Hai Lin, Guisheng Zhai, Panos J.Antsaklis, “Set-Valued Observer Design for a Class of Uncertain Linear Systems with Persistent Disturbance,” Proceedings of the 2003 American Control Conference, pp. 1902-1907, Denver, Colorado, June 4-6, 2003.

# Set-Valued Observer Design for a Class of Uncertain Linear Systems with Persistent Disturbance[1]

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**Abstract**

In this paper, a class of linear systems affected by both parameter variations and additive disturbances is considered. The problem of designing a set-valued state observer, which estimates a region containing the real state for each time being, is investigated. The techniques for designing the observer are based on positive invariant set theory. By constructing a set-induced Lyapunov function, it is shown that the estimation error exponentially converges to a given compact set with an assigned rate of convergence.

## 1 Introduction

In control theory and engineering, it is often desirable to obtain full state information for control or diagnostic purpose. Therefore it is not surprising that the synthesis of a state observer has been of considerable interest in classical system theory, see for example [11] and the references therein. The original theory of the state observer involves the asymptotic reconstruction of the state by using exact knowledge of inputs and outputs [9]. However, the real processes are often affected by disturbances and noises. Therefore, the design procedures of state observers were later extended to include the cases when disturbances and/or measurement noises were present. These generalizations may be roughly divided into two main groups. The first group relies on the stochastic control approaches, which are based on probabilistic models of the disturbances and noises. The stochastic approach provides optimal state estimation based on the probabilistic models of the exogenous signals. Unfortunately, in many cases, no information about the disturbances (in the deterministic or statistical sense) is available, and it can only be assumed that they are bounded in a compact set. Alternatively, disturbances and noise are dealt with in the framework of robust control. Under such framework, optimal state estimation that minimizes the induced-norm from exogenous noises to estimation errors is often considered. For example in [12], an $l^1$ optimal estimation problem was studied for a class of time varying discrete-time systems with process disturbance and measurement noise, and a set-valued observer, whose centers provide optimal estimates in the sense of $l^\infty$-induced norm, was designed. The optimal $l^\infty$-induced norm estimation problem was also considered in [14]. There also exist results for $\mathcal{H}^\infty$ optimal estimation problems, see for example [10].

In the previous work on observer design as mentioned above, deterministic dynamics were assumed, where there is no parameter variation in the model. However, it is known that we only have partial knowledge of almost all practical systems. In addition, the system parameters are often subject to unknown, possibly time-varying, perturbations. Therefore it is of practical importance to deal with systems with uncertain parameters. This consideration leads to the robust estimation problem, where robustness is with respect to not only exogenous signals but also model uncertainties. There are some results for the robust estimation problem from a variety of different approaches, see for example [2, 1, 7] and references therein. In [2], the structure features of robust observers in the presence of arbitrary small parameter perturbations were studied from a sensitivity standpoint. A similar problem was considered in [1], where a technique for designing robust observers for perturbed linear systems was presented. In [7], the robust $l^1$ estimation with plant uncertainties and external disturbance inputs was studied, and the estimator was applied to robust $l^1$ fault detection. The techniques in [7] were based on the mixed structured singular value theory. There were also investigations into developing robust estimators using parametric quadratic Lyapunov theory, see for example [8].

In this paper, we deal with a class of uncertain linear systems affected by both parameter variations and exterior disturbances. The problem studied is the design of a set-valued state observer, which constructs a set of possible state values based on measured outputs and inputs. The techniques used in this paper are based on positive invariant set theory and set-induced Lyapunov functions. By constructing a set-induced Lyapunov function, we can guarantee the ultimate bound-

[1]The partial support of the National Science Foundation (NSF ECS99-12458 & CCR01-13131), and of the DARPA/ITO-NEST Program (AF-F30602-01-2-0526) is gratefully acknowledged.

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edness and convergence rate of the estimation error. The work is inspired by the success of set-induced Lyapunov function together with positive invariant set theory in the fields of robust stability analysis, stabilization, constrained regulation etc, see [5, 3]. For a general review of the set invariance theory, see for example [6].

This paper is organized as follows. In Section 2, a mathematical model for uncertain linear systems is described, and the observer design problem is formulated. Section 3 contains the necessary background from invariant set theory, and the definitions of positive $\mathcal{D}$-invariance and strong positive $\mathcal{D}$-invariance are introduced. The approaches to the observer design and its implementation are described in Section 4. The convergence and ultimate boundedness of the estimation error are shown in Section 5. In Section 6, a numerical example is given. Finally, concluding remarks are presented.

Following the notation of [5], we use the letters $\mathcal{E}, \mathcal{P}, \mathcal{S} \cdots$ to denote sets. $\partial\mathcal{P}$ stands for the boundary of set $\mathcal{P}$, and $int\{\mathcal{P}\}$ its interior. For any real $\lambda \geq 0$, the set $\lambda\mathcal{S}$ is defined as $\{x = \lambda y,\ y \in \mathcal{S}\}$. The term C-set stands for a convex and compact set containing the origin in its interior.

## 2 Problem Formulation

In this paper, we consider linear discrete-time systems described by the difference equation

$$x(t+1) = A(w)x(t) + B(w)u(t) + Ed(t), \quad t \in \mathbb{Z}^+ \tag{2.1}$$

and continuous-time systems represented by the differential equation

$$\dot{x}(t) = A(w)x(t) + B(w)u(t) + Ed(t), \quad t \in \mathbb{R}^+ \tag{2.2}$$

with the measured output

$$y(t) = Cx(t) \tag{2.3}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathcal{U} \subset \mathbb{R}^m$, $d(t) \in \mathcal{D} \subset \mathbb{R}^r$, and $A(w) \in \mathbb{R}^{n\times n}$, $B(w) \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$. Assume that $\mathcal{U}$ and $\mathcal{D}$ are C-sets, and that the entries of $A(w)$ and $B(w)$ are continuous function of $w \in \mathcal{W}$, where $\mathcal{W} \subset \mathbb{R}^v$ is an assigned compact set.

For this system, we are interested in determining the state $x(t)$ based on the measured output $y(t)$ and control signal $u(t)$. Because of the uncertainty and disturbance, we can not estimate the state $x(t)$ exactly. Therefore, it is only reasonable to estimate a region in which the real state is contained. The problem being addressed can be formulated as follows:

**Problem:** *Given the system with the measured output $y(t)$ and input $u(t)$, find $\mathcal{X}(t)$ such that $x(t) + e(t) \in \mathcal{X}(t)$, and assure that the estimation error $e(t)$ is uniformly ultimately bounded in a given C-set, $\mathcal{E}$, with an assigned rate of convergence.*

Here, *uniformly ultimately bounded* in $\mathcal{E}$ means that for any initial value of the estimation error $e(t_0) \notin \mathcal{E}$, $\exists T \geq t_0$ such that for all $t \geq T$, $e(t) \in \mathcal{E}$. The meaning of the convergence rate will be explained later in Section 5. Our methodology for computing the observer that guarantees uniformly ultimate boundedness of the estimation error is based on *positive invariant sets* and *set-induced Lyapunov functions*, which will be derived in the following sections. For systems with linearly constrained uncertainties, it is shown that such observer and Lyapunov functions may be derived by numerically efficient algorithms involving polyhedral sets.

## 3 Positive Disturbance Invariance

In this section, let us first consider the following discrete-time system

$$x(t+1) = A(w)x(t) + Ed(t) \tag{3.1}$$

where $d(t)$ is contained in a C-set $\mathcal{D}$.

**Definition 3.1** A set $\mathcal{S}$ in the state space is said to be *positive $\mathcal{D}$-invariant (PDI)* for this system if for every initial condition $x(0) \in \mathcal{S}$ we have that $x(t) \in \mathcal{S}$, $t \geq 0$, for every admissible disturbance $d(t) \in \mathcal{D}$ and every admissible parameter variation $w(t) \in \mathcal{W}$.

In the particular case when $\mathcal{D} = \{0\}$, the positive $\mathcal{D}$-invariance is equivalent to the positive invariance [6]. For the counterpart of continuous-time systems, we have corresponding definitions for invariant set and positive $\mathcal{D}$-invariance.

In this paper, it is important to consider an index of the convergence speed of the state estimation error. For such purpose, we need to introduce the following definitions.

**Definition 3.2** Let $\mathcal{S}$ be a compact set with nonempty interior in the state space. $\mathcal{S}$ is said to be *strongly positive $\mathcal{D}$-invariant*(SPDI) for system (3.1), if for every initial condition $x(0) \in \mathcal{S}$ and for every disturbance sequence $d(t) \in \mathcal{D}$ and every admissible parameter variation $w(t) \in \mathcal{W}$ with $t = 0, 1, \cdots$, we have that $x(t) \in int\{\mathcal{S}\}$ for $t \geq 0$.

If no disturbances are present, namely $\mathcal{D} = \{0\}$, we shall refer to this property as strong positive invariance(SPI). In the discrete-time case, the strong positive invariance of $\mathcal{S}$ is equivalent to the contractivity. Note that similar definitions are given in [4] for deterministic linear systems. Next, we introduce the following notation for system (3.1):

$$post(x, \mathcal{W}, \mathcal{D}) = \{x' : x' = A(w)x + Ed;\ \forall w \in \mathcal{W}, d \in \mathcal{D}\}$$

It can be shown that a set $\mathcal{S}$ is *strongly positive $\mathcal{D}$-invariant* if and only if $\exists\lambda$, $0 < \lambda < 1$, such that $\mathcal{S}$ is *$\lambda$-contractive*, i.e. for any $x \in \mathcal{S}$, $post(x, \mathcal{W}, \mathcal{D}) \subset \lambda\mathcal{S}$.

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In the following, we shall assume that $\mathcal{D}$ and $\mathcal{S}$ are convex and compact polyhedrons containing the origin, and in addition, $\mathcal{S}$ contains the origin in its interior. A polyhedral set $\mathcal{S}$ in $\mathbb{R}^n$ can be represented by a set of linear inequalities

$$\mathcal{S} = \{x \in \mathbb{R}^n : f_i^T x \leq \theta_i,\ i = 1, \cdots, s\} \tag{3.2}$$

and for brevity, we denote $\mathcal{S}$ as $\{x : Fx \leq \theta\}$, where $\leq$ is with respect to componentwise. Let $vert\{\mathcal{S}\}$ stand for the vertices of a polytope $\mathcal{S}$. Consider the vector $\delta$ whose components are

$$\delta_i = \max_{d \in \mathcal{D}} f_i E d, \quad i = 1, \cdots, s \tag{3.3}$$

The vector $\delta$ incorporates the effects of the disturbance $d(t)$. In the discrete-time case, the following results hold. Note that similar results were given in [4] for deterministic linear systems. The extensions to uncertain dynamics are not difficult, so the details of proof is omitted here for space limit.

**Lemma 3.1** The polyhedral region $\mathcal{S} = \{x \in \mathbb{R}^n : Fx \leq \theta\}$ is PDI for system (3.1), *if and only if* for every vertex of $\mathcal{S}$, $v_j \in vert\{\mathcal{S}\}$, $j = 1, \cdots, r$, we have

$$FA(w)v_j \leq \theta - \delta, \quad \forall w \in \mathcal{W} \tag{3.4}$$

Similarly, we can derive the following result for SPDI.

**Corollary 3.1** The polyhedral region $\mathcal{S} = \{x \in \mathbb{R}^n : Fx \leq \theta\}$ is SPDI for system (3.1), *if and only if* $\exists 0 < \lambda < 1$, such that $\forall v_j \in vert\{\mathcal{S}\}$, $j = 1, \cdots, r$, we have

$$FA(w)v_j \leq \lambda\theta - \delta, \quad \forall w \in \mathcal{W} \tag{3.5}$$

We consider now a continuous-time system of the form

$$\dot{x}(t) = A(w)x(t) + Ed(t) \tag{3.6}$$

Similarly, we can introduce PDI, SPDI concepts for system (3.6). The use of invariant sets allows us to extend results for the discrete-time case to continuous-time systems by introducing the Euler approximating system (EAS) [4] as follows:

$$x(t+1) = [I + \tau A(w)]x(t) + \tau Ed(t) \tag{3.7}$$

It has been proven in [4] that: $\mathcal{S}$ is a SPDI region for a deterministic continuous-time system *if and only if* $\mathcal{S}$ is a SPDI region for its corresponding Euler approximating system for some $\tau > 0$. Similarly, we can derive the following proposition for uncertain continuous-time system (3.6) with polytopic constrains.

**Proposition 3.1** The polyhedral region $\mathcal{S} = \{x \in \mathbb{R}^n : f_i^T x \leq \theta_i,\ i = 1, \cdots, s\}$ is SPDI for (3.6), *if and only if* $\exists 0 < \lambda < 1$, for some $\tau > 0$, and for every vertices of $\mathcal{S}$, $v_j \in vert\{\mathcal{S}\}$, $j = 1, \cdots, r$,

$$Fv_j + \tau FA(w)v_j \leq \lambda\theta - \tau\delta, \quad \forall w \in \mathcal{W} \tag{3.8}$$

In the next section, we will design an observer based on the SPDI and its properties discussed above.

## 4 Observer Design

For simplicity, we only consider the observer design for discrete-time case in this section, namely for the system described by (2.1) and (2.3). The extension of these results to the continuous-time case is immediate, and it is illustrated through an example in Section 6.

We consider a full state observer of the form

$$\bar{x}(t+1) = (A(w) - LC)\bar{x}(t) + B(w)u(t) + Ly(t) \tag{4.1}$$

Assume an admissible disturbance sequence $d_s(t) \in \mathcal{D}$ and an admissible parameter variation sequence $w_s(t) \in \mathcal{W}$. The corresponding real state trajectory is denoted as $x_s(t)$ for such $w_s(t)$ and $d_s(t)$. At every time step $t$, the state region estimate of the observer, $\mathcal{X}(t)$, contains a state estimation $\bar{x}_s(t)$, which corresponds to the specified disturbance sequence $d_s(t)$ and parameter variation sequence $w_s(t)$. Then the estimation error for $x_s(t)$ is $e_s(t) = \bar{x}_s(t) - x_s(t)$ which satisfies $e_s(t+1) = (A(w_s) - LC)e_s(t) - Ed_s(t)$. Considering all possible $w(t) \in \mathcal{W}$ and $d(t) \in \mathcal{D}$, we can describe the behavior of the estimation error $e(t) = \bar{x}(t) - x(t)$ by the equation

$$e(t+1) = (A(w) - LC)e(t) - Ed(t) \tag{4.2}$$

Our design objective is to ultimately bound the error $e(t)$ in a given compact set $\mathcal{E}$ for every admissible disturbance $d(t) \in \mathcal{D}$ and parameter uncertainty $w(t) \in \mathcal{W}$.

Let $\mathcal{E}$ be a given convex and compact polyhedral set containing the origin in its interior. We assume that $\mathcal{E}$ can be represented as $\mathcal{E} = \{e : Fe \leq \theta\}$, and also assume that the vertices of $\mathcal{E}$ are known. Otherwise a procedure is needed to calculate the vertices of $\mathcal{E}$.

In the discrete-time case, let us assume that $\mathcal{E}$ is SPDI for some given $0 < \lambda < 1$ with respect to state estimation error equation (4.2). Therefore the matrix $L$ fulfills the following constraints

$$[A(w) - LC]v_j - Ed \in \lambda\mathcal{E},$$

for $\forall v_j \in vert\{\mathcal{E}\}$ and $\forall w \in \mathcal{W}$. It is known that in practice uncertainties often enter linearly in the system model and they are linearly constrained. To handle this particular but interesting case, we consider the class of polyhedral sets. Such sets have been considered in the literature concerning the control of systems with input and state constraints, see for example [5, 6]. Their main advantage is that they are suitable for computation. If we assume polytopic uncertainty, i.e. $[A(w), B(w)] = \sum_{k=1}^r w_k[A_k, B_k]$, $w_k \geq 0$, $\sum_{k=1}^r w_k = 1$, then the above constraints can be written as

$$f_i[\sum_{k=1}^r w_k A_k - LC]v_j \leq \lambda\theta_i - \delta_i,$$

$\forall v_j \in vert\{\mathcal{E}\}$, $\forall w_k \in [0,1]$, and $\sum_{k=1}^{r} w_k = 1$, where $\delta_i = \max_{d\in\mathcal{D}}(-f_i E d)$. Because of linearity and convexity, it is equivalent to only considering the vertices of $A(w)$, i.e.

$$f_i[A_k - LC]v_j \le \lambda\theta_i - \delta_i,$$

$\forall v_j \in vert\{\mathcal{E}\}$, $\forall i = 1,\cdots,s$, and $\forall k = 1,\cdots,r$. For brevity, we write

$$F[A_k - LC]v_j \le \lambda\theta - \delta \qquad (4.3)$$

which holds for $\forall v_j \in vert\{\mathcal{E}\}$, $\forall k = 1,\cdots,r$. And $\delta$ has components as $\delta_i$. We see that the observer design problem is solved if the sets of the linear inequalities in the unknown $L$ derived above have a feasible solution. The feasibility of the above linear inequalities is guaranteed by the assumption that $\mathcal{E}$ is SPDI for some given $0 < \lambda < 1$ with respect to system (4.2), and vise versa.

In conclusion, the existence of the set-valued state observer (4.1), whose state estimation error ultimately bounds in a specified region $\mathcal{E}$ [1], is equivalent to the feasibility of the linear inequalities in (4.3), and it is also equivalent to the condition that $\mathcal{E}$ is SPDI for some given $0 < \lambda < 1$ with respect to system (4.2).

Note that the observer is set-valued, or, in other words, it estimates the region in which the real state stays. The observer maps set $\mathcal{X}(t)$ to another set $\mathcal{X}(t+1)$ as time progresses. Our next question is how to implement the set-valued observer in practice.

Consider the initial set $\mathcal{X}(t_0)$ as a polytope, whose vertices, $\bar{x}^i(t_0)$, $i = 1,\cdots,n$, are known. For any $\bar{x}(t_0) \in \mathcal{X}(t_0)$, we have $\bar{x}(t_0) = \sum_{i=1}^{n} \alpha_i \bar{x}^i(t_0)$, where $\alpha_i \ge 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. The next corresponding estimated state $\bar{x}(t_1)$, in fact a set, is given by:

$$\bar{x}(t_1) = (A(w) - LC)\bar{x}(t_0) + B(w)u(t_0) + Ly(t_0)$$

which is just the linear transformation of $\bar{x}(t_0)$. Note that the linear transformation of a polytope is still a polytope. In addition

$$\begin{aligned}
\bar{x}(t_1) &= (A(w) - LC)\bar{x}(t_0) + B(w)u(t_0) + Ly(t_0) \\
&= (A(w) - LC)\sum_{i=1}^{n} \alpha_i \bar{x}^i(t_0) + B(w)u(t_0) + Ly(t_0) \\
&= \sum_{i=1}^{n} \alpha_i[(A(w) - LC)\bar{x}^i(t_0) + B(w)u(t_0) + Ly(t_0)]
\end{aligned}$$

If we assume polytopic uncertainty, i.e. $[A(w), B(w)] = \sum_{k=1}^{r} w_k[A_k, B_k]$, $w_k \ge 0$, $\sum_{k=1}^{r} w_k = 1$, then the implementation of the set-valued observer can be further simplified as:

$$\begin{aligned}
\bar{x}(t_1) &= \sum_{i=1}^{n} \alpha_i[(A(w) - LC)\bar{x}^i(t_0) + B(w)u(t_0) + Ly(t_0)] \\
&= \sum_{i=1}^{n} \alpha_i\{\sum_{k=1}^{r} w_k[A_k - LC, B_k]\begin{bmatrix} \bar{x}^i(t_0) \\ u(t_0) \end{bmatrix} + Ly(t_0)\} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{r} \alpha_i\{w_k[A_k - LC, B_k]\begin{bmatrix} \bar{x}^i(t_0) \\ u(t_0) \end{bmatrix} + Ly(t_0)\} \\
&= \sum_{i,k=1}^{n,r} \alpha_i w_k\{[A_k - LC, B_k]\begin{bmatrix} \bar{x}^i(t_0) \\ u(t_0) \end{bmatrix} + Ly(t_0)\} \\
&= \sum_{j=1}^{n\times r} \beta_j \bar{x}^j(t_1)
\end{aligned}$$

where $j = (i-1)\times n + k$, $\bar{x}^j(t_1)$ is the corresponding estimated state corresponding to the vertices $\bar{x}^i(t_0)$ under the vertices matrix $[A_k, B_k]$. Also $\beta_j = (\alpha_i \times w_k) \ge 0$ and $\sum_{j=1}^{n\times r} \beta_j = 1$. Therefore for the case of polytopic uncertainty, the implementation of the observer only needs to consider the finite vertices of state matrices , i.e. $(A_k, B_k)$ for $k = 1,\cdots,r$, and the finite vertices of the $\mathcal{X}(t)$, i.e. $\bar{x}^i(t)$ for $i = 1,\cdots,n$. In summary, the formulation of the observer can be described as the following:

$$\begin{aligned}
\bar{x}^{(1,1)}(t+1) &= (A_1 - LC)\bar{x}^1(t) + B_1u(t) + Ly(t) \\
\bar{x}^{(1,2)}(t+1) &= (A_1 - LC)\bar{x}^2(t) + B_1u(t) + Ly(t) \\
&\cdots \\
\bar{x}^{(1,n)}(t+1) &= (A_1 - LC)\bar{x}^n(t) + B_1u(t) + Ly(t) \\
\bar{x}^{(2,1)}(t+1) &= (A_2 - LC)\bar{x}^1(t) + B_2u(t) + Ly(t) \\
&\cdots \\
\bar{x}^{(2,n)}(t+1) &= (A_2 - LC)\bar{x}^n(t) + B_2u(t) + Ly(t) \\
&\cdots\cdots \\
\bar{x}^{(r,1)}(t+1) &= (A_r - LC)\bar{x}^1(t) + B_ru(t) + Ly(t) \\
&\cdots \\
\bar{x}^{(r,n)}(t+1) &= (A_r - LC)\bar{x}^n(t) + B_ru(t) + Ly(t)
\end{aligned}$$

And $\mathcal{X}(t+1) = conv\{\bar{x}^{(1,1)}(t+1),\cdots,\bar{x}^{(r,n)}(t+1)\}$, where $conv\{\cdot\}$ stands for the convex hull.

## 5 Convergence of the Estimation Error

In this section, we will study the uniformly ultimate boundedness of the estimation error $e(t) = \bar{x}(t) - x(t)$, which satisfies

$$e(t+1) = (A(w) - LC)e(t) - Ed(t) \qquad (5.1)$$

Our objective is to show that the error $e(t)$ is uniformly ultimately bounded in some C-set $\mathcal{E}$ for every admissible disturbance $d(t) \in \mathcal{D}$ and parameter uncertainty $w(t) \in \mathcal{W}$. For this purpose, we introduce the following concepts. Note that these concepts have previously appeared in [5, 4] and also in the references therein.

[1] The convergence issue of the state estimation error will be discussed in the next section based on the set-induced Lyapunov functions.

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A function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is said to be a *gauge function* if

1. $\Psi(x) \geq 0$, $\Psi(x) = 0 \Leftrightarrow x = 0$;
2. for $\mu > 0$, $\Psi(\mu x) = \mu \Psi(x)$;
3. $\Psi(x + y) \leq \Psi(x) + \Psi(y)$, $\forall x, y \in \mathbb{R}^n$.

A gauge function is convex and it defines a distance of $x$ from the origin which is linear in any direction. A gauge function $\Psi$ is 0-symmetric, that is $\Psi(-x) = \Psi(x)$, if and only if $\Psi$ is a norm.

If $\Psi$ is a gauge function, we define the closed set (possibly empty) $\bar{N}[\Psi, \xi] = \{x \in \mathbb{R}^n : \Psi(x) \leq \xi\}$. On the other hand, the set $\bar{N}[\Psi, \xi]$ is a C-set for all $\xi > 0$. Any C-set $\mathcal{S}$ induces a gauge function $\Psi_{\mathcal{S}}(x)$ (Known as Minkowski function of $\mathcal{S}$), which is defined as $\Psi(x) \doteq \inf\{\mu > 0 : x \in \mu\mathcal{S}\}$. Therefore a C-set $\mathcal{S}$ can be regarded as the unit ball $\mathcal{S} = \bar{N}[\Psi, 1]$ of a gauge function $\Psi$ and $x \in \mathcal{S} \Leftrightarrow \Psi(x) \leq 1$.

**Lemma 5.1** [5] If $\mathcal{E}$ is SPDI (or PDI if $\lambda = 1$) set for system (5.1) with convergence index $\lambda \leq 1$, then $\mu\mathcal{E}$ is so for all $\mu \geq 1$.

**Lemma 5.2** A C-set $\mathcal{E}$ is SPDI set for system (5.1) with convergence index $\lambda < 1$ *if and only if* there exists a gauge function $\Psi(e)$ such that the unit ball $\bar{N}[\Psi, 1] \subset \mathcal{E}$ and, if $e \notin int\{\bar{N}[\Psi, 1]\}$, then $\Psi(post(e, w, d)) \leq \lambda\Psi(e)$ for all $w \in \mathcal{W}$ and $d \in \mathcal{D}$ (or. equivalently, $\bar{N}[\Psi, \mu]$ is $\lambda$-contractive for all $\mu \geq 1$).

According to the above two lemmas, we can derive the following theorem about the uniformly ultimate boundedness of the estimation error $e(t)$.

**Theorem 5.1** The observation error $e(t)$ for the observer designed in the previous section is uniformly ultimate bounded with convergence rate $0 < \lambda < 1$ (or. $\beta = \frac{1-\lambda}{\tau}$) in the given C-set $\mathcal{E}$, *if and only if* the inequalities (4.3) are feasible. In addition,

$$x(t) \in \mathcal{X}(t) \oplus \mathcal{E} \tag{5.2}$$

for $t$ large enough, where $\oplus$ stands for the Minkowski sum.

*Proof* : $\mathcal{E}$ is a C-set, and let $\psi(e) = \Psi_{\mathcal{E}}(e)$ be its Minkowski functional. For any $e \in \mathbb{R}^n$, we have $\psi(e(t + 1)) \leq \lambda\psi(e(t))$ for all $e(t) \notin int\{\mathcal{E}\}$, because of linear inequalities (4.3) and according to Lemma 5.1. Then $\psi(e)$ is a Lyapunov function for system (5.1), which is uniquely generated from the target set $\mathcal{E}$ for any fixed $\lambda$. Such a function has been named Set-induced Lyapunov Function (SILF). Then the existence of the Lyapunov function implies the exponential convergence of the estimation error to $\mathcal{E}$ according to Lemma 5.2. The exponential convergence is in the sense that $\psi(e(t + 1)) \leq \lambda\psi(e(t))$ in discrete-time case or $\psi(e(t + \delta)) \leq e^{-\beta\delta}\psi(e(t))$ in continuous-time case (where $\beta = \frac{1-\lambda}{\tau}$). Also for any initial value of the estimation error $e(t_0)$, $\exists T \geq t_0$ such that for all $t \geq T$, $e(t) \in \mathcal{E}$ and $x(t) + e(t) \in \mathcal{X}(t)$. Therefore, $x(t) \in \mathcal{X}(t) \oplus \mathcal{E}$, where $\oplus$ stands for the Minkowski sum. This completes the proof. □

In the following, we will focus on linear constrained case for computational tractability. Let $\mathcal{E}$ be a polyhedral C-set for which the following plane description is given:

$$\mathcal{E} = \{e : \ f_i e \leq \theta_i, \ \theta_i > 0, \ i = 1, \cdots, s\} \tag{5.3}$$

or synthetically, $\mathcal{E} = \{e : \ Fe \leq \theta\}$. We call a polyhedral function the Minkowski function of a polyhedral C-set. This function has expression

$$\Psi(e) = \max_{1 \leq i \leq s} \{f_i e\} \tag{5.4}$$

We consider elements of the above form as candidate Lyapunov functions.

## 6 Numerical Example

Consider the following continuous-time uncertain systems:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -0.1 & w \\ -1 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ w \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

We assume that the time varying uncertain parameter $w$ is subjected to the constraint $1 \leq w \leq 2$, and the continuous disturbance $d(t)$ is bounded by $d \in \mathcal{D} = \{d : -0.05 \leq d \leq 0.05\}$. We consider a full state observer of the form (4.1):

$$\dot{\bar{x}}(t) = (\begin{bmatrix} -0.1 & w \\ -1 & -0.1 \end{bmatrix} - L \begin{bmatrix} 0 & 1 \end{bmatrix})\bar{x}(t) + \begin{bmatrix} 0 \\ w \end{bmatrix} u(t) + Ly(t)$$

Then, we know that the estimation error $e(t)$ satisfies

$$\dot{e}(t) = (\begin{bmatrix} -0.1 & w \\ -1 & -0.1 \end{bmatrix} - L \begin{bmatrix} 0 & 1 \end{bmatrix})e(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t)$$

we assume the specified set $\mathcal{E} = \{e \in R^2 : \|e\|_\infty \leq 0.1\}$. Our problem is to design the matrix $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$, such that the estimation error $e(t)$ exponentially converges to $\mathcal{E}$.

Using (3.7) with $\tau = 1$, we obtain the EAS system for estimation error as follows:

$$e(t+1) = (\begin{bmatrix} 0.9 & w \\ -1 & 0.9 \end{bmatrix} - L \begin{bmatrix} 0 & 1 \end{bmatrix})e(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t)$$

Then, using (4.3) for the above EAS system with $\lambda = 0.8$, we obtain

$$(\begin{bmatrix} 0.9 & w \\ -1 & 0.9 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix})e_i \leq 0.8 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \min_{d \in \mathcal{D}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t)$$

where $e_i$ corresponding to the four vertices of $\mathcal{E}$, that is $e_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$, $e_3 = \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix}$, and $e_4 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$. Solving the above inequalities with $w_1 = 1$ and $w_2 = 2$, we get the following conditions with respect to $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$.

$$-5.05 \leq l_1 \leq 8.05, \quad -6.05 \leq l_2 \leq 7.85$$

For example, we select $L = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then the set-valued observer has the form:

$$\dot{\bar{x}}(t) = \begin{bmatrix} -0.1 & w-1 \\ -1 & -1.1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ w \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y(t)$$

According to the discussion in Section 4, the implementation of the set-valued observer only needs to consider the evolution of the vertices of $\mathcal{X}(t)$ with respect to the case $w = 1, 2$. If we assume that the initial value of the observer is (or outer-approximated by) a polytope, $\mathcal{X}_0$, then at each time-being the output of the observer is also a polytope, $\mathcal{X}(t)$, which is the convex hull of the mapping of the vertices of $\mathcal{X}_0$ under the case of $w = 1, 2$. And $x(t) \in \mathcal{X}(t) \oplus \mathcal{E}$ for $t$ large enough, where $\oplus$ stands for the Minkowski sum. The estimation error $e(t)$ satisfies

$$\dot{e}(t) = \begin{bmatrix} -0.1 & w-1 \\ -1 & -1.1 \end{bmatrix} e(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t)$$

Following Section 5, we take the set-induced Lyapunov function from $\mathcal{E} = \{e \in R^2 : \|e\|_\infty \leq 0.1\}$ as $\Psi(x) = \max_{1\leq i\leq s}\{f_i e\} = \|e\|_\infty$. We know $\mathcal{E}$ is $\lambda$-contractive ($\lambda = 0.8$) by the above design procedure, so by theorem 5.1, the estimation is uniformly ultimately bounded in $\mathcal{E}$ with rate $\beta = \frac{1-\lambda}{\tau} = 0.2$.

## 7 Concluding Remarks

In this paper, we developed a set-valued state observer for a class of uncertain linear systems affected by both parameter variation and persistent disturbance. The design procedure proposed assures that the estimation error will be ultimately bounded within a given convex and compact set containing the origin with an assigned rate of convergence.

A necessary and sufficient condition for the existence of the observer in form of (4.1) was derived. However, the answer to the existence problem is not satisfactory, because the feasibility of these linear inequalities in (4.3), namely the condition that $\mathcal{E}$ is SPDI for some given $0 < \lambda < 1$ with respect to system (4.2), is still a problem. Some stronger results need to be obtained for the existence problem. In addition, a method should be developed to specify the parameter $\tau$ in the Euler approximating system (3.7).

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