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A Dynamic Programming Approach for Optimal Control of Switched Systems

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Abstract

In optimal control problems of switched systems, in general, one needs to find both optimal continuous inputs and optimal switching sequences, since the system dynamics vary before and after every switching instant. In a previous paper, we proved that an optimal control problem can be posed as a two stage optimization problem under some additional assumptions. In general, the two stage optimization problem is still difficult to solve analytically. In this paper, we develop a search algorithm to explore the solution of the two stage optimization problem and find useful suboptimal solutions. This algorithm is motivated by the idea of dynamic programming which studies the value functions. The algorithm is used to determine suboptimal solutions for general switched linear quadratic problems.

1 Introduction

A switched system is a hybrid system that consists of several subsystems and a switching law indicating the active subsystem at each time instant. Examples of switched systems can be found in chemical processes, automotive systems, and electrical circuit systems, etc.

Optimal control problems of switched systems and hybrid systems have been attracting recently researchers from various fields in science and engineering and several new results have appeared in the literature. Some of them are primarily theoretical (see, e.g., [3, 12, 13, 16]). For example, Sussmann has proved a maximum principle for such problems in [13]. In [3, 16], Capuzzo Dolcetta and Yong studied systems with switchings using dynamic programming approaches and proved the existence and uniqueness of viscosity solutions. However, there were no efficient and constructive methodologies suggested in these papers for finding optimal solutions and there is a significant gap between theoretical results and their applications to real-world examples.

The advent of high speed computers and efficient nonlinear optimization and search techniques has led to methodologies on solving hybrid optimal control problems (see e.g., [1, 2, 6, 8, 9, 11, 14]). For example, in [1], general formulations and algorithms for optimal control of hybrid systems were given. In [9], a novel algorithm using constrained differential dynamic programming was proposed for a class of discrete-time hybrid-state systems. It is worth mentioning that because there are many different models and optimal control objectives for hybrid systems, these papers often differ greatly in their problem formulations and approaches. Switched systems, on the other hand, tend to be described by similar models and similar optimal control problem formulations have appeared in the literature. (e.g, [6, 8, 9, 11, 14, 15]). For an optimal control problem of a switched system, one needs to find both an optimal continuous input and an optimal switching sequence since the system dynamics vary before and after every switching instant. The problem of determining optimal solutions to such problems are in general very difficult. Most of the methods in the literature that we are aware of are based on some discretization of continuous-time space and/or discretization of state space into grids and use search methods for the resultant discrete model to find optimal/suboptimal solutions. But the discretization approaches may lead to computational combinatoric explosions and the solutions obtained may not be accurate enough. In view of this, in this paper, we will explore methods that are not based on discretizations.

In a previous paper [15], we have proved that an optimal control problem can be formulated as a two stage optimization problem under some additional assumptions. In this paper, we will further investigate the two stage approach and develop numerical algorithms. We first translate the two stage optimization into a general algorithm. Then we focus on the first part of the algorithm (i.e., Step 1) and develop search algorithms for finding the optimal continuous input and optimal switching instants under the assumption that the number of switchings and the order of active subsystems are already given. This actually gives us a suboptimal control for the original optimal control problem. Note that in many practical situations, we only need to study problems with fixed number of switchings and fixed order of active subsystems (e.g., the speeding up of a power train only requires switchings from gear 1 to 2 to 3 to 4), so Step 1 itself is worth the attention as in such cases our solutions are indeed optimal. We will use an approach based on dynamic programming motivated by the first and second-order method in [4, 5] to obtain the first and second derivatives of the value function with respect to the switching instants. Then, by using nonlinear search methods, we will find optimal solutions. (Note that in [4, 5], the authors introduced a method to deal with discontinuities, but the method is only applicable to fuel optimal and bang-bang types of control problems.) Specifically, we will apply the method to general switched linear quadratic problems. Note that some of the computations encountered in general optimal control problems can be avoided in such kind of systems.

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2 Problem Formulation

2.1 Switched Systems

Switched systems

Definition 2.1 (Switched System) *A switched system is a tuple $\mathcal{S} = (\mathcal{F}, \mathcal{D})$ where*
- *$\mathcal{F} = \{f_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n,\ i \in I\}$ with f_i describing the vector field for the ith subsystem $\dot{x} = f_i(x, u, t)$. $I = \{1, 2, \cdots, M\}$ is the set of indices of subsystems.*
- *$\mathcal{D} = (I, E)$ is a simple finite state machine which can also be viewed as a directed graph. I is the set of indices as defined above. Here I serves as the set of discrete states indexing the subsystems. $E \subseteq I \times I - \{(i,i) | i \in I\}$ is a collection of events. If an event $e = (i, j)$ takes place, the switched system will switch from subsystem i to j.*

In view of Definition 2.1, a switched system is a collection of subsystems which are related by a switching logic restricted by $\mathcal{D}$. The continuous state x and the continuous input u satisfy $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. If a particular switching law has been specified, then the switched system can be described as

$$\dot{x}(t) = f_{i(t)}(x(t), u(t), t) \tag{1}$$

$$i(t) = \varphi(x(t), i(t^-), t), \tag{2}$$

where $\varphi : \mathbb{R}^n \times I \times \mathbb{R} \to I$ determines the active subsystem at time t. Note that (1)-(2) are used as the definition of switched systems in some of the literature. Here we adopt Definition 2.1 rather than (1)-(2) because in design problems, in general, φ is not defined a priori and it is a designer's task to find a switching law. A salient feature of a switched system is that its continuous state x does not exhibit jumps at switching instants.

Switching sequences

For a switched system $\mathcal{S}$, the inputs of the system consist of both a continuous input $u(t), t \in [t_0, t_f]$ and a switching sequence. We define a switching sequence as follows.

Definition 2.2 (Switching Sequence) *For a switched system $\mathcal{S}$, a switching sequence σ in $[t_0, t_f]$ is defined as*

$$\sigma = ((t_0, e_0), (t_1, e_1), (t_2, e_2), \cdots, (t_K, e_K)), \tag{3}$$

with $0 \le K < \infty$, $t_0 \le t_1 \le t_2 \le \cdots \le t_K \le t_f$, and $e_0 = i_0 \in I$, $e_k = (i_{k-1}, i_k) \in E$ for $k = 1, 2, \cdots, K$. (If $K = 0$, $\sigma = ((t_0, e_0))$.)

We define $\Sigma_{[t_0,t_f]} = \{\sigma\text{'s in } [t_0, t_f]\}$. We also define an untimed switching sequence as

$$\sigma^U = (e_0, e_1, e_2, \cdots, e_K). \tag{4}$$

A switching sequence σ as defined above indicates that, if $t_k < t_{k+1}$, then subsystem i_k is active in $[t_k, t_{k+1})$ ($[t_K, t_f]$ if $k = K$); if $t_k = t_{k+1}$, then i_k is switched through at instant t_k ('switched through' means that the system switches from subsystem i_{k-1} to i_k and then to i_{k+1} all at instant t_k). For a switched system to be well-behaved, we generally exclude the undesirable *Zeno* phenomenon, i.e., infinitely many switchings in finite amount of time. Hence in Definition 2.2, we only allow nonZeno sequences which switch at most a finite number of times in $[t_0, t_f]$, though different sequences may have different numbers of switchings. We specify $\sigma \in \Sigma_{[t_0,t_f]}$ as a discrete input to a switched system.

2.2 An Optimal Control Problem

Problem 2.1 *For a switched system $\mathcal{S} = (\mathcal{F}, \mathcal{D})$, find a switching sequence $\sigma \in \Sigma_{[t_0,t_f]}$ and an input $u \in \mathcal{U} = \{$piecewise continuous function u on $[t_0, t_f]$ with $u(t) \in U \subseteq \mathbb{R}^m$, $\forall t \in [t_0, t_f]\}$ such that the cost functional*

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt \tag{5}$$

is minimized, where t_0, t_f and $x(t_0) = x_0$ are given, $\psi : \mathbb{R}^n \to \mathbb{R}$, $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$.

Problem 2.1 is a fixed final time, free final state problem. Although for a continuous-time system $\dot{x} = f(x, u, t)$ there have been many methods for finding solutions for optimal control problems such as the calculus of variations method, the maximum principle and the Hamilton-Jacobi-Bellman equation, etc., the involvement of σ makes the dynamics of the system vary in $[t_0, t_f]$, so the problem is in general difficult to solve.

2.3 Two Stage Optimization

In general, we need to find an optimal control input (σ^*, u^*) for Problem 2.1 such that

$$J(\sigma^*, u^*) = \min_{\sigma \in \Sigma_{[t_0,t_f]},\ u \in \mathcal{U}} J(\sigma, u). \tag{6}$$

Notice that for any given switching sequence σ, Problem 2.1 reduces to a conventional optimal control problem for which we only need to find an optimal continuous input u so as to minimize $J_\sigma(u) = J(\sigma, u)$. In a previous paper [15], we have proved the following lemma which provides a way to formulate (6) into a two stage optimization problem.

Lemma 2.1 *For Problem 2.1, if*
(1). an optimal solution (σ^*, u^*) exists and
(2). for any given switching sequence σ, there exists a corresponding $u^* = u^*(\sigma)$ such that $J_\sigma(u)$ is minimized, then the following equation holds

$$\min_{\sigma \in \Sigma_{[t_0,t_f]},\ u \in \mathcal{U}} J(\sigma, u) = \min_{\sigma \in \Sigma_{[t_0,t_f]}} \min_{u \in \mathcal{U}} J(\sigma, u). \tag{7}$$

The right hand side of (7) is a two stage optimization problem and the following two stage optimization method can be adopted for solving it.

Two stage optimization method

Stage 1: Fixing σ, solve the inner minimization problem.
Stage 2: Regarding the optimal cost for each σ as a function $J_1 = J_1(\sigma) = \min_{u \in \mathcal{U}} J(\sigma, u)$, minimize J_1 with respect to $\sigma \in \Sigma_{[t_0,t_f]}$.

We can implement the above method by the following algorithm.

Two Stage Algorithm

Step 1. (a) Fix the total number of switchings to be K and the order of active subsystems, let the minimum value of J with respect to u be a function of the K switching instants, i.e., $J_1 = J_1(t_1, t_2, \cdots, t_K)$ for $K \geq 0$, and then find J_1.

(b) Minimize J_1 with respect to $t_1, t_2, \cdots, t_K$.

Step 2. (a) Vary the order of active subsystems to find an optimal solution under K switchings.

(b) Vary the number of switchings K to find an optimal solution for Problem 2.1.

The above algorithm has high computational costs. In practice, we usually find suboptimal solutions for optimal control problems with fixed number of switchings and fixed order of active subsystems by using Step 1 (In fact, in many practical situations we only need to study such problems.). Note that in Step 1, we consider the case of fixed number of switchings with fixed order of active subsystems. Step 1 can further be separated into two substeps (a) and (b); Step 1(a) finds the optimal cost and optimal continuous input with fixed switching instants, and Step 1(b) searches for optimal switching instants.

In the following, we will focus on Step 1 of the above algorithm and search for both an optimal continuous inputs and optimal switching instants, given fixed number of switchings and fixed order of active subsystems . In general, explicit representation of J_1 is difficult to obtain, therefore we need to use optimization methods that do not require explicit representation of J_1 to develop numerical methods.

3 A Dynamic Programming Approach

In the following we develop a search algorithm for finding the optimal switching instants for Step 1 in the algorithm so as to minimize J_1. This is motivated by the idea of the dynamic programming by studying the value functions. Note that in the following, the value functions we use may not be the optimal value functions under fixed switching sequences, except that in Section 4, they are indeed optimal for fixed switching sequences. Although some equations we will use is similar to the equations obtainable by the dynamic programming approach, our purpose is only to use these equations to find the derivatives of the value function with regard to the switching instants and so the combinatoric explosion issue in dynamic programming will not be of concern here.

We assume that the number of switchings is K and the order of subsystems is $i_0, i_1, \cdots, i_K$. In other words, the switching sequences we are interested in are of the form

$$\sigma = ((t_0, i_0), (t_1, (i_0, i_1)), \cdots, (t_K, (i_{K-1}, i_K))), \qquad (8)$$

where $t_0, t_1, \cdots, t_K$ are yet unknown. We will search for optimal switching instants $t_1, \cdots, t_K$ (i.e., Step 1(b)).

In [5], based on a differential dynamical programming approach, the authors developed a method to deal with discontinuities and used it in fuel optimal and bang-bang type of control problems. Motivated by their approach, first we will develop our search algorithm and identify the difficulties of the approach for solving general switched optimal control problems. In the next section, we will then focus on a special class of switched systems, i.e, the general switched linear quadratic problems, for which many difficulties can be avoided.

Let us assume that we have a nominal control input $u(t), t \in [t_0, t_f]$ and nominal switching instants $t_1, t_2, \cdots, t_K$ (if possible, choose $u(\cdot)$ to be an optimal input corresponding to the current values of switching instants). Assume $u(\cdot)$ does not vary and we want to optimize the cost by only varying the switching instants, the value function V^0 at t_0 (may not be the optimal value function) will then depend on $x_0, t_0, t_1, \cdots, t_K$ only. Similarly, the value function V^i at t_i will depend on $x(t_i), t_i, t_{i+1}, \cdots, t_K$ only.

For simplicity of notation, let's now consider the case of a single switching. The result for K switchings can be similarly derived. In the following derivation, we write a function with a superscript 0 whenever we consider the function for $t \in [t_0, t_1)$ and write a function with a superscript 1 whenever we consider the function for $t \in [t_1, t_f]$. Also we assume that all functions with superscript 0 (resp. 1) are evaluated at t_1- (resp. t_1+) if they are to be evaluated at t_1. First of all, it is not difficult to see that

$$V^0(x_0, t_0, t_1) = V^1(x(t_1), t_1) + \int_{t_0}^{t_1} L^0(x, u, t)dt$$

Now if t_1 has a small variation dt_1, we have

$$\begin{aligned}
& V^0(x_0, t_0, t_1 + dt_1) \\
= \; & V^1(x(t_1 + dt_1), t_1 + dt_1) + \int_{t_0}^{t_1+dt_1} L^0(x, u, t)dt \\
= \; & V^1(x(t_1), t_1) + \int_{t_0}^{t_1} L^0(x, u, t)dt + V_x^1 dx(t_1) \\
& + V_{t_1}^1 dt_1 + L^0 dt_1 + \frac{1}{2}(dx(t_1))^T V_{xx}^1 dx(t_1) \\
& + \frac{1}{2} V_{t_1 t_1}^1 dt_1^2 + dt_1 V_{t_1 x}^1 dx(t_1) + \frac{1}{2} dt_1 L_x^0 dx(t_1) \\
& + \frac{1}{2} dt_1 L_u^0 du(t_1) + \frac{1}{2} L_t^0 dt_1^2 + o(dt_1^2) \qquad (9)
\end{aligned}$$

where

$$\begin{aligned}
dx(t_1) &= f^0 dt_1 + \frac{1}{2}(f_t^0 + f_x^0 f^0 + f_u^0 \dot{u}^0) dt_1^2 \\
& \quad + o(dt_1^2) \qquad (10) \\
du(t_1) &= \dot{u}^0 dt_1 + o(dt_1) \qquad (11)
\end{aligned}$$

By substituting (10) and (11) into (9), we obtain

$$\begin{aligned}
& V^0(x_0, t_0, t_1 + dt_1) \\
= \; & V^0(x_0, t_0, t_1) + (V_x^1 f^0 + V_{t_1}^1 + L^0) dt_1 \\
& + \frac{1}{2}[(f^0)^T V_{xx}^1 f^0 + V_x^1 (f_t^0 + f_x^0 f^0 + f_u^0 \dot{u}^0) \\
& + 2V_{t_1 x}^1 f^0 + V_{t_1 t_1}^1 + L_x^0 f^0 + L_u^0 \dot{u}^0 + L_t^0] dt_1^2 + o(dt_1^2) \\
\triangleq \; & V^0(x_0, t_0, t_1) + V_{t_1}^0 dt_1 + \frac{1}{2} V_{t_1 t_1}^0 dt_1^2 + o(dt_1^2) \qquad (12)
\end{aligned}$$

Now since $V^1(x(t_1), t_1)$ is the value function for fixed $u(\cdot)$, we have the relationship

$$V_{t_1}^1 + V_x^1 f^1 + L^1 = 0 \qquad (13)$$

By differentiating (13), we can further have the following relations.

$$
\begin{aligned}
V^1_{t_1x} &= -(f^1)^T V^1_{xx} - V^1_x f^1_x - L^1_x & (14)\\
V^1_{t_1t_1} &= -V^1_{t_1x} f^1 - V^1_x f^1_t - L^1_t - (V^1_x f^1_u + L^1_u)\dot{u}^1 \\
&= (f^1)^T V^1_{xx} f^1 + (V^1_x f^1_x + L^1_x) f^1 - V^1_x f^1_t - L^1_t \\
&\quad -(V^1_x f^1_u + L^1_u)\dot{u}^1 & (15)
\end{aligned}
$$

With the help of (13), (14), (15), we can write $V^0_{t_1}$ and $V^0_{t_1t_1}$ in the following forms.

$$
\begin{aligned}
V^0_{t_1} &= L^0 - L^1 + V^1_x(f^0 - f^1) & (16)\\
V^0_{t_1t_1} &= (f^0 - f^1)^T V^1_{xx}(f^0 - f^1) - (V^1_x f^1_x + L^1_x)(f^0 \\
&\quad - f^1) + (V^1_x(f^0_x - f^1_x) + L^0_x - L^1_x) f^0 \\
&\quad + V^1_x(f^0_t - f^1_t) + L^0_t - L^1_t + (V^1_x f^0_u + L^0_u)\dot{u}^0 \\
&\quad -(V^1_x f^1_u + L^1_u)\dot{u}^1 & (17)
\end{aligned}
$$

Once we know $V^0_{t_1}$, $V^0_{t_1t_1}$, we can use the first or second-order search method of nonlinear programming to optimize the switching instant (see [10]). However, there are two difficulties associated with this approach.

(a). These conditions are derived under the assumption that $u(\cdot)$ is not varying when optimizing (for at least one iteration). Yet in most cases, when the switching instants vary, the control input would vary correspondingly. Therefore, this approach can only give us the optimal switching instants for the nominal $u(\cdot)$.

(b). In general, $V^1_x, V^1_{xx}, \dot{u}^0, \dot{u}^1$ can only be determined after significant computational effort has been made.

Problem (a) may be addressed by updating the $u(\cdot)$ to be the corresponding optimal input for the new switching instant at each new iteration. However, in this way, we still may not be able to find the optimal control input. For problem (b), we can find out the values for V^1 at $(x(t_1), t_1)$ by integration and obtain a numerical approximation of V^1_x, (resp. V^1_{xx}) by observing the variation of V^1 (resp V^1_{xx}) with respect to small variation of x. Similarly we can also find approximations for $\dot{u}^0, \dot{u}^1$. Another way to deal with problem (b) is to use the differential dynamic programming approach introduced in [7]. With the above discussions in mind, our algorithm for a single switching can be summarized as follows.

A Second-Order Search Algorithm

Step 1. Choose nominal $u(\cdot)$ and switching instant t_1.

Step 2. Fix $u(\cdot)$ and calculate $V^0_{t_1}$ and $V^0_{t_1t_1}$.

Step 3. Update $t_1^{new} = t_1 - (V^0_{t_1t_1})^{-1} V^0_{t_1}$ (If t_1^{new} is outside $[t_0, t_f]$, enforce it to be on the appropriate boundary of $[t_0, t_f]$).

Step 4. Update $u(\cdot)$ by finding optimal (or suboptimal) control input for the new switching sequence.

Step 5. Repeat Step 2 to Step 4 until $\|(V^0_{t_1t_1})^{-1} V^0_{t_1}\|_2$ is smaller than a given tolerance value $\epsilon > 0$.

4 Application to General Switched Linear Quadratic Problems

In this section, we consider a special class of optimal control problems for switched systems, i.e., general switched linear quadratic problems. For this class of problems, the two difficulties mentioned in Section 3 can be avoided. First of all, we state the problem as follows.

Problem 4.1 *For a switched system $\mathcal{S} = (\mathcal{F}, \mathcal{D})$, with linear subsystems $\dot{x} = A_i x + B_i u, i = 1, 2, \cdots, M$ and a given untimed switching sequence $\sigma^U = (i_0, (i_0, i_1), \cdots, (i_{K-1}, i_K))$, find the optimal switching instants $t_1, \cdots, t_K$ and optimal control input $u(\cdot)$ such that the cost functional in general quadratic form*

$$
\begin{aligned}
J &= \frac{1}{2}x(t_f)^T Q_f x(t_f) + M_f x(t_f) + L_f + \int_{t_0}^{t_f}(\frac{1}{2}x^T Q x \\
&\quad + x^T V u + \frac{1}{2}u^T R u + M x + N u + W)dt \qquad (18)
\end{aligned}
$$

is minimized, where t_0, t_f and $x(t_0) = x_0$ are given, $Q_f, Q \geq 0$ and $R > 0$.

Note that for general quadratic control of a linear system $\dot{x} = Ax + Bu$, we can use the dynamic programming approach to obtain the following results.

The optimal value function is

$$V^*(x,t) = \frac{1}{2}x^T P(t) x + S(t)x + T(t) \qquad (19)$$

where $P(t) = P^T(t)$ and

$$
\begin{aligned}
-\dot{P}(t) &= Q + P(t)A + A^T P(t) - (P(t)B \\
&\quad + V)R^{-1}(B^T P(t) + V^T) & (20)\\
-\dot{S}(t) &= M + S(t)A - (N + S(t)B)R^{-1}(B^T P(t) \\
&\quad + V^T) & (21)\\
-\dot{T}(t) &= W - \frac{1}{2}(N + S(t)B)R^{-1}(B^T S^T(t) \\
&\quad + N^T) & (22)
\end{aligned}
$$

and the optimal control is in the feedback form

$$u(x(t), t) = -K(t)x(t) - E(t) \qquad (23)$$

where

$$
\begin{aligned}
K(t) &= R^{-1}(B^T P(t) + V^T) & (24)\\
E(t) &= R^{-1}(B^T S^T(t) + N^T) & (25)
\end{aligned}
$$

Focusing now on the general switched linear quadratic problem, assume that for any nominal switching instants, we always choose $u(\cdot)$ in the form (23) which is optimal in this case. We can derive $V^0_{t_1}$ and $V^0_{t_1t_1}$ by modifying the derivation in Section 3. And this time we choose the nominal $K(\cdot)$ and $E(\cdot)$ rather than $u(\cdot)$ to be fixed at each iteration of optimization (but be updated after the iteration). This can give us the flexibility of letting $u(\cdot)$ vary as a function of x since u depends on x now (see (23)).

The derivation of $V_{t_1}^0$ and $V_{t_1t_1}^0$ is as follows. Equations (9)-(13) remains unchanged, while (14) and (15) will become

$$\begin{aligned} V_{t_1x}^1 &= -(f^1)^T V_{xx}^1 - V_x^1 f_x^1 - L_x^1 \\ &\quad -(V_x^1 f_u^1 + L_u^1) u_x^1 \qquad (26) \\ V_{t_1t_1}^1 &= -V_{t_1x}^1 f^1 - V_x^1 f_t^1 - L_t^1 - (V_x^1 f_u^1 + L_u^1) u_t^1 \\ &= (f^1)^T V_{xx}^1 f^1 + (V_x^1 f_x^1 + L_x^1) f^1 - V_x^1 f_t^1 - L_t^1 \\ &\quad -(V_x^1 f_u^1 + L_u^1) \dot{u}^1 \qquad (27) \end{aligned}$$

And consequently the form of $V_{t_1}^0$ is still the same as equation (16)and

$$\begin{aligned} & V_{t_1t_1}^0 \\ = \; & (f^0 - f^1)^T V_{xx}^1 (f^0 - f^1) - (V_x^1 f_x^1 + L_x^1)(f^0 - f^1) \\ & +(V_x^1 (f_x^0 - f_x^1) + L_x^0 - L_x^1) f^0 \\ & +V_x^1 (f_t^0 - f_t^1) + L_t^0 - L_t^1 + (V_x^1 f_u^0 + L_u^0) \dot{u}^0 \\ & -(V_x^1 f_u^1 + L_u^1)(2 u_x^1 f^0 + \dot{u}^1). \qquad (28) \end{aligned}$$

It can now be seen from the expressions of $V_{t_1}^0$ and $V_{t_1t_1}^0$ that all terms necessary for the evaluation of them are readily available..

$$\begin{aligned} V_x^1 &= x^T P^1 + S^1 \qquad (29) \\ V_{xx}^1 &= P^1 \qquad (30) \\ \dot{u}^0 &= -\dot{K}^0 x - K^0 f^0 - \dot{E}^0 \qquad (31) \\ \dot{u}^1 &= -\dot{K}^1 x - K^1 f^1 - \dot{E}^1 \qquad (32) \\ u_x^1 &= -K^1 \qquad (33) \end{aligned}$$

where the functions with superscript 0 (resp. 1) are evaluated at t_1- (resp. t_1+); x is continuous at t_1; $\dot{K}^0, \dot{K}^1, \dot{E}^0, \dot{E}^1$ can be represent as functions of P, S by substituting (20) and (21) into the differentiation of (24) and (25).

Now that we have the expressions for $V_{t_1}^0$ and $V_{t_1t_1}^0$, we can modify the algorithm in Section 3 to find the optimal switching instants. Note here $K(\cdot)$ and $E(\cdot)$ are assumed to be fixed at each iteration, but $u(\cdot)$ varies as a function of x, therefore this algorithm is better than the one in Section 3. And for this specific problem, we can easily obtain the forms and values of $V_x^1, V_{xx}^1, \dot{u}^0, \dot{u}^1, u_x^1$ which avoids difficulty (b) at the end of Section 3.

Example 4.1 *Consider a switched system consisting of subsystem 1:*

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

and subsystem 2:

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$$

Assume that $t_0 = 0$, $t_f = 2$ and the system switches once at $t = t_1$ ($0 \le t_1 \le 2$) from subsystem 1 to 2. We want to find an optimal switching instant t_1 and an optimal input u such that $x(0) = [1\ 1]^T$ and $x(2)$ is close to $[e\ e]^T$ and the cost functional $J = \int_0^2 u^2(t)dt$ is minimized.

For this problem, we add to J a penalty term $\frac{1}{2}[(x_1(2) - e)^2 + (x_2(2) - e)^2]$ and then consider the expanded cost functional J_{exp}. By using the second-order search algorithm, we find that the optimal switching instant is $t_1 = 1.0189$ and the corresponding optimal cost is 0.0063. The corresponding state trajectory and control input $u(\cdot)$ are shown in Figure 1(a),(b). This numerical solution is close to the theoretical optimal solution $t_1^{opt} = 1$, $J_{exp}^{opt} = 0$ and $u^{opt} \equiv 0$.

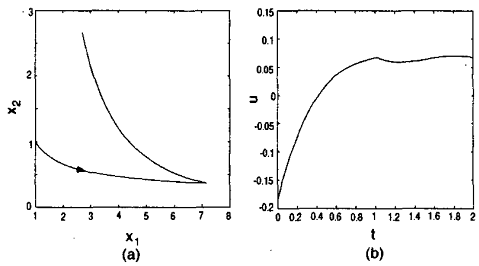

Figure 1: Example 4.1 (a) The state trajectory (b) The control $u(t)$.

Example 4.2 *Consider a switched system consisting of subsystem 1:*

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

and subsystem 2:

$$\dot{x} = \begin{bmatrix} 0.5 & 5.3 \\ -5.3 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

and subsystem 3:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 1 to 2 and at $t = t_2$ from subsystem 2 to 3 ($0 \le t_1 \le t_2 \le 3$). We want to find optimal switching instants t_1, t_2 and an optimal input u such that $x(0) = [4\ 4]^T$ and $x(3)$ is close to $[-4.1437\ 9.3569]^T$ and the cost functional $J = \int_0^2 u^2(t)dt$ is minimized.

For this problem, we add to J a penalty term $[(x_1(3) + 4.1437)^2 + (x_2(3) - 9.3569)^2]$ and then consider the expanded cost functional J_{exp}. By using the second-order search algorithm with initial values $t_1 = 0.8$, $t_2 = 1.8$, after 24 iterations we find that the optimal switching instant is $t_1 = 1.0187$, $t_2 = 2.0318$ and the corresponding optimal cost is 0.0515. The corresponding state trajectory and control input $u(\cdot)$ are shown in Figure 2(a),(b). This numerical solution is close to the theoretical optimal solution $t_1^{opt} = 1$, $t_2^{opt} = 2$, $J_{exp}^{opt} = 0$ and $u^{opt} \equiv 0$.

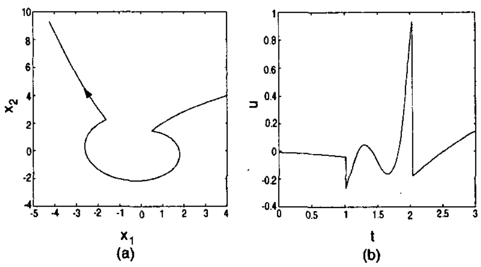

Figure 2: Example 4.2 (a) The state trajectory (b) The control $u(t)$.

5 Conclusion

In this paper, a search algorithm is developed to find optimal switching instants. The method is motivated by the method in [4, 5]. The difficulties of the application of the algorithm are pointed out. For the special class of general switched linear quadratic problems, some of the difficulties can be addressed efficiently as we show in Section 4.

References

[1] M. S. Branicky, S. K. Mitter, *Algorithms for optimal hybrid control*, In Proceedings of the 34th IEEE Conference on Decision and Control, pp. 2661-2666, 1995.

[2] M. S. Branicky, R. Hebbar, G. Zhang, *A fast marching algorithm for hybrid systems*, In Proceedings of the 38th IEEE Conference on Decision and Control, pp. 4897-4902, 1999.

[3] I. Capuzzo Dolcetta and L. C. Evans, *Optimal switching for ordinary differential equations*, SIAM J. Control and Optim., Vol. 22, No. 1, pp. 143-161, January 1984.

[4] P. Dyer, S.R. McReynolds, *On optimal control problems with discontinuities*, J. of Math. Anal. and Appl., **23**, pp. 585-603, 1968.

[5] P. Dyer, S.R. McReynolds, *The computation and theory of optimal control*, Academic Press, 1970.

[6] S. Hedlund, A. Rantzer, *Optimal control of hybrid systems*, In Proceedings of the 38th IEEE Conference on Decision and Control, pp. 3972-3977, 1999.

[7] D. H. Jacobson, *Second-order and second-variation methods for determining optimal control: a comparative study using differential dynamic programming*, Int. J. of Control, **7**(2), pp. 175-196, 1968.

[8] M. Johansson, *Piecewise Linear Control Systems*, Ph.D. Dissertation, Lund Inst. of Tech., Sweden, 1999.

[9] J. Lu, L. Liao, A. Nerode, J. H. Taylor, *Optimal control of systems with continuous and discrete states*, In Proceedings of the 32nd IEEE Conference on Decision and Control, pp. 2292-2297.

[10] S. Nash, A. Sofer, *Linear and nonlinear programming*, McGraw-Hill Company, 1996.

[11] P. Riedinger, C. Zanne, F. Kratz, *Time optimal control of hybrid systems*, In Proceedings of the 1999 American Control Conference, pp. 2466-2470.

[12] T. I. Seidman, *Optimal control for switching systems*, In Proceedings of the 21st Annual Conference on Information Sciences and Systems, pp. 485-489, Baltimore, Maryland, 1987.

[13] H. Sussmann, *A maximum principle for hybrid optimal control problems*, In Proceedings of the 38th IEEE Conference on Decision and Control, 1999.

[14] L. Y. Wang, A. Beydoun, J. Cook, J. Sun, I. Kolmanovsky, *Optimal hybrid control with applications to automotive powertrain systems*, in LNCIS 222, Springer-Verlag, 1997.

[15] X. Xu, P. J. Antsaklis, *Optimal control of switched systems: new results and open problems*, In Proceedings of the 2000 American Control Conference, pp. 2683-2687.

[16] J. Yong, *Systems governed by ordinary differential equations with continuous, switching and impulse controls*, Appl. Math. Optim., **20**(1989), pp. 223-235.