmatrix} \); \( \xi_{P_T} = \begin{bmatrix} s & 0 \\ -s^2 & s^3 \end{bmatrix} \) row proper with row degrees 1, 3 the infinite zero orders of \( P \) (Prop. 4). The Smith McMillan form at infinity is given by (13) with

\[
\begin{align*}
\lambda_p &= \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix}, \quad B_1 = \xi_{\lambda_p}^{-1} \begin{bmatrix} 1 & 0 \\ s^{-2}/s & 1 \end{bmatrix} \\
B_2 &= \bar{P}_{F,G} = \begin{bmatrix} 6s^2/(s+1)(s+3) & 8s^2/(s+2)(s+4) \end{bmatrix} \\
\end{align*}
\]

Note that \( H_p(1/\lambda) = \begin{bmatrix} -1 & -2 & 3 \end{bmatrix} \) (Prop. 5)

If \( P \) as above but with \( s \) on the second row numerators then

\[
\xi_p = \begin{bmatrix} s & 0 \\ -s^2 & 2s^2 & s^3 \end{bmatrix}
\]

which is not row proper. Interchange rows of \( P \), that is

\[
C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

Then \( \xi_{C_F} = \begin{bmatrix} -s^2 & 2s & s^3 \end{bmatrix} \) which is row proper with row degrees the infinite zero orders of \( P \) (Lemma 6, Prop. 4).

**REFERENCES**


