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FEEDBACK CONTROLLER PARAMETERIZATIONS: FINITE HIDDEN MODES AND CAUSALITY

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ABSTRACT

Parameterizations of feedback controllers are derived in a unifying way, using polynomial matrix internal descriptions, and the important design issues of causality and hidden modes are clarified.

1. INTRODUCTION

The design of multivariable control systems can often be simplified if an appropriate parameterization of the feedback controller is used to incorporate important design objectives such as internal stability. Youla et al. [1] were the first to introduce a parameterization of all stabilizing controllers for linear multivariable systems. Since then, knowledge of this parameterization has been increased [2,16] and has been extended to more general classes of systems [3,4]; alternative controller parameterizations have also been introduced [5-8,17].

If feedback controller parameterizations are to be used effectively to control a system, the issues of causality and hidden modes must be clarified. In particular, note that certain parameterizations, the ones more closely related to the internal description of the plant (eg. Youla's), offer good control of the closed loop eigenvalues but might lead to a nonproper controller; other parameterizations involving rational matrices can easily solve the properness problem but they have less direct control over the closed loop eigenvalues and so can result in hidden modes and high order compensators (eg. using Zames' parameter [5,9] and proper, stable matrices or generalized polynomials [3,4,10].)

In this paper, controller parameterizations are derived in a unifying way, using polynomial matrix descriptions. In Proposition 1, all stabilizing controllers are characterized using parameters K [1] and (D_k,N_k) [2]. Alternative internal stability conditions are derived in Proposition 2; some of these conditions can be used as alternative definitions for internal stability [3,5]. Theorem 3 is fundamental in deriving known as well as novel parameterizations involving rational matrices as parameters; note that the use of internal de-

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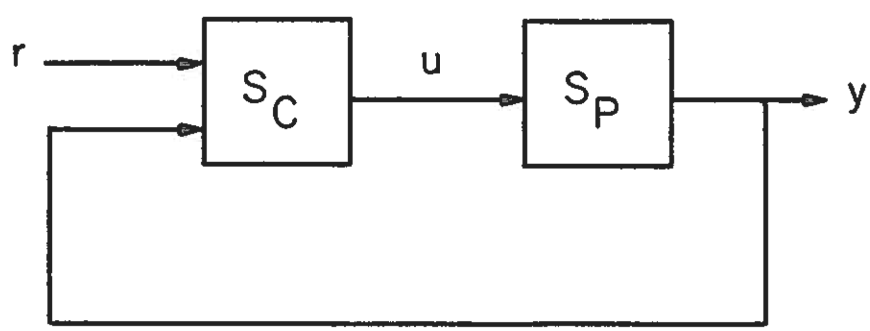
If feedback controller parameterizations are to be used effectively to control a system, the issues of causality and hidden modes must be clarified. In particular, note that certain parameterizations, the ones more closely related to the internal description of the plant (eg. Youla's), offer good control of the closed loop eigenvalues but might lead to a nonproper controller; other parameterizations involving rational matrices can easily solve the properness problem but they have less direct control over the closed loop eigenvalues and so can result in hidden modes and high order compensators (eg. using Zames' parameter [5,9] and proper, stable matrices or generalized polynomials [3,4,10].)

In this paper, controller parameterizations are derived in a unifying way, using polynomial matrix descriptions. In Proposition 1, all stabilizing controllers are characterized using parameters K [1] and (D_k,N_k) [2]. Alternative internal stability conditions are derived in Proposition 2; some of these conditions can be used as alternative definitions for internal stability [3,5]. Theorem 3 is fundamental in deriving known as well as novel parameterizations involving rational matrices as parameters; note that the use of internal descriptions in the analysis allows the detailed study of the internal structure of the feedback system when rational parameters are used to characterize all stabilizing controllers. The relation of stabilizing controllers to observers of the state is established in Proposition 4. Proposition 5 characterizes all stabilizing controllers using rational parameters, and Corollaries 5.1 and 5.2 study the special cases of stable and nonsingular plants; Corollary 5.3 introduces an additional test for internal stability. Causality of the controller is then discussed and methods to obtain proper controllers when using parameterizations are introduced; proper controllers are obtained in Proposition 6 working over stable and proper matrices [3,4]. The hidden modes of the feedback system are fully characterized and they are identified as the uncontrollable and/or unobservable modes in the case of the single degree of freedom feedback configurations {G,I;P} and {I,H;P}; this is done in terms of the transfer matrices of the plant and the controller, and also in terms of the parameters characterizing the stabilizing controllers. The discussion on hidden modes makes possible the characterization of the closed loop eigenvalues in terms of poles of loop quantities, which in turn leads to additional tests for internal stability. Finally in Proposition 7 a parameterization of all stabilizing controllers is employed to achieved desired command/output-response and command/control-response when {G,I;P} or {I,H;P} feedback configurations are used to compensate the plant.

2. MAIN RESULTS

Consider



where S_P is the given plant and S_C the controller. Assume S_P, S_C controllable and observable, and let their transfer matrices be given by

$$y = Pu \quad , \quad u = [-C_y, \ C_r]\begin{bmatrix} y \\ r \end{bmatrix} \tag{1}$$

where y is the output, u the control input and r the command input. If the feedback loop is well defined, that is if $|I + C_yP| \neq 0$, the closed loop transfer matrix T between y and r is given by

$$T = PM \tag{2}$$

where

$$M \stackrel{\Delta}{=} (I + C_y P)^{-1} C_r \ . \tag{3}$$

Notice that T characterizes the command/output-response $y = Tr$, while the command/control-response $u = Mr$ is characterized by M.

We are interested in causal controllers which internally stabilize the system. The input r does not affect internal stability; and for such studies it will be taken to be zero. In addition, we are interested in the hidden modes of the system. These are affected by C_r. We shall study the hidden modes of the following single degree of freedom feedback systems:

$\{G,I;P\}$ where $u = Ge$, $e = r - y$ ($C_y = G$, $C_r = G$). This is the error or unity feedback configuration with compensator G in the feedforward path.

$\{I,H;P\}$ where $u = -Hy + r$ ($C_y = H$, $C_r = I$) with compensator H in the feedback path.

2.1 <u>Internal Stability</u>. Let $u = -Cy$ ($r = 0$, $C = C_y$ for notational convenience) and consider the controllable and observable internal operator polynomial matrix descriptions [11]:

$$S_P : \underline{D}\underline{z} = \underline{N}u,\ y = \underline{z} \qquad ; (\underline{D},\underline{N})\ \ell p \tag{4}$$

$$S_C : \underline{D}_c \underline{z}_c = -\underline{N}_c y,\ u = \underline{z}_c \ ; (\underline{D}_c,\underline{N}_c)\ \ell p \tag{5}$$

(ℓp or rp will be used for left or right prime polynomial matrices). Then

$$\underline{A}\begin{bmatrix} \underline{z}_c \\ \underline{z} \end{bmatrix} = 0\ ,\ y = [0 \quad I]\begin{bmatrix} \underline{z}_c \\ \underline{z} \end{bmatrix} \quad ; \quad \underline{A} \stackrel{\Delta}{=} \begin{bmatrix} \underline{D}_c & \underline{N}_c \\ -\underline{N} & \underline{D} \end{bmatrix} \tag{6}$$

is the closed loop internal description. If the dual descriptions

$$S_P : Dz = u\ ,\ y = Nz \qquad ; \quad (D,N)\ rp \tag{7}$$

$$S_C : D_c z_c = -y\ ,\ u = N_c z_c \quad ; \quad (D_c,N_c)\ rp \tag{8}$$

are used, then

$$A\begin{bmatrix} z \\ z_c \end{bmatrix} = 0\ ,\ y = [N \quad 0]\begin{bmatrix} z \\ z_c \end{bmatrix} \quad ; \quad A \stackrel{\Delta}{=} \begin{bmatrix} D & -N_c \\ N & D_c \end{bmatrix}. \tag{9}$$

Note that, in transform terms, which we shall use hereafter unless otherwise noted,

$$P = \underline{D}^{-1}\underline{N} = ND^{-1} \quad , \quad C = \underline{D}_c^{-1}\underline{N}_c = N_c D_c^{-1} \tag{10}$$

are prime factorizations of P and C corresponding to the above internal descriptions. Furthermore,

$$\underline{A}\,A = \begin{bmatrix} \underline{D}_k & 0 \\ 0 & D_k \end{bmatrix} \tag{11}$$

where

$$\underline{D}_k \overset{\Delta}{=} \underline{D}_c D + \underline{N}_c N \quad , \quad D_k \overset{\Delta}{=} \underline{D}D_c + \underline{N}N_c \; . \tag{12}$$

Notice that other operator systems, such as

$$\underline{D}_k z = 0 \; , \; y = Nz \quad ; \quad D_k z_c = 0 \quad , \quad y = -D_c z_c \tag{13}$$

are also closed loop internal descriptions, equivalent to (6) and (9). $|A|$, $|\underline{A}|$, $|D_k|$ and $|\underline{D}_k|$ are therefore alternative expressions for the closed loop characteristic polynomial; and they are equal within a multiplicative constant. Note that similar results can be derived using matrix identities to evaluate $|\underline{A}|$ and $|A|$ from (6) and (9).

Definition. The feedback loop is well defined if the closed loop internal description is well defined, that is if $|\underline{A}| \neq 0$.

Notice that

$$(I + CP) = \underline{D}_c^{-1}\, \underline{D}_k D^{-1} \tag{14}$$

which, in view of the assumption that $|\underline{D}_c|$, $|D| \neq 0$ and the fact that $|\underline{A}| = k|\underline{D}_k|$, directly implies the known result, namely:

The feedback loop is well defined if and only if $|I + CP| \neq 0$.

Definition. The closed loop system is internally stable if $\underline{A}^{-1}$ exists and is stable.

Since the roots of $|\underline{A}|$ are the closed loop eigenvalues, the system will be internally stable when all eigenvalues lie in the open left half of the s-plane (continuous system) or inside the open unit disc in the z-plane (discrete system).

We shall now parametrically characterize all stabilizing controllers C. Here we shall follow the development in [2]:

Consider a unimodular matrix U so that $U\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and of the form:

$$U = \begin{bmatrix} x_1 & x_2 \\ -\underline{N} & \underline{D} \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} D & -\underline{x}_2 \\ N & \underline{x}_1 \end{bmatrix} . \tag{15}$$

It is known [2,11] that such a matrix U does exist. Postmultiply $\underline{A}$ of (6) by U^{-1} and premultiply A of (9) by U to obtain:

$$\underline{A}\, U^{-1} = \begin{bmatrix} \underline{D}_k & \underline{N}_k \\ 0 & I \end{bmatrix}, \quad U\, A = \begin{bmatrix} I & -N_k \\ 0 & D_k \end{bmatrix} \tag{16}$$

where $\underline{N}_k$, N_k are polynomial matrices. In view of the definitions of $\underline{A}$ and A, (16) directly implies that

$$[\underline{D}_c \; \underline{N}_c] = [\underline{D}_k \; \underline{N}_k]\, U \tag{17}$$

and

$$\begin{bmatrix} -N_c \\ D_c \end{bmatrix} = U^{-1} \begin{bmatrix} -N_k \\ D_k \end{bmatrix} . \tag{18}$$

Since U is unimodular, it is clear that $(\underline{D}_c,\underline{N}_c)$ ℓp $((N_c,D_c)$ rp) if and only if $(\underline{D}_k,\underline{N}_k)$ ℓp $((N_k,D_k)$ rp). Furthermore, if the product $(\underline{A}U^{-1})(UA)$ is determined using (16), then in view of (11),

$$\underline{D}_k N_k = \underline{N}_k D_k . \tag{19}$$

As it has been shown in [2,12], (17) and (18) characterize any and all solutions of equations (12) where $(D,N,\underline{D}_k)$, $(\underline{D},\underline{N},D_k)$ are given and $\underline{N}_k, N_k$ are arbitrary polynomial matrices of appropriate dimensions. Furthermore, in the system context, $\underline{D}_k$ and $\underline{N}_k$ must satisfy $|\underline{D}_k| \neq 0$, $|\underline{D}_k x_1 - \underline{N}_k \underline{N}| = |\underline{D}_c| \neq 0$ for the loop and controller to be well defined; and $\underline{D}_k^{-1}$ must be stable for internal stability. In view of (10), it follows that [2]:

<u>Proposition 1</u>. Any and all stabilizing controllers are given by

$$\begin{aligned} C &= (\underline{D}_k x_1 - \underline{N}_k \underline{N})^{-1}(\underline{D}_k x_2 + \underline{N}_k \underline{D}) = (x_1 - K\underline{N})^{-1}(x_2 + K\underline{D}) \\ &= (\underline{x}_2 D_k + D N_k)(\underline{x}_1 D_k - N N_k)^{-1} = (\underline{x}_2 + DK)(\underline{x}_1 - NK)^{-1} \end{aligned} \tag{20}$$

where $(\underline{D}_k,\underline{N}_k)$ $((N_k,D_k))$ are any polynomial matrices with appropriate dimensions such that $\underline{D}_k^{-1}$ (D_k^{-1}) is stable and $|\underline{D}_k x_1 - \underline{N}_k \underline{N}| \neq 0$ $(|\underline{x}_1 D_k - N N_k| \neq 0)$ or, alternatively, K is any stable rational matrix such that $|x_1 - K\underline{N}| \neq 0$ $(|\underline{x}_1 - NK| \neq 0)$.

Note that

$$K = \underline{D}_k^{-1} \underline{N}_k = N_k D_k^{-1} \tag{21}$$

which shows that the poles of K are the desired closed loop eigenvalues. It should be noted that the parameter K and the expression $C = (\underline{x}_2 + DK)(\underline{x}_1 - NK)^{-1}$ were first introduced in [1] using an alternative method.

Proposition 1 parametrically characterizes all stabilizing feedback controllers. The parameters are either the polynomial matrices D_k, N_k ($\underline{D}_k, \underline{N}_k$) or the rational matrix K. Note that this parameterization requires the knowledge of prime polynomial matrix factorizations of the plant transfer matrix P. It is clear that using (20), the designer has control over the closed loop internal descriptions; furthermore, complete and arbitrary closed loop eigenvalue assignment is easy to achieve by appropriately choosing D_k ($\underline{D}_k$) or K. The problem of causality is addressed in a later section of this paper.

Internal stability in the feedback loop can also be determined directly from the transfer matrices of the plant P and the controller C without using internal descriptions. In particular, let α_P, α_C denote the characteristic polynomials of P, C respectively and

$$S_1 \stackrel{\Delta}{=} (I + PC)^{-1} \quad , \quad S_2 \stackrel{\Delta}{=} (I + CP)^{-1} ,$$
$$Q \stackrel{\Delta}{=} C\, S_1 = S_2\, C \,. \tag{22}$$

Note that $\underline{D}(I + PC)y = \underline{D}S_1^{-1}y = 0$ and $\underline{D}_c(I + CP)u = \underline{D}_c S_2^{-1}u = 0$, when interpreted in an operator sense.

Proposition 2. The following statements are equivalent.

(a) The closed loop system is internally stable.

(b) $\alpha_P\alpha_C\, |S_1^{-1}| = \alpha_P\alpha_C\, |S_2^{-1}|$ is Hurwitz.

(c) The zero polynomial of $\begin{bmatrix} I & C \\ -P & I \end{bmatrix}$ is Hurwitz.

(d) $\begin{bmatrix} S_2 & -Q \\ PS_2 & S_1 \end{bmatrix}$ is stable.

Proof: (b) $S_2^{-1} = I + CP = \underline{D}_c^{-1}\underline{D}_k D^{-1}$ and $\alpha_P = |D|$, $\alpha_C = |\underline{D}_c|$. Then $\alpha_P\alpha_C\, |S_2^{-1}| = |\underline{D}_k| = k\, |\underline{A}|$ which is Hurwitz by definition. Also $|S_2^{-1}| = |I + CP| = |I + PC| = |S_1^{-1}|$.

To show (c), note that $\begin{bmatrix} I & C \\ -P & I \end{bmatrix} = \begin{bmatrix} \underline{D}_c & 0 \\ 0 & \underline{D} \end{bmatrix}^{-1} \underline{A}$, which is a ℓp factorization. The zero polynomial is $|\underline{A}|$ and therefore Hurwitz. (d) The zero polynomial in (c) is the characteristic polynomial of the inverse system $\begin{bmatrix} I & C \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I+CP)^{-1} & -C(I+PC)^{-1} \\ P(I+CP)^{-1} & (I+PC)^{-1} \end{bmatrix}$, which is the matrix in (d). ΔΔΔ

Note that (b) is a well known stability test, while the matrix in (d) has been used by Zames and Desoer in [3,5] and earlier papers to define stable feedback systems. Here internal stability is defined using internal descriptions from which the results of Proposition 2 are easily derived as alternative tests for internal stability.

The feedback controller parameterizations of Proposition 1 are closely related to internal system descriptions; and it is clear that they can be used to solve directly the design problems of eigenvalue assignment and stabilization. However, if the main design objective is to obtain desired input/output maps between certain signals in the loop, then it may be more convenient to use alternative controller parameterizations directly related to those maps. The following basic feedback compensation theorem introduces such parameterizations and establishes their relation to the internal system descriptions.

<u>Theorem 3.</u> Given a plant $y = Pu$ with $P = ND^{-1}(=\underline{D}^{-1}\underline{N})$ prime polynomial matrix factorizations, and controller $u = -Cy$, the closed loop system is internally stable if and only if

$$C = \underline{L}_2^{-1}\underline{L}_1 \quad (C = L_1 L_2^{-1}) \tag{23}$$

where $\underline{L}_1$, $\underline{L}_2$ (L_1, L_2) are stable rational matrices with $|\underline{L}_2|$ $(|L_2|) \neq 0$ which satisfy

$$\underline{L}_2 D + \underline{L}_1 N = I \quad (\underline{D} L_2 + \underline{N} L_1 = I) \ . \tag{24}$$

Furthermore, if

$$[\underline{L}_2 \ \underline{L}_1] = \underline{D}_k^{-1} \ [\underline{D}_c \ \underline{N}_c] \ \left(\begin{bmatrix} L_2 \\ L_1 \end{bmatrix} = \begin{bmatrix} D_c \\ N_c \end{bmatrix} D_k^{-1} \right) \tag{25}$$

are prime polynomial matrix factorizations, the closed loop internal description is given in operator terms by

$$\underline{D}_k z = 0, \ y = Nz \ (D_k z_c = 0, \ y = -D_c z_c). \tag{13}$$

Proof: The part in parentheses will not be shown as it follows in a similar way. Let $[\underline{L}_2 \ \underline{L}_1]$ stable satisfy (23), (24) and write a ℓp factorization as in (25). Then $\underline{D}_c D + \underline{N}_c N = \underline{D}_k$ where $(\underline{D}_c \ \underline{N}_c)$ ℓp, $\underline{D}_c^{-1}$ and $\underline{D}_k^{-1}$ exist with $\underline{D}_k^{-1}$ stable. Therefore the feedback loop with controller $C = \underline{L}_2^{-1}\underline{L}_1 = \underline{D}_c^{-1}\underline{N}_c$ is well defined, and it is internally stable with internal description (13). Assume now that the closed loop system is internally stable, that is $\underline{D}_k^{-1}$ in (13) exists and is stable. (17) implies that $\underline{D}_k^{-1} \ [\underline{D}_c \ \underline{N}_c] = [I \ K] \ U$; note that $(\underline{D}_k, [\underline{D}_c \ \underline{N}_c])$ is ℓp since $(\underline{D}_c, \underline{N}_c)$ is ℓp. Let $\underline{L}_2 = \underline{D}_k^{-1}\underline{D}_c$, $\underline{L}_1 = \underline{D}_k^{-1}\underline{N}_c$; the rest easily follows. $\Delta\Delta\Delta$

The stable rational matrices $\underline{L}_1$, $\underline{L}_2$, which satisfy (24) and characterize all stabilizing output controllers, also characterize all <u>state observers</u>.

Proposition 4. $[\underline{L}_2\ \underline{L}_1]\begin{bmatrix} u \\ y \end{bmatrix}$, where $\underline{L}_1$, $\underline{L}_2$ are stable, is a partial state observer if and only if $\underline{L}_2D + \underline{L}_1N = I$.

Proof: In view of the plant description (7) $Dz=u$ $y=Nz$, $\underline{L}_2u + \underline{L}_1y = (\underline{L}_2D + \underline{L}_1N)z$ which is equal to the partial state z if and only if $\underline{L}_2D + \underline{L}_1N = I$. ΔΔΔ

Observers of linear functionals of the state of the form Fz can also be easily derived as follows:

Let D be column proper (reduced) and let F be such that the column degrees of D, $\partial_{ci}D > \partial_{ci}F$; F is a desired state feedback matrix ($u = Fz$) [11]. Let $D_F = D-F$; then in view of (24)

$$(I - D_F\underline{L}_2)D + (-D_F\underline{L}_1)N = F$$

which implies

$$(I - D_F\underline{L}_2)u + (-D_F\underline{L}_1)y = Fz.$$

This means that $[I - D_F\underline{L}_2, -D_F\underline{L}_1]$ is an observer with output Fz. Furthermore, if

$$[I - D_F\underline{L}_2, -D_F\underline{L}_1] = L^{-1}[K_1, K_2]$$

is a left prime factorization, then

$$K_1D + K_2N = LF$$

which is precisely the relation used in [11] to derive linear state feedback realizations via an observer compensation scheme.

These results establish the exact relation between the factors of stabilizing compensators C and observers of the state of the plant. Also note that if $\underline{L}_1$ is chosen as discussed in a later section to guarantee C proper, then the corresponding observer of Fz derived above will also be proper.

If the output feedback configuration {G,H;P}, where $u = G(r - HY)$, is used to realize linear state feedback then the appropriate choices for G and H are [13]:

$$G = (L - K_1)^{-1}L = (D_F\underline{L}_2)^{-1} \quad , \quad H = -L^{-1}K_2 = D_F\underline{L}_1.$$

It is clear that the feedback path compensator H is stable; however, the feedforward path compensator G is stable if and only if $\underline{L}_2^{-1}$ is stable (D_F^{-1} is chosen to be stable) that is, if and only if $\bar{C}$, in Theorem 3, is stable. Therefore, stabilizing the plant P via a stable controller C is a problem in precisely the same spirit as that of realizing a state feedback control law via {G,H;P} compensation with G and H stable. Furthermore note that in view of Theorem 3, the above choice for G and H stabilize the plant; this is an alternative proof of the known result, namely that any coprimely represented P can be stabilized with the aid of an observer.

It is of interest to notice that any stabilizing controller C of the plant P can be written, in view of Theorem 3, as a product of two factors; one of the factors is stable ($\underline{L}_1$) while the other ($\underline{L}_2^{-1}$) has stable transmission zeros [18]. This observation also implies that $P\underline{L}_2^{-1}$ (or $P(D_F\underline{L}_2)^{-1}$) can be stabilized by a stable controller $\underline{L}_1$ ($D_F\underline{L}_1$).

In view of Theorem 3, a number of controller parameterizations can now be readily derived. In particular, we have the following.

Proposition 5. Any and all stabilizing controllers are given by:

(a) $C = \underline{L}_2^{-1}\underline{L}_1 \; (= L_1 L_2^{-1})$ (23)

where $\underline{L}_2, \underline{L}_1$ (L_2, L_1) satisfy (24);

(b) $C = S_2^{-1}Q \; (= QS_1^{-1})$ (26)

where $D^{-1}[S_2 \; Q]$ $(S_1\underline{D}^{-1}, Q\underline{D}^{-1})$ stable with $|S_2|$ ($|S_1|$) $\neq 0$ satisfying $S_2 + QP = I$ $(S_1 + PQ = I)$; (27)

(c) $C = [(I - \underline{L}_1 N)D^{-1}]^{-1}\underline{L}_1$ (28)

where $(I - \underline{L}_1 N)D^{-1}$, $\underline{L}_1$ stable with $|I - \underline{L}_1 N| \neq 0$;

(d) $C = Q(I - PQ)^{-1}$ (29)

where $(I - PQ)\underline{D}^{-1}$, $Q\underline{D}^{-1}$ stable with $|I - PQ| \neq 0$.

Proof: (a) is clear in view of Theorem 3. $S_2 + QP = I$ can be written as $(D^{-1}S_2)D + (D^{-1}Q)N = I$ which in view of (a) directly implies (b). If $\underline{L}_2$, S_2 are expressed in terms of $\underline{L}_1$, Q respectively (c) and (d) are derived; note that the dual of (c) and (d) are also true. ΔΔΔ

It should be noted that these parameterizations are related to internal descriptions of the closed loop system via (25). In this way, the effect of the particular choice for the parameter on the closed loop eigenvalues and, in general, on the closed loop internal description can be determined.

When C is expressed in terms of Q, S_2, S_1 as above, (22) are satisfied. These parameters are important design maps related to feedback and response properties. For example, $S_1 = (I + PC)^{-1}$ is the well known comparison sensitivity matrix which provides a measure of the effect of the parameter variations in P on the output y. Expressing the internal stability criteria directly in terms of these maps can provide significant insight in design. Note that the parameter Q is the parameter introduced by Zames in [5] using an alternative method; furthermore $\underline{L}_1$ is in the case of error feedback configuration the design parameter X discussed by Sain, et. al. [6-8]. Notice that there is a one-to-one correspondence between $\underline{L}_1$ or Q and the stabilizing compensators C given by (28) or (29).

In Propositions 1 and 5 all stabilizing controllers have been parametrically characterized. The parameters involved are of course related and their exact relation is easily derived to be:

$$\begin{aligned} \underline{L}_2 &= D^{-1}S_2 = x_1 - K\underline{N} \quad (L_2 = S_1\underline{D}^{-1} = \underline{x}_1 - NK), \\ \underline{L}_1 &= D^{-1}Q = x_2 + K\underline{D} \quad (L_1 = Q\underline{D}^{-1} = \underline{x}_2 + DK). \end{aligned} \tag{30}$$

In view of these relations, additional parameterizations involving combinations of these parameters (e.g. S_1 and K) can also be obtained.

When the plant P is unstable it is clear that the conditions (c) and (d) impose restrictions on the structure of the parameters $\underline{L}_1$ and Q. When the plant is stable, the conditions on the parameters in Proposition 5 are simplified [5,8]:

Corollary 5.1. If P is stable, any and all stabilizing controllers are given by:

(a) $C = [(I - \underline{L}_1 N)D^{-1}]^{-1}\underline{L}_1$

$$\text{where } \underline{L}_1 \text{ is stable with } |I - \underline{L}_1 N| \neq 0; \tag{31}$$

(b) $C = Q(I - PQ)^{-1}$

$$\text{where } Q \text{ is stable with } |I - PQ| \neq 0. \tag{32}$$

Proof: When P is stable, D^{-1} is stable. Note that the dual of (a) and (b) are also true. ΔΔΔ

When P is stable, these parameterizations are simple to use because, for internal stability, the only restriction imposed on the parameter is that it must be stable; furthermore, as it will be shown in the section on causality, if P is proper and Q or $D\underline{L}_1$ are chosen to be strictly proper, then C will be proper. Q and (32) are used in [5,9] where it is assumed that the plant P is proper and stable; Q is then chosen to satisfy additional design requirements.

When P is square and nonsingular, that is $|P| \neq 0$, additional parameterizations can be derived:

Corollary 5.2. If $|P| \neq 0$, any and all stabilizing controllers are given by:

(a) $C = \underline{L}_2^{-1}(I - \underline{L}_2 D)N^{-1}$

$$\text{where } \underline{L}_2,\ (I - \underline{L}_2 D)N^{-1} \text{ are stable with } |\underline{L}_2| \neq 0. \tag{33}$$

(b) $C = (S_1 P)^{-1}(I - S_1)$

$$\text{where } N^{-1}S_1 P,\ N^{-1}(I - S_1) \text{ are stable with } |S_1| \neq 0. \tag{34}$$

Proof: The proof of (a) is similar to (c) of Proposition 5. (b) can be shown directly: notice that $[N^{-1}S_1P]D + [N^{-1}(I - S_1)]N = I$ and $I + PC = I + P(S_1P)^{-1}(I - S_1) = S_1^{-1}$ which in view of Theorem 3 implies the result. The dual of (a) and (b) are also true. ΔΔΔ

If P has no zeros in the closed right half s-plane (or outside the closed unit disc) then N^{-1} is stable and stabilizing controllers can be derived by choosing any stable $\underline{L}_2$ or S_1, with S_1P stable, in (a) and (b) above; this is true for stable or unstable plants P. However, using these parameterizations, proper C is more difficult to obtain. Nevertheless, Corollary 5.2 points to the fact that if P has stable zeros then it can be easily stabilized via a, not necessarily proper, controller C.

Corollaries 5.1 and 5.2 are examples of cases where special properties of P are used to derive alternative parameterizations. It is clear that for a given plant, additional parameterizations could be derived depending on the particular properties of P.

In the following corollary an internal stability test is presented:

<u>Corollary 5.3.</u> The closed loop system is internally stable if and only if $|I + CP| = |I + PC| \neq 0$ and

$$D^{-1}\ [(I + CP)^{-1},\ (I + CP)^{-1}C]\ (\begin{bmatrix}(I + PC)^{-1} \\ C(I + PC)^{-1}\end{bmatrix}\ \underline{D}^{-1})\ \text{stable} \tag{35}$$

Proof: See (d) of Proposition 5 and (22).

Notice that in (d) of Proposition 2, four rational matrices must be stable for internal stability. In Corollary 5.3, by using the denominator D ($\underline{D}$) of the plant P, internal stability depends on the stability of only two rational matrices.

2.2 <u>Causality.</u> We are interested in proper controllers C. In view of

$$C = Q(I - PQ)^{-1} \tag{29}$$

if Q is proper and $(I - PQ)$ at (∞) is finite and nonsingular (that is $(I - PQ)$ and its inverse are proper, or $(I - PQ)$ is biproper) then C is proper. If in addition P is proper then any strictly proper Q satisfies the requirements and results, by (29), in a strictly proper controller C.

It is therefore clear that proper stabilizing controllers C are derived from (d) of Proposition 5 if the additional requirement of Q proper and $(I - PQ)$ biproper is added. Using the dual of (d) the equivalent conditions Q proper and $(I - QP)$ biproper are derived. These conditions on Q are easily translated into conditions on $\underline{L}_1$ (L_1) using (30) to give $D\underline{L}_1$ proper and $(I - N\underline{L}_1)$ biproper; in this way (c) of Proposition 5 can also be used to obtain proper stabilizing controllers C.

In view of (30)

$$K = (\underline{L}_1 - x_2)\underline{D}^{-1} = (D^{-1}Q - x_2)\underline{D}^{-1}. \tag{36}$$

For any Q or $\underline{L}_1$ which satisfies the causality conditions or, with a proper P for simplicity, for any Q strictly proper, the corresponding K, if used in (20) of Proposition 1 will give a proper C. For stability, K must also be stable or Q must satisfy the conditions in Proposition 5 unless P is stable, in which case Q stable suffices (Corollary 5.1). It is apparent that it is more difficult to choose K (or $\underline{D}_k,\underline{N}_k$), than Q or $\underline{L}_1$, to guarantee causality of the controller. In general, for C proper, the order of K (or $\partial|\underline{D}_k|$ in (12)) must be higher than the order of P by an amount which depends on the structure of N and D [8]. This can be seen from the known result [14] namely that all closed loop eigenvalues can be arbitrarily assigned with a controller C of order min $(\mu-1,\nu-1)$ where μ and ν are the controllability and observability indices of P; this result implies that for $\partial|\underline{D}_k|$ in (12) large enough proper C does exist. Note that for particular (N,D) it might be possible to derive lower order proper C, that is proper solution $C = \underline{D}_c^{-1}\underline{N}_c$ of (12) might exist for lower $\partial|\underline{D}_k|$; this is the case when for example eigenvalue assignment via constant output feedback is possible.

Proper C can also be achieved if one works over the proper and stable matrices [3,4]. This is shown here to be a direct consequence of Theorem 3.

Proposition 6. Let P be strictly proper. Any and all proper stabilizing controllers are given by

$$C = L_2'^{-1}\, L_1' \tag{37}$$

where $L_2'\, D' + L_1'\, N' = I$, $|L_2'| \neq 0$, $P = N'\, D'^{-1}$ with all (') matrices proper and stable.

Proof: If $L_2'D' + L_1'N' = I$, then $L_2' + L_1'\, P = D'^{-1}$ which implies D'^{-1} proper, that is D' biproper. Also, if C is proper then $D' + CN' = L_2'^{-1}$ which implies that $L_2'^{-1}$ is proper, that is L_2' biproper. Let $\begin{bmatrix} D' \\ N' \end{bmatrix} = \begin{bmatrix} D \\ N \end{bmatrix} D_1^{-1}$ [6-7] and let $D_1^{-1}\,[L_2'\ L_1'] = [\underline{L}_2,\underline{L}_1]$; then $\underline{L}_2D + \underline{L}_1N = I$ which in view of Theorem 3 implies that the proper $C = L_2'^{-1}\, L_1' = \underline{L}_2^{-1}\underline{L}_1$ is a stabilizing controller. Assume that the proper $C = \underline{L}_2^{-1}\underline{L}_1$ satisfies the conditions of Theorem 3. Let D_1 be a polynomial matrix such that D_1^{-1} is stable, $D_1\underline{L}_2$ is biproper and $D_1\underline{L}_1$ is proper. Note that such a matrix exists because P strictly proper and C proper imply $D\underline{L}_1$ proper, $D\underline{L}_2 = I - D\underline{L}_1P$ biproper (see causality arguments).

Now $\underline{L}_2D + \underline{L}_1 N = I$ implies $(D_1\underline{L}_2)(DD_1^{-1}) + (D_1\underline{L}_1)(ND_1^{-1}) = I$ from which $(D_1\underline{L}_2) + (D_1\underline{L}_1) P = (DD_1^{-1})^{-1}$ which implies that (DD_1^{-1}) is biproper; in view of P proper, (ND_1^{-1}) is also proper. If $[L_2', L_1'] = D_1[\underline{L}_2 \; \underline{L}_1]$ and $D' = DD_1^{-1}$, $N' = ND_1^{-1}$ the result is obtained. ΔΔΔ

All solutions of $L_2'D' + L_1'N' = I$ over the proper and stable matrices are given by

$$[L_2'L_1'] = [x_1' - K'\underline{N}' \quad x_2' + K'\underline{D}'] \tag{38}$$

with K' any proper and stable matrix corresponding to parameter K in (20) [3,4,12]. Proper stabilizing controllers C are easily obtained by appropriately choosing K'. Note that in view of the above proof and (25), the closed loop eigenvalues are the zeros of $|\underline{D}_k|$ where

$$D_1^{-1} [x_1' - K'\underline{N}' \quad x_2' + K'\underline{D}'] = \underline{D}_k^{-1} [\underline{D}c \; \underline{N}c] \tag{39}$$

a left prime polynomial factorization with $\begin{bmatrix} D' \\ N' \end{bmatrix} = \begin{bmatrix} D \\ N \end{bmatrix} D_1^{-1}$ [6-7].

This shows that the closed loop eigenvalues (zeros of $|\underline{D}_k|$) depend on x_1', x_2' in addition to K'. Clearly in this case a desired set of closed loop eigenvalues cannot be achieved easily via K'. Note that similar difficulty is encountered when the parameter Q [5,9] or the λ-generalized polynomials [10] are used. In the latter method, a transformation $\lambda = 1/(s+a)$ is introduced to transform the prbper transfer matrices into polynomial matrices in λ; in this way causality is easily achieved but multiple closed loop eigenvalues at $-a$ tend to appear. The discussion on hidden modes sheds more light upon this issue.

2.3. Hidden Modes. Hidden modes are those modes of the closed loop system which do not appear as poles of the closed loop transfer matrix. They correspond to closed loop eigenvalues which are uncontrollable and/or unobservable. If the internal stability conditions are satisfied, it is clear that unstable hidden modes do not exist in the loop. It is possible, however, to have stable hidden modes which might cause undesirable signal behavior in the loop; furthermore these hidden modes unnecessarily increase the order of the controller [8].

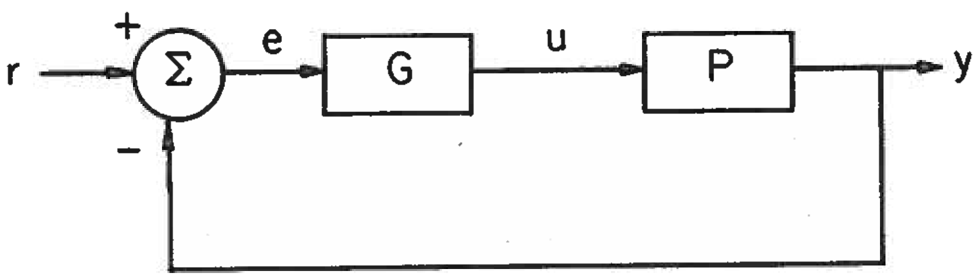
The unobservable eigenvalues can be determined using the internal operator description (13) since an output y has already been specified. They are the zeros of the determinant of a greatest right divisor (grd) of $(\underline{D}_k, N)$ [11]. This grd is equal to a grd of $(\underline{D}_cD, N)$ since

$$\begin{bmatrix} \underline{D}_k \\ N \end{bmatrix} = \begin{bmatrix} I & \underline{N}_c \\ 0 & I \end{bmatrix} \begin{bmatrix} \underline{D}_cD \\ N \end{bmatrix} \tag{40}$$

which implies, in view of $N(\underline{D}_cD)^{-1} = P\underline{D}_c^{-1}$, that the unobservable eigenvalues are exactly those poles of C which cancel in the product PC.

If parameterizations involving D_k or K and (20) are used, one can easily control the number of unobservable hidden modes. However, when parameterizations involving $\underline{L}_1$, Q and so forth, as in Proposition 5, are used, it is not as clear how this can be achieved. This is studied next. The unobservable eigenvalues are exactly those poles of $\underline{L}_1$ which cancel in $N\underline{L}_1$. This can be shown using (25) as follows: The unobservable eigenvalues are those zeros of $|D_k|$ which cancel in ND_k^{-1} or in $ND_k^{-1}[\underline{D}_c\ \underline{N}_c] = N[(I - \underline{L}_1 N)D^{-1}\ \underline{L}_1]$, since $(D_k, (\underline{D}_c, \underline{N}_c))$ are prime and D_k^{-1} contains all poles of $\underline{L}_1$. The only poles which cancel in $N(I - \underline{L}_1 N)D^{-1} = (I - N\underline{L}_1)P$ with N, are the poles of $\underline{L}_1$ which cancel $N\underline{L}_1$. Furthermore note that in view of (23) all the unobservable eigenvalues appear as poles of C.

To discuss uncontrollable eigenvalues we must specify an input r. Consider first unity or error feedback:



$$\{G,I;P\}: \quad u = Ge,\ e = r-y.$$

In view of (1-3), $C_y = C = C_r = G$ and

$$T = PM_G \quad , \quad M_G = (I + GP)^{-1}G \tag{41}$$

where T (y=Tr) is the closed loop transfer matrix and M_G ($u=M_G r$) characterizes the control action u. Using the internal operator description $Dz = u$, $y = Nz$ and $\underline{D}_c \underline{z}_c = \underline{N}_c e$, $u = \underline{z}_c$ for the plant P and the controller G respectively with $e = r-y$, the closed loop internal description is

$$D_k z = \underline{N}_c r \quad , \quad y = Nz \tag{42}$$

with $D_k = \underline{D}_c D + \underline{N}_c N$ as in (12).

The uncontrollable eigenvalues are the zeros of the determinant of a greatest common left divisor ($g\ell d$) of $(D_k, \underline{N}_c)$. This $g\ell d$ is equal to a $g\ell d$ of $(\underline{D}_c D, \underline{N}_c)$ since

$$[D_k\ \underline{N}_c] = [\underline{D}_c D\ \underline{N}_c] \begin{bmatrix} I & 0 \\ N & I \end{bmatrix} \tag{43}$$

which implies, in view of $(\underline{D}_c D)^{-1}\underline{N}_c = D^{-1}G$, that in the {G,I;P} feedback configuration the <u>uncontrollable eigenvalues are exactly those poles of P which cancel in the product PG.</u> Note that, as it was shown above, the unobservable eigenvalues are exactly those poles of G which cancel in the product PG.

When (20) and parameters $\underline{D}_k$ or K are used then one can easily control the number of uncontrollable hidden modes. However, if parameters $\underline{L}_1$, Q and so forth are used, the issue becomes quite complicated and it is treated below. First note that, as it was shown above, the unobservable eigenvalues are those poles of $\underline{L}_1$ which cancel in $N\underline{L}_1$. It is now shown that <u>the uncontrollable eigenvalues are exactly those poles of P which do not cancel in</u> $(I - \underline{L}_1 N)D^{-1}$:

In view of (25), $[(I - \underline{L}_1 N)D^{-1}\ \underline{L}_1] = \underline{D}_k^{-1}\ [\underline{D}_c, \underline{N}_c]$ is ℓp. Let $\underline{L}_1 = D_1^{-1}N_1$ be a left prime factorization. Then $[\underline{D}_c\ \underline{N}_c] = \underline{D}_k\ D_1^{-1}$ $[(D_1 - N_1 N)D^{-1},\ N_1]$. Since $\underline{N}_c$ is a polynomial matrix, $\underline{D}_k = D_2 D_1$ and $\underline{N}_c = D_2 N_1$; also $\underline{D}_c = D_2(D_1 - N_1 N)D^{-1}$. Clearly the poles of D^{-1} (of P) which do not cancel in $(D_1 - N_1 N)$ must cancel D_2 completely since $\underline{D}_c$ is a polynomial matrix and $(\underline{D}_c, \underline{N}_c)$ are ℓp. This shows the result because D_2 is a $g\ell d$ of $(\underline{D}_k, \underline{N}_c)$ and it contains the uncontrollable eigenvalues. Note that the uncontrollable eigenvalues appear in the numerator $\underline{N}_c$ of the controller G.

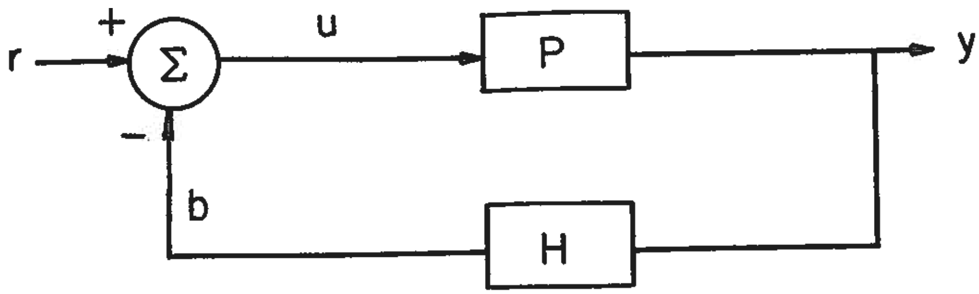
It is clear that in view of $\underline{L}_1 = D^{-1}Q$, the parameter Q can also be used to characterize the hidden modes. If (25) is expressed in terms of Q then

$$D^{-1}[I - QP \quad Q] = \underline{D}_k^{-1}\ [\underline{D}_c\ \underline{N}_c]\ \ \ell p. \tag{44}$$

It follows that if Q is chosen to be proper and stable as in the case of stable P [5,9] then at least all of the poles of P will tend to appear as uncontrollable hidden modes in the loop thus increasing the complexity of the controller. In order to eliminate the possibly undesirable stable hidden modes (poles of P) and simplify the controller, the designer must use the above results on hidden modes, which impose additional structural restrictions on Q (or $\underline{L}_1$) similar to the ones imposed when P is unstable. This of course reduces the ease of implementation of this parameterization.

Notice that if $G = \underline{L}_2^{-1}\ \underline{L}_1$ in (41), then $M_G = D\underline{L}_1$ (=Q) and $T = N\underline{L}_1$; that is $\underline{L}_1$ is the <u>design matrix</u> X (z=Xr) introduced in [6,7] and used in [8] to show similar results.

We consider next another of a variety of possible feedback configurations which have come to be known as single degree of freedom types. The previous case, of course, which is output error feedback, is probably the most frequently studied example in the class. However, the case following is also of considerable interest, and has been commonly studied in the classical literature. In addition, it plays an important role in the applications, due to the use of sensors in the feedback path. Of course, the two cases under study here do not exhaust the set of possibilities; but they are representative of the techniques by which such problems may be approached. Consider, then, the following diagram:



$\{I,H;P\} : u = -Hy + r.$

In view of (1-3), $C_y = C = H$, $C_r = I$ and

$$T = PM_H \quad , \quad M_H = (I + HP)^{-1} \ . \tag{45}$$

Using the internal operator descriptions $Dz = u$, $y = Nz$ and $\underline{D}_c z_c = \underline{N}_c y$, $b = z_c$ for the plant P and the controller H respectively with $u = r-b$, the closed loop internal description is

$$\underline{D}_k z = \underline{D}_c r \quad , \quad y = Nz \tag{46}$$

with $\underline{D}_k$ as in (12). Proceeding similarly, the uncontrollable eigenvalues are the zeros of the determinant of a gℓd of $(\underline{D}_k,\underline{D}_c)$ or of $(\underline{N}_c N,\underline{D}_c)$ which implies, in view of $\underline{D}_c^{-1}\underline{N}_c N = HN$ that they are exactly those poles of H which cancel in the product HP. Note that, as it was shown above, the unobservable eigenvalues are those poles of H which cancel in the product PH.

If parameters $(\underline{L}_1,\underline{L}_2)$ are used, it was shown that the unobservable eigenvalues are those poles of $\underline{L}_1$ which cancel in $N\underline{L}_1$. It is now shown that, in the $\{I,H;P\}$ feedback configuration, the uncontrollable eigenvalues are those poles of $\underline{L}_1$ which cancel in $\underline{L}_1 N$:

Let $\underline{L}_2 = D_1^{-1}N_1$ be a left prime factorization. Then in view of (25), $\underline{D}_c = \underline{D}_k\underline{L}_2 = \underline{D}_k D_1^{-1}N_1$ which implies that $\underline{D}_k = D_2D_1$ and $\underline{D}_c = D_2N_1$. D_2 is a gld of $(\underline{D}_k,\underline{D}_c)$ and it contains all the uncontrollable eigenvalues. Note that $\underline{N}_c = \underline{D}_k\underline{L}_1 = D_2D_1\underline{L}_1$; since $\underline{N}_c$ is a polynomial matrix ℓp to $\underline{D}_c$, the poles of $\underline{L}_1$ must cancel all zeros of $|D_2|$. Now the gℓd of $(\underline{N}_c N,\underline{D}_c)$ contains all the zeros of $|D_2|$. Therefore all the poles of $\underline{L}_1$ which correspond to the uncontrollable eigenvalues must cancel with N in $\underline{N}_c N = D_2D_1\underline{L}_1 N$. Note that the uncontrollable eigenvalues appear in the denominator $\underline{D}_c$ of the controller H.

It is clear that, in view of $\underline{L}_1 = D^{-1}Q$, the parameter Q can also be used to characterize the hidden modes. Notice that if $H = \underline{L}_2^{-1}\underline{L}_1$ in (45) then $M_H = D\underline{L}_2$ and $T = N\underline{L}_2$, that is $\underline{L}_2$ is the design matrix X [6-8]; furthermore $\underline{L}_1 = XH$ and $Q = M_H H = D\underline{L}_1$. For no hidden modes, $\underline{L}_1$ must be chosen so that no cancellations take place in $\underline{L}_1 N$ or in $N\underline{L}_1$.

In view of the above discussion on hidden modes, the closed loop eigenvalues can be easily described in terms of poles of certain transfer matrices in the loop. This provides additional insight and leads to additional tests for internal stability. The results presented below can be shown either by using the hidden modes results derived above for the single degree of freedom feedback configurations or by using internal descriptions and derivations similar to the ones used in the above proofs:

Consider the $\{G,I;P\}$ feedback configuration. The closed loop eigenvalues are:

(i) The poles of $PG(I + PG)^{-1} = T$ and the poles of P and G which cancel in PG (uncontrollable and unobservable eigenvalues).

(ii) The poles of $(I + PG)^{-1} = S_1$ and the poles of P and G which cancel in PG. Notice that $S_1 = I - T$.

(iii) The poles of $(I + GP)^{-1} = S_2$ and the poles of P and G which cancel in GP.

(iv) The poles of $G(I + PG)^{-1} = M_G$ and the poles of P which cancel in PG and GP.

(v) The poles of $P(I + GP)^{-1}$ and the poles of G which cancel in PG and GP.

Notice that (ii) implies that the closed loop system is internally stable if and only if $(I + PG)^{-1}$ is stable and no unstable cancellations take place in PG; this is the main result of [15].

(vi) The poles of X and the poles of P which cancel in PG (uncontrollable eigenvalues). Note that the poles of X which cancel in NX are the unobservable eigenvalues; the poles of P which cancel in GP are the poles of X which cancel in DX.

As a direct consequence of Theorem 3 and the relation between X and $\underline{L}_1$, $\underline{L}_2$ the following important result is presented, an alternative proof of which was given in [8].

Proposition 7

$\begin{bmatrix} T \\ M \end{bmatrix} = \begin{bmatrix} N \\ D \end{bmatrix} X$ can be realized with internal stability via:

(a) $\{G,I;P\}$ compensation if and only if

$$X,\ (I - XN)D^{-1} \text{ stable and } |I - XN| \neq 0; \tag{47}$$

then $G = [(I - XN)D^{-1}]^{-1}X = M(I - PM)^{-1}$;

(b) $\{I,H;P\}$ compensation if and only if
X stable, $|X| \neq 0$ and there exists stable $\underline{X}$ such that

$$XD + \underline{X}N = I; \tag{48}$$

then $H = X^{-1}\underline{X}$

Proof In (a) $X = \underline{L}_1$ and in (b) $X = \underline{L}_2$, $\underline{X} = \underline{L}_1 = XH$ of Theorem 3. This proves the result. If (25) is used the corresponding internal description can be derived. ΔΔΔ

In (b), if P^{-1} is stable then an appropriate H (not necessarily proper) always exists; also note that H is stable if and only if X^{-1} is stable. Furthermore, note that in view of the causality discussion above, if P is proper then for DX strictly proper G is proper and for $D\underline{X}$ strictly proper (DX biproper) H is proper.

It should be noted that the comparison sensitivity matrix is $S_1 = S = (I + PG)^{-1} = I - T = I - NX$. Notice that the poles of S are the poles of T. The zeros of S are the poles of PG, that is: all of the poles of P except the uncontrollable hidden modes and all of the poles of G except the unobservable hidden modes [8].

Proposition 7 is an example of expressing the conditions for internal stability directly in terms of the design parameter of interest; here X characterizes the command/output-reponse $y = Tr$ and the command/control-response $u = Mr$. The internal stability conditions can also be expressed in terms of other design parameters, such as Q or the comparison sensitivity matrix S_1, as it was shown above, and the designer must choose the parameterization which best fits his or her data and design objectives. In general, when designing using controller parameterizations, constraints must be imposed on the parameters so that other design objectives in addition to internal stability are attained. (eg. [1,5,8,9,13]).

3. CONCLUSIONS

In this paper, a number of stabilizing controller parameterizations were presented. Certain parameterizations are closely related to polynomial matrix internal descriptions and they allow complete control of the closed loop eigenvalues, but they might lead to nonproper controllers. Other parameterizations involving rational matrices can easily solve the properness problem but they have less direct control over the closed loop eigenvalues and so can result in (stable) hidden modes and high order compensators. Internal polynomial matrix descriptions were used in the analysis and the relation of all the parameterizations to the internal structure of the feedback system was established. Tests for internal stability were also presented throughout the paper.

The theory of all stabilizing controllers as presented here can be used to derive other parameterizations as well as additional tests for internal stability. Having established their strengths and weaknesses the designer can choose the parameterization which best suits the given data and the design objectives.

4. ACKNOWLEDGEMENT

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