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An Approach to Optimal Control of Switched Systems with Internally Forced Switchings[1]

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Abstract

This paper provides an approach to optimal control of switched systems with internally forced switchings (IFS). For such systems, one can only control the continuous input. But when the system state trajectory evolves under the continuous control, a switching sequence will be generated implicitly. Many practical problems only involve optimization where the number of switchings and the sequence of active subsystems are specified. This is the stage 1 optimization we study in detail in this paper. In our previous papers, we proposed an approach for solving optimal control problems for switched systems with externally forced switchings (EFS). In this paper, we extend such an approach to problems with IFS. The approach first transcribes a stage 1 problem into an equivalent problem parameterized by the switching instants and then the values of the derivatives of the optimal cost with respect to the switching instants are obtained based on the solution of a two point boundary value differential algebraic equation formed by the state, costate, stationarity, boundary equations and the equations for the state and the costate at the switching instants, along with their differentiations. With the knowledge of the derivatives, nonlinear optimization methods can then be applied to find the implicitly-generated optimal switching instants along with the corresponding continuous input. An example is shown to illustrate the results.

1 Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law that orchestrates the active subsystem at each time instant. Switchings can be classified as externally forced switchings (EFS) and internally forced switchings (IFS). Many real-world processes such as chemical processes, automotive systems, and electrical circuit systems, etc., can be modeled as switched systems.

Optimal control problems for switched systems, which are one of the most challenging and important classes of problems for such systems, have attracted the attention of researchers recently. Many literature results have appeared for problems with EFS only (see e.g., [3, 5, 7, 8]). In our previous papers [11, 12], we also proposed an approach to such problems. However, theoretical or practical results for optimal control of switched systems with IFS have rarely be reported in the literature (see e.g., [1, 4, 6, 9]; [1, 6] deal with discrete-time problems and [4, 9] provide some theoretical results). The difficulty in solving such problems lies in the fact that the switching sequence is generated implicitly along with the systems state trajectory evolution.

In this paper, we study optimal control problems for switched systems with IFS. A conceptual algorithm is first presented which relates the IFS problems and the EFS problems. Many practical problems only involve optimization where the number of switchings and the sequence of active subsystems are specified. We call such problems stage 1 problems similar to the EFS case. The main idea for solving such stage 1 problems is as follows. We first regard an IFS problem as an EFS problem with additional state constraints at the switching instants and solve the implicitly-generated optimal switching sequence along with the corresponding control input and then verify the results back in the IFS case. In order to find the implicitly-generated optimal switching sequence, the derivatives of the optimal cost with respect to the switching instants need to be known. It is shown how the approach in [11, 12] can be extended to such problems to obtain such derivative values. Note here at each switching instant, the system's state must be restricted to a switching hypersurface (which is not required for EFS problems in [11, 12]). Our approach first transcribes a stage 1 problem into an equivalent problem parameterized by the switching instants and then the values of the derivatives are obtained based on the solution of a two point boundary value differential algebraic equation (DAE) formed by the state, costate, stationarity, boundary equations and the equations for the state and the costate at the switching instants, along with their differentiations.

2 Problem Formulation

2.1 Switched Systems with IFS

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching logic among them. According to the different nature

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that a switching might be generated, we classify switchings into externally forced switchings (EFS) and internally forced switchings (IFS) (see Chapter 2 of [10] for more details). In this paper, we focus on switched systems with IFS only which are defined as follows.

Definition 2.1 (Switched System with IFS) *A switched system with IFS is a 3-tuple $\mathcal{S} = (\mathcal{D}, \mathcal{F}, \mathcal{L})$ where*
- $\mathcal{D} = (I, E)$ *is a directed graph indicating the discrete structure of the system. The node set* $I = \{1, 2, \cdots, M\}$ *is the set of indices for subsystems. The directed edge set* E *is a subset of* $I \times I - \{(i,i)|i \in I\}$ *which contains valid internal events. If an event* $e = (i_1, i_2)$ *takes place, the system switches from subsystem* i_1 *to* i_2.
- $\mathcal{F} = \{f_i : X_i \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n | i \in I\}$ *where* f_i *describes the vector field for the* i*-th subsystems* $\dot{x} = f_i(x, u, t)$. *Here* $X_i \subseteq \mathbb{R}^n$ *is the state constraint set for the* i*-th subsystem.*
- $\mathcal{L} = \{\Gamma_e | \Gamma_e \subseteq \mathbb{R}^n, \emptyset \neq \Gamma_e \subseteq X_{i_1} \cap X_{i_2}, e = (i_1, i_2) \in E\}$ *provides a logic constraint that relates the continuous state and mode switchings. When the state trajectory intersects* Γ_e, $e = (i_1, i_2)$ *at subsystem* i_1, *the event* e *must be triggered and the system is forced to switch to subsystem* i_2. *Furthermore, in this paper we consider* Γ_e *in the form of* $\Gamma_e = \{x | \gamma_e(x) = 0, \gamma_e : \mathbb{R}^n \to \mathbb{R}^{l_e}\}$. □

In view of Definition 2.1, a switched system $\mathcal{S}$ with IFS is a collection of subsystems $\mathcal{F}$ related by a switching logic described by $\mathcal{D}$ and $\mathcal{L}$. The only control input for such a system is its continuous input. Although one can only directly control $\mathcal{S}$ by the continuous input $u(t), t \in [t_0, t_f]$, a switching sequence will be generated implicitly along with the evolution of the system state trajectory. We define a switching sequence as follows.

Definition 2.2 (Switching Sequence) *For a switched system* $\mathcal{S}$, *a switching sequence* σ *in* $[t_0, t_f]$ *is defined as*

$$\sigma = ((t_0, i_0), (t_1, e_1), (t_2, e_2), \cdots, (t_K, e_K)), \tag{2.1}$$

with $0 \leq K < \infty$, $t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f$, *and* $i_0 \in I$, $e_k = (i_{k-1}, i_k) \in E$ *for* $k = 1, 2, \cdots, K$.

We define $\Sigma_{[t_0,t_f]} \triangleq \{\sigma\text{'s in } [t_0, t_f]\}$. □

A switching sequence σ as defined above indicates that subsystem i_k is active in $[t_k, t_{k+1})$. Note that in Definition 2.2, we only allow nonZeno sequences which switch at most a finite number of times in $[t_0, t_f]$, though different sequences may have different numbers of switchings.

2.2 An Optimal Control Problem

Now we formulate the optimal control problem we will study in this paper. In the sequel we denote $\mathcal{U}_{[t_0,t_f]} \triangleq \{u | u \in C_p[t_0, t_f], u(t) \in \mathbb{R}^m\}$; i.e., $\mathcal{U}_{[t_0,t_f]}$ is the set of all piecewise continuous functions for $t \in [t_0, t_f]$ that take values in $\mathbb{R}^m$.

Problem 2.1 (Systems with IFS) *Consider a switched system* $\mathcal{S}$ *with IFS. Given a fixed time interval* $[t_0, t_f]$, *find a continuous input* $u \in \mathcal{U}_{[t_0,t_f]}$ *such that the corresponding continuous state trajectory* x *departs from a given initial state* $x(t_0) = x_0$ *with initial active subsystem* i_0 *and meets an* $(n - l_f)$*-dimensional smooth manifold* $S_f = \{x | \phi_f(x) = 0, \phi_f : \mathbb{R}^n \to \mathbb{R}^{l_f}\}$ *at* t_f *and*

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t)\, dt \tag{2.2}$$

is minimized. □

Problem 2.1 is a basic optimal control problem with a fixed end-time where the final state is on a smooth manifold. In the sequel, we assume that f, L, ϕ_f, ψ possess enough smoothness properties for our derivations.

3 A Conceptual Algorithm for Stage 1 Problem

In [11, 12], we proposed a two stage optimization methodology and a two stage algorithm for optimal control problems of switched systems with EFS (note that a similar hierarchical decomposition method was developed independently in [4]; see [11] for comments on the similarity and difference between our method and that in [4]). In particular, we focused on stage 1 optimization where the number of switchings and the sequence of active subsystems are given and decompose it into the following two substages.

Stage 1(a). Fix the total number of switchings to be K and the sequence of active subsystems and let the minimum value of J with respect to u be a function of the K switching instants, i.e., $J_1 = J_1(\hat{t})$ for $K \geq 0$ (here $\hat{t} \triangleq (t_1, t_2, \cdots, t_K)^T$ and $\hat{t} \in T \triangleq \{(t_1, \cdots, t_K)^T | t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f\}$). Find J_1.

Stage 1(b). Minimize J_1 with respect to $\hat{t}$.

The following conceptual algorithm was then proposed in [11, 12] for stage 1 problems.

Algorithm 3.1 (A Conceptual Algorithm)

(1). Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.

(2). By solving an optimal control problem (stage 1(a)), find $J_1(\hat{t}^j)$.

(3). Find $\frac{\partial J_1}{\partial \hat{t}}(\hat{t}^j)$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}(\hat{t}^j)$.

(4). Use the gradient projection method or the constrained Newton's method (if $\frac{\partial^2 J_1}{\partial \hat{t}^2}(\hat{t}^j)$ is known) to update $\hat{t}^j$ to be $\hat{t}^{j+1}$ (see [2] for more on the methods). Set the iteration index $j = j + 1$.

(5). Repeat Steps (2), (3), (4) and (5), until a prespecified termination condition is satisfied. □

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As pointed in [11, 12], the difficulty for Algorithm 3.1 lies on how to obtain $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$. To address this difficulty, an approach based on the solution of a two point boundary value differential algebraic equation (DAE) was proposed in these papers for deriving the values of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$.

In this paper, we focus on *stage 1 problems* for switched systems with IFS in which we need to find an optimal continuous input and optimal switching instants if a given number of switchings and a given sequence of active subsystems are prespecified. Note that many practical IFS problems are in fact stage 1 problems. For example, the speeding-up of an automatic transmission automobile only requires switchings from gear 1 to 2 to 3 to 4 (although the switchings cannot be externally forced by the driver). It can be seen that the decomposition of stage 1 into two substages and the conceptual algorithm 3.1 are still applicable to problems with IFS. However, we point out that the IFS problems are more difficult due to the following reasons. First, the state x must be in the set Γ_e when event e takes place; this puts more constraints on the problem. Second, the switching sequences can depend on the continuous input in a complicated way (note that for problems with EFS, the switching sequence and the continuous input are independent and can be generated separately). To address these difficulties, in this paper, we propose the following idea of a method which leads to an approach based on an extension of the results for EFS problems.

Method 3.1 (A Method for IFS Problems)

1. Denote in a redundant fashion that an optimal solution to the IFS problem contains both an optimal continuous input and an optimal switching sequence (starting at subsystem i_0), i.e., regard an IFS problem as an EFS problem with additional state constraints at the switching instants. Solve the corresponding EFS problem.

2. Verify the validity of the solution for the IFS problem (i.e., if the system under the continuous input can generate the corresponding switching sequence). □

Step 1 in the above method can be solved using extensions of the methodology in [11]. Note that such an extension must address the additional requirement in an IFS problem which demands that the system's state to be restricted to a switching hypersurface at each switching instant (it is not required for EFS problems in [11]).

Necessary Conditions for Stage 1(a)

Stage 1(a) is in essence a conventional optimal control problem which seeks the minimum value J with respect to u under a given switching sequence $\sigma = ((t_0, i_0), (t_1, e_1), \cdots, (t_K, e_K))$. The only difference between stage 1(a) and most of the problems in many optimal control texts is that in stage 1(a), the system dynamics changes with respect to different time intervals. However, it is not difficult to use the calculus of variations techniques to prove the following necessary conditions.

Theorem 3.1 (Nec. Cond. for Stage 1(a)) *Consider the stage 1(a) problem for Problem 2.1. Assume that subsystem k is active in $[t_{k-1}, t_k)$ for $1 \leq k \leq K$ and subsystem $K+1$ in $[t_K, t_{K+1}]$ where $t_{K+1} = t_f$. Also assume that $x \in \Gamma_k = \{x | \gamma_k(x) = 0, \gamma_k : \mathbb{R}^n \to \mathbb{R}^{l_k}\}$ at the switching instant t_k. Let $u \in \mathcal{U}_{[t_0,t_f]}$ be a continuous input such that the corresponding continuous state trajectory x departs from a given initial state $x(t_0) = x_0$ and meets $S_f = \{x | \phi_f(x) = 0, \phi_f : \mathbb{R}^n \to \mathbb{R}^{l_f}\}$ at t_f. In order that u be optimal, it is necessary that there exists a vector function $p(t) = [p_1(t), \cdots, p_n(t)]^T$, $t \in [t_0, t_f]$, such that the following conditions hold*

(a). For almost any $t \in [t_0, t_f]$ the following state and costate equations hold

$$\frac{dx(t)}{dt} = [\frac{\partial H}{\partial p}(x(t), p(t), u(t), t)]^T \quad (3.1)$$

$$\frac{dp(t)}{dt} = -[\frac{\partial H}{\partial x}(x(t), p(t), u(t), t)]^T, \quad (3.2)$$

where $H(x, p, u, t) \triangleq L(x, u, t) + p^T f_k(x, u, t)$ for $t \in [t_{k-1}, t_k)$ ($k = K+1$ for $t \in [t_K, t_f]$).

(b). For almost any $t \in [t_0, t_f]$, the stationarity condition holds

$$0 = [\frac{\partial H}{\partial u}(x(t), p(t), u(t), t)]^T. \quad (3.3)$$

(c). At t_f, the function p satisfies

$$p(t_f) = [\frac{\partial \psi}{\partial x}(x(t_f))]^T + [\frac{\partial \phi_f}{\partial x}(x(t_f))]^T \lambda \quad (3.4)$$

where λ is an l_f-dimensional vector.

(d). At any t_k, $k = 1, 2, \cdots, K$, we have

$$p(t_k+) - p(t_k-) + (\frac{\partial \gamma_k}{\partial x}(x(t_k)))^T \nu_k = 0 \quad (3.5)$$

where ν_k is an l_k-dimensional vector.

Proof: See Chapter 6 of [10]. □

The above necessary conditions will be used in Section 5 in the development of a method for finding $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$. In general, it is difficult or even impossible to find an analytical expression of $J_1(\hat{t})$ using the above conditions. The reason is that conditions (a)-(d) present a two point boundary value DAE which, in most cases, cannot be solved analytically. However, the above DAE can be solved efficiently using many numerical methods (e.g., shooting methods).

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4 An Equivalent Problem Formulation Based on Parameterization of the Switching Instants

In the following two sections, an approach to stage 1 optimization based on equivalent transcription is presented (this is an extension of the approach in [11, 12]). In this section, we transcribe a stage 1 problem into an equivalent optimal control problem parameterized by the unknown switching instants. For simplicity of notation, in the followings, we concentrate on the case of two subsystems where subsystem 1 is active in the interval $t \in [0, t_1)$ and subsystem 2 is active in the interval $t \in [t_1, t_f]$ (t_1 is the switching instant to be determined). We also assume that $S_f = \mathbb{R}^n$ (for general S_f, we can introduce Lagrange multipliers and develop similar methods). We consider the following stage 1 problem.

Problem 4.1 *For a switched system*

$$\begin{aligned} \dot{x} &= f_1(x,u,t),\ 0 \le t < t_1, &(4.1)\\ \dot{x} &= f_2(x,u,t),\ t_1 \le t \le t_f, &(4.2) \end{aligned}$$

find an optimal switching instant t_1 and an optimal continuous input $u(t)$, $t \in [t_0, t_f]$ such that $x(t) \in Int(X_1)$ for $t \in [t_0, t_1)$, $x(t) \in Int(X_2)$ for $t \in (t_1, t_2]$, $x(t_1) \in \Gamma_1$ and the cost functional

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x,u,t)\ dt \qquad (4.3)$$

is minimized. Here t_0, t_f and $x(t_0) = x_0$ are given. □

We transcribe Problem 4.1 into an equivalent problem as follows. We introduce a state variable x_{n+1} corresponding to the switching instant t_1. Let x_{n+1} satisfy

$$\begin{aligned} \frac{dx_{n+1}}{dt} &= 0 &(4.4)\\ x_{n+1}(0) &= t_1 &(4.5) \end{aligned}$$

Next a new independent time variable τ is introduced. A piecewise linear correspondence relationship between t and τ is established as follows.

$$t(\tau) = \begin{cases} t_0 + (x_{n+1} - t_0)\tau, & 0 \le \tau \le 1 \\ x_{n+1} + (t_f - x_{n+1})(\tau - 1), & 1 \le \tau \le 2. \end{cases} \qquad (4.6)$$

Clearly, $\tau = 0$ corresponds to $t = t_0$, $\tau = 1$ to $t = t_1$, and $\tau = 2$ to $t = t_f$.

By introducing x_{n+1} and τ, Problem 4.1 can be transcribed into the following equivalent problem.

Problem 4.2 (An Equivalent Problem) *For a system with dynamics*

$$\begin{aligned} \frac{dx(\tau)}{d\tau} &= (x_{n+1} - t_0) f_1(x,u,t(\tau)) &(4.7)\\ \frac{dx_{n+1}}{d\tau} &= 0 &(4.8) \end{aligned}$$

in the interval $\tau \in [0, 1)$ and

$$\begin{aligned} \frac{dx(\tau)}{d\tau} &= (t_f - x_{n+1}) f_2(x,u,t(\tau))) &(4.9)\\ \frac{dx_{n+1}}{d\tau} &= 0 &(4.10) \end{aligned}$$

in the interval $\tau \in [1, 2]$, find an optimal x_{n+1} and an optimal $u(\tau)$, $\tau \in [0, 2]$ such that $x(\tau) \in Int(X_1)$ for $\tau \in [0, 1)$, $x(\tau) \in Int(X_2)$ for $\tau \in (1, 2]$, $x(\tau) \in \Gamma_1$ for $\tau = 1$ and the cost functional

$$\begin{aligned} J &= \psi(x(2)) + \int_0^1 (x_{n+1} - t_0) L(x,u,t(\tau))\ d\tau \\ &\quad + \int_1^2 (t_f - x_{n+1}) L(x,u,t(\tau))\ d\tau \qquad (4.11) \end{aligned}$$

is minimized. Here t_f, $x(0) = x_0$ are given. □

Remark 4.1 Problems 4.2 and 4.1 are equivalent in the sense that an optimal solution for Problem 4.2 is an optimal solution for Problem 4.1 by a proper change of independent variables as in (4.6) and by regarding $x_{n+1} = t_1$, and vice versa. □

Remark 4.2 Problem 4.2 is conventional because it has fixed time instant when the system dynamics change. In fact, because x_{n+1} is an unknown constant throughout $\tau \in [0, 2]$, it can be regarded as a conventional optimal control problem with an unknown parameter x_{n+1}. In the sequel, we regard it as an optimal control problem parameterized by x_{n+1} with cost (4.11) and subsystems (4.7) and (4.9). □

5 A Method Based on Solving a Boundary Value Differential Algebraic Equation

In this section, based on Problem 4.2, we develop a method which can give us the value of $\frac{dJ_1}{dt_1}$. It is based on the solution of a two point boundary value differential algebraic equation (DAE) formed by the state, costate, stationarity, boundary equations and the equations for the state and the costate at the switching instants for Problem 4.2, along with their differentiations with respect to the parameter x_{n+1}. In the sequel, we denote $\frac{\partial L}{\partial x}$, $\frac{\partial L}{\partial u}$ as row vectors and we denote $\frac{\partial f}{\partial x}$ as an $n \times n$ matrix whose (i_1, i_2)-th element is $\frac{\partial f_{i_1}}{\partial x_{i_2}}$. Similar notations apply to $\frac{\partial H}{\partial x}$, $\frac{\partial H}{\partial u}$, $\frac{\partial f}{\partial u}$, etc.

Consider the equivalent Problem 4.2, define

$$\begin{aligned} \tilde{f}_1(x,u,\tau,x_{n+1}) &\triangleq (x_{n+1} - t_0) f_1(x,u,t(\tau)), &(5.1)\\ \tilde{f}_2(x,u,\tau,x_{n+1}) &\triangleq (t_f - x_{n+1}) f_2(x,u,t(\tau)), &(5.2)\\ \tilde{L}_1(x,u,\tau,x_{n+1}) &\triangleq (x_{n+1} - t_0) L(x,u,t(\tau)), &(5.3)\\ \tilde{L}_2(x,u,\tau,x_{n+1}) &\triangleq (t_f - x_{n+1}) L(x,u,t(\tau)). &(5.4) \end{aligned}$$

Regarding x_{n+1} as a parameter, we can denote the optimal state trajectory as $x(\tau, x_{n+1})$. We define the parameterized Hamiltonian as

$$H(x,p,u,\tau,x_{n+1}) \triangleq \begin{cases} \tilde{L}_1(x,u,\tau,x_{n+1}) + p^T \tilde{f}_1(x,u,\tau,x_{n+1}), \\ \quad \text{for } \tau \in [0,1), \\ \tilde{L}_2(x,u,\tau,x_{n+1}) + p^T \tilde{f}_2(x,u,\tau,x_{n+1}), \\ \quad \text{for } \tau \in [1,2]. \end{cases} \qquad (5.5)$$

Assume that a parameter x_{n+1} is given, then we can apply Theorem 3.1 to Problem 4.2. The necessary conditions (a) and (b) provides us with

$$\text{State eq: } \frac{\partial x}{\partial \tau} = \left(\frac{\partial H}{\partial p}\right)^T = \tilde{f}_1(x,u,\tau,x_{n+1}) \qquad (5.6)$$

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$$\text{Costate eq: } \frac{\partial p}{\partial \tau} = -(\frac{\partial H}{\partial x})^T = -(\frac{\partial \tilde{f}_1}{\partial x})^T p - (\frac{\partial \tilde{L}_1}{\partial x})^T \quad (5.7)$$

$$\text{Stationarity eq: } 0 = (\frac{\partial H}{\partial u})^T = (\frac{\partial \tilde{f}_1}{\partial u})^T p + (\frac{\partial \tilde{L}_1}{\partial u})^T \quad (5.8)$$

in $\tau \in [0,1)$ and

$$\text{State eq: } \frac{\partial x}{\partial \tau} = (\frac{\partial H}{\partial p})^T = \tilde{f}_2(x,u,\tau,x_{n+1}) \quad (5.9)$$

$$\text{Costate eq: } \frac{\partial p}{\partial \tau} = -(\frac{\partial H}{\partial x})^T = -(\frac{\partial \tilde{f}_2}{\partial x})^T p - (\frac{\partial \tilde{L}_2}{\partial x})^T \quad (5.10)$$

$$\text{Stationarity eq: } 0 = (\frac{\partial H}{\partial u})^T = (\frac{\partial \tilde{f}_2}{\partial u})^T p + (\frac{\partial \tilde{L}_2}{\partial u})^T \quad (5.11)$$

in $\tau \in [1,2]$. Note that the p and u corresponding to the optimal solution are also functions of τ and x_{n+1}. Therefore, in the following, we denote them as $p = p(\tau, x_{n+1})$ and $u = u(\tau, x_{n+1})$.

From the necessary condition (c) of Theorem 3.1, we obtain the boundary conditions

$$\begin{aligned} x(0,x_{n+1}) &= x_0, && (5.12)\\ p(2,x_{n+1}) &= (\frac{\partial \psi}{\partial x}(x(2,x_{n+1})))^T. && (5.13) \end{aligned}$$

The necessary condition (d) tells us that $p(\tau, x_{n+1})$ has a discontinuity at $\tau = 1$ for fixed x_{n+1} (note this is different from EFS problems), i.e.,

$$p(1+,x_{n+1}) = p(1-,x_{n+1}) - (\frac{\partial \gamma_1}{\partial x}(x(1,x_{n+1})))^T \nu_1(x_{n+1}). \quad (5.14)$$

Moreover, here we also require that

$$\gamma_1(x(1,x_{n+1})) = 0. \quad (5.15)$$

(5.6)-(5.8), (5.9)-(5.11) along with conditions (5.12)-(5.15) form a two point boundary value DAE with discontinuities which is parameterized by x_{n+1}. For each given x_{n+1}, the DAE can be solved using numerical methods. Now assume that we have solved the above DAE and obtain the optimal $x(\tau, x_{n+1})$, $p(\tau, x_{n+1})$ and $u(\tau, x_{n+1})$, we then have the optimal value of J which is a function of the parameter x_{n+1}

$$J_1(x_{n+1}) = \psi(x(2,x_{n+1})) + \int_0^1 \tilde{L}_1(x,u,\tau,x_{n+1})\, d\tau + \int_1^2 \tilde{L}_2(x,u,\tau,x_{n+1})\, d\tau. \quad (5.16)$$

Differentiating J_1 with respect to x_{n+1} provides us with

$$\begin{aligned} \frac{dJ_1}{dx_{n+1}} &= \frac{\partial \psi(x(2,x_{n+1}))}{\partial x} \frac{\partial x(2,x_{n+1})}{\partial x_{n+1}} + \int_0^1 [L(x,u,t(\tau)) \\ &+ (x_{n+1} - t_0)(\frac{\partial L}{\partial x}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial L}{\partial u}\frac{\partial u}{\partial x_{n+1}} + \tau \frac{\partial L}{\partial t})]\, d\tau \\ &+ \int_1^2 [-L(x,u,t(\tau)) + (t_f - x_{n+1})(\frac{\partial L}{\partial x}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial L}{\partial u}\frac{\partial u}{\partial x_{n+1}} \\ &+ (2-\tau)\frac{\partial L}{\partial t})]\, d\tau. \end{aligned} \quad (5.17)$$

So we need to obtain the function $\frac{\partial x(\tau,x_{n+1})}{\partial x_{n+1}}$ and $\frac{\partial u(\tau,x_{n+1})}{\partial x_{n+1}}$ (here we assume that x_{n+1} is fixed) in order to obtain the value $\frac{dJ_1}{dx_{n+1}}$. By differentiating the above equations (5.6)-(5.8) and (5.9)-(5.11) with respect to x_{n+1}, we obtain

$$\begin{aligned} \frac{\partial}{\partial \tau}(\frac{\partial x}{\partial x_{n+1}}) &= \frac{\partial}{\partial x_{n+1}}(\frac{\partial x}{\partial \tau}) = f_1 + (x_{n+1} - \\ &- t_0)(\frac{\partial f_1}{\partial x}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x_{n+1}} + \tau\frac{\partial f_1}{\partial t}), \end{aligned} \quad (5.18)$$

$$\begin{aligned} \frac{\partial}{\partial \tau}(\frac{\partial p}{\partial x_{n+1}}) &= -\frac{\partial}{\partial x_{n+1}}(\frac{\partial p}{\partial \tau}) = -(\frac{\partial f_1}{\partial x})^T p - (\frac{\partial L}{\partial x})^T - (x_{n+1} \\ &- t_0)[(\frac{\partial f_1}{\partial x})^T \frac{\partial p}{\partial x_{n+1}} + (p^T \frac{\partial^2 f_1}{\partial x^2}\frac{\partial x}{\partial x_{n+1}})^T + (p^T \frac{\partial^2 f_1}{\partial x \partial u}\frac{\partial u}{\partial x_{n+1}})^T \\ &+ \tau (p^T \frac{\partial^2 f_1}{\partial x \partial t})^T \frac{\partial^2 L}{\partial x^2}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial^2 L}{\partial x \partial u}\frac{\partial u}{\partial x_{n+1}} + \tau \frac{\partial^2 L}{\partial x \partial t}], \end{aligned} \quad (5.19)$$

$$\begin{aligned} 0 &= (\frac{\partial f_1}{\partial u})^T p + (\frac{\partial L}{\partial u})^T + (x_{n+1} - t_0)[(\frac{\partial f_1}{\partial u})^T \frac{\partial p}{\partial x_{n+1}} \\ &+ (p^T \frac{\partial^2 f_1}{\partial u \partial x}\frac{\partial x}{\partial x_{n+1}})^T + (p^T \frac{\partial^2 f_1}{\partial u^2}\frac{\partial u}{\partial x_{n+1}})^T + \tau (p^T \frac{\partial^2 f_1}{\partial u \partial t})^T \\ &+ \frac{\partial^2 L}{\partial u \partial x}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial^2 L}{\partial u^2}\frac{\partial u}{\partial x_{n+1}} + \tau \frac{\partial^2 L}{\partial u \partial t}], \end{aligned} \quad (5.20)$$

for $\tau \in [0,1)$ and

$$\begin{aligned} \frac{\partial}{\partial \tau}(\frac{\partial x}{\partial x_{n+1}}) &= \frac{\partial}{\partial x_{n+1}}(\frac{\partial x}{\partial \tau}) = -f_2 + (t_f \\ &- x_{n+1})(\frac{\partial f_2}{\partial x}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x_{n+1}} + (2-\tau)\frac{\partial f_2}{\partial t}), \end{aligned} \quad (5.21)$$

$$\begin{aligned} \frac{\partial}{\partial \tau}(\frac{\partial p}{\partial x_{n+1}}) &= -\frac{\partial}{\partial x_{n+1}}(\frac{\partial p}{\partial \tau}) = (\frac{\partial f_2}{\partial x})^T p + (\frac{\partial L}{\partial x})^T - (t_f \\ &- x_{n+1})[(\frac{\partial f_2}{\partial x})^T \frac{\partial p}{\partial x_{n+1}} + (p^T \frac{\partial^2 f_2}{\partial x^2}\frac{\partial x}{\partial x_{n+1}})^T + (p^T \frac{\partial^2 f_2}{\partial x \partial u}\frac{\partial u}{\partial x_{n+1}})^T \\ &+ (2-\tau)(p^T \frac{\partial^2 f_2}{\partial x \partial t})^T + \frac{\partial^2 L}{\partial x^2}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial^2 L}{\partial x \partial u}\frac{\partial u}{\partial x_{n+1}} \\ &+ (2-\tau)\frac{\partial^2 L}{\partial x \partial t}], \end{aligned} \quad (5.22)$$

$$\begin{aligned} 0 &= -(\frac{\partial f_2}{\partial u})^T p - (\frac{\partial L}{\partial u})^T + (x_{n+1} - t_0)[(\frac{\partial f_2}{\partial u})^T \frac{\partial p}{\partial x_{n+1}} \\ &+ (p^T \frac{\partial^2 f_2}{\partial u \partial x}\frac{\partial x}{\partial x_{n+1}})^T + (p^T \frac{\partial^2 f_2}{\partial u^2}\frac{\partial u}{\partial x_{n+1}})^T + (2-\tau)(p^T \frac{\partial^2 f_2}{\partial u \partial t})^T \\ &+ \frac{\partial^2 L}{\partial u \partial x}\frac{\partial x}{\partial x_{n+1}} + \frac{\partial^2 L}{\partial u^2}\frac{\partial u}{\partial x_{n+1}} + (2-\tau)\frac{\partial^2 L}{\partial u \partial t}], \end{aligned} \quad (5.23)$$

for $\tau \in [1,2]$.

In the above equations, $\frac{\partial^2 f_1}{\partial x^2}$ is an $n \times n \times n$ array whose (j_1, j_2, j_3) element is $\frac{\partial^2 f_{1,j_1}}{\partial x_{j_2} \partial x_{j_3}}$ and the notation $p^T \frac{\partial^2 f_1}{\partial x^2}\frac{\partial x}{\partial x_{n+1}}$ denotes an $1 \times n$ row vector which has its j_2-th element as $\sum_{j_1=1}^n \sum_{j_3=1}^n p_{j_1} \frac{\partial^2 f_{1,j_1}}{\partial x_{j_2} \partial x_{j_3}} \frac{\partial x_{j_3}}{\partial x_{n+1}}$ where f_{1,j_1} is the j_1-th element of f_1, p_{j_1} is the j_1-th element of p and x_{j_2} is the j_2-th element of x. Other notations can be interpreted similarly (see [10] for details).

Differentiating (5.12), (5.13) and (5.14) with respect to x_{n+1}, we obtain

$$\frac{\partial x(0,x_{n+1})}{\partial x_{n+1}} = 0, \quad (5.24)$$

$$\frac{\partial p(2,x_{n+1})}{\partial x_{n+1}} = \frac{\partial^2 \psi(x(2,x_{n+1}))}{\partial x^2}\frac{\partial x(2,x_{n+1})}{\partial x_{n+1}} \quad (5.25)$$

$$\begin{aligned} \frac{\partial p}{\partial x_{n+1}}(1+,x_{n+1}) &= \frac{\partial p}{\partial x_{n+1}}(1-,x_{n+1}) \\ &- (\nu_1^T(x_{n+1})\frac{\partial^2 \gamma_1}{\partial x^2}(x(1,x_{n+1}))\frac{\partial x}{\partial x_{n+1}})^T \\ &- (\frac{\partial \gamma_1}{\partial x}(x(1,x_{n+1})))^T \frac{d\nu_1}{dx_{n+1}}(x_{n+1}) \end{aligned} \quad (5.26)$$

where $\nu_1^T \frac{\partial^2 \gamma_1}{\partial x^2}\frac{\partial x}{\partial x_{n+1}}$ denotes an $1 \times n$ row vector which has its j_2-th element as $\sum_{j_1=1}^{l_1} \sum_{j_3=1}^n \nu_{1,j_1} \frac{\partial^2 \gamma_{1,j_1}}{\partial x_{j_2} \partial x_{j_3}} \frac{\partial x_{j_3}}{\partial x_{n+1}}$. The differentiation of (5.15) is

$$\frac{\partial \gamma_1}{\partial x}(x(1,x_{n+1}))\frac{\partial x}{\partial x_{n+1}}(1,x_{n+1}) = 0. \quad (5.27)$$

It can now be observed that (5.6)-(5.8), (5.9)-(5.11) and (5.18)-(5.20), (5.21)-(5.23) along with the boundary conditions (5.12), (5.13) and (5.24), (5.25) and with the equations for the costate and the state at the switching instant (5.14)-(5.15), (5.26)-(5.27) form a two point boundary value DAE

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for $x(\tau, x_{n+1})$, $p(\tau, x_{n+1})$, $u(\tau, x_{n+1})$, $\nu_1(x_{n+1})$ and $\frac{\partial x(\tau,x_{n+1})}{\partial x_{n+1}}$, $\frac{\partial p(\tau,x_{n+1})}{\partial x_{n+1}}$, $\frac{\partial u(\tau,x_{n+1})}{\partial x_{n+1}}$, $\frac{d\nu_1(x_{n+1})}{dx_{n+1}}$. By solving them and substitute the result into (5.17), we can obtain $\frac{dJ_1}{dx_{n+1}}$.

Remark 5.1 The approach developed in this section can be extended in a straightforward manner to the case of several subsystems and more than one switchings. The value of $\frac{d^2 J_1(t_1)}{dt_1^2}$ can also be similarly obtained. See Chapters 8 and 9 of [10] for details. □

Remark 5.2 Note that in the solution process of the two point boundary value DAE, we have not enforced the requirement $x(\tau, x_{n+1}) \in \mathrm{Int}(X_1)$ for $\tau \in [0, 1)$ and $x(\tau, x_{n+1}) \in \mathrm{Int}(X_2)$ for $\tau \in (1, 2]$. However, after a solution has been found, we need to verify these conditions for the result. This is the second step of Method 3.1 in Section 3 which verifies the validity of the solution. □

6 An Example

Now we illustrate the effectiveness of the approach proposed in this paper using an example.

Example 6.1 Consider a switched system with IFS consisting of

$$\text{subsystem 1: } \dot{x} = x + 2u, \tag{6.1}$$
$$\text{subsystem 2: } \dot{x} = 0.5x + u. \tag{6.2}$$

Assume that $t_0 = 0$, $t_f = 2$ and the system state starts at $x(0) = 1$ following subsystem 1 ($X_1 = \{x \in \mathbb{R} | x \le e\}$ and $X_2 = \{x \in \mathbb{R} | x \ge e\}$). Assume that upon hitting the set $\gamma_1 = \{x \in \mathbb{R} | x = e\}$, the system switches from subsystem 1 to 2. Also assume there is only one switching which takes place at time t_1 ($0 \le t_1 \le 2$). Find an optimal input u such that the cost functional $J = \frac{1}{2}(x(2) - e^{1.5})^2 + \frac{1}{2}\int_0^2 u^2(t)\, dt$ is minimized.

We use the approach developed in this paper and solve this stage 1 problem. Choose an initial nominal $t_1 = 1.3$. We find that the optimal switching instant is $t_1 = 0.9997$ and the corresponding optimal cost is 1.0092×10^{-7} after 6 iterations. The corresponding $\nu_1 = 9.7655 \times 10^{-5}$. The corresponding continuous control and state trajectory are shown in Figure 1. Note that the theoretical optimal solutions for this problem are $u^{opt} \equiv 0$ and $J^{opt} = 0$ (the corresponding $t_1^{opt} = 1$), so the result we obtain is quite accurate. □

7 Conclusion

In this paper, an approach to optimal control of switched systems with IFS is proposed. We extend our earlier approach for EFS problems to IFS problems. The approach is based on solving a two point boundary value DAE formed by the state, costate, stationarity, boundary equations and the equations for the state and the costate at the switching instants, along with their differentiations. Derivatives of the optimal cost with respect to the switching instants can be obtained accurately and therefore nonlinear optimization algorithms can be used to find the implicitly-generated optimal switching instants along with the corresponding continuous input. Further research topics include the incorporation of the state constraints in stage 1(a) in our approach, so as to guarantee the validity of the method (therefore eliminate the verification process).

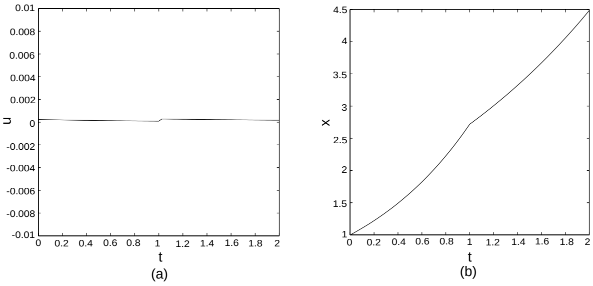

Figure 1: Example 6.1: (a) The control input. (b) The state trajectory $x(t)$.

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