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Optimal Control of Hybrid Autonomous Systems with State Jumps[1]

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Abstract

In this paper, optimal control problems for hybrid autonomous systems with state jumps are studied. In particular, we focus on problems in which a prespecified sequence of active subsystems is given and propose an approach to finding the optimal switching instants. Specifically, the derivatives of the cost with respect to the switching instants are derived and nonlinear optimization techniques are used to locate the optimal switching instants. Using the approach, accurate numerical values of local optimal solutions can be obtained. An example illustrates the approach.

1 Introduction

A hybrid system is a dynamic system that involves both continuous and discrete event dynamics. The continuous dynamics is usually described by subsystem differential/difference equations and the discrete event dynamics is described by switching laws. Discontinuous jumps of continuous states may occur when the system switches from one subsystem to another. Examples of hybrid systems can be found in chemical processes, automotive systems, and electrical circuit systems, etc.

In this paper, we study optimal control problems for a class of hybrid systems in which each subsystem is autonomous and state jumps are present at switching instants. We develop an effective approach for finding accurate numerical values of local optimal solutions for such problems. In particular, we focus on problems in which a prespecified sequence of active subsystems is given. Such problems arise naturally in multimodal control and in logic-based control systems whose controllers are switched among given controllers. Nonlinear autonomous subsystems and performance costs which are not necessarily quadratic are considered in the paper. We note that the cost is actually a function of the switching instants and use constrained nonlinear optimization to locate the optimal switching instants. To apply nonlinear optimization techniques, we need to determine the values of the derivatives of the cost with respect to the switching instants. An approach is proposed for their derivations and is presented in detail. One of the main results of the paper is Theorem 3.1 which makes possible the calculation of accurate values of the derivatives. Some related computational issues are also addressed in the paper.

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This paper extends our earlier results in [8]. In this paper, we focus on hybrid autonomous systems with state jumps which are an important class of hybrid systems, as opposed to switched systems without jumps in [8]. It is worth noting that most of the available literature results on numerical solutions of hybrid systems optimal control problems are for discrete-time hybrid systems [1, 6], or based on the discretizations of time and/or state spaces [5, 7]. However, the discretization approaches may lead to combinatoric explosions and the solutions obtained may not be accurate enough. Unlike these results, the problems we consider here are for continuous-time systems and our approach is not based on discretizations; hence our approach can provide us with accurate values of local minimums. The closest literature result to ours, as far as we are aware of, is [4] which presents closed-loop solutions to infinite horizon optimal control problems for switched linear autonomous systems. However, we point out that our approach can deal with finite horizon problems with nonlinear subsystems, and with costs that are not necessarily quadratic, as opposed to infinite horizon problems with linear subsystems and quadratic costs in [4]. In view of the above, we believe our results are new and contribute to the understanding and the solution of optimal control problems of hybrid systems.

2 Problem Formulation

We consider the following *hybrid autonomous systems with state jumps*. The hybrid system consists of autonomous subsystems (i.e., without continuous input)

$$\dot{x} = f_i(x), \quad f_i : \mathbb{R}^n \to \mathbb{R}^n, \; i \in I = \{1, 2, \cdots, M\} \tag{2.1}$$

and whenever the system dynamics switches from subsystem i_k to subsystem i_{k+1}, a discontinuous jump of the state x will occur, which are described by a function

$$x(t_k^+) = \gamma^{i_k, i_{k+1}}\left(x(t_k^-)\right) \tag{2.2}$$

where $x(t_k^+)$ and $x(t_k^-)$ are the righthand limit and lefthand limit of the state x at t_k, respectively.

The state trajectory evolution of such a system can be controlled by choosing appropriate switching sequences. A *switching sequence* in $[t_0, t_f]$ is defined as

$$\sigma = \left((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K)\right), \tag{2.3}$$

with $0 \le K < \infty$, $t_0 \le t_1 \le \cdots \le t_K \le t_f$, and $i_k \in I$, $k = 0, 1, \cdots, K$. σ indicates that subsystem i_k is active in $[t_k, t_{k+1})$.

In the following, we assume without loss of generality that a prespecified sequence of active subsystems is given as $(1, 2, \cdots, K, K+1)$, i.e., subsystem k is active in $[t_{k-1}, t_k)$.

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We can always do this by relabeling the subsystem indices and even expanding the collection of subsystems (i.e., two subsystems may refer to the same actual subsystem). Under this assumption, we denote the state jump function at the k-th switching as γ^k.

Problem 2.1 (Optimal Control Problem) *Consider a hybrid autonomous system with state jumps, which consists of subsystems $f_i(x)$, $i \in I$. Assume that a prespecified sequence of active subsystems $(1,2,\cdots,K,K+1)$ is given. Find optimal switching instants $t_1,\cdots,t_K (t_0 \le t_1 \le \cdots \le t_K \le t_f)$ such that the corresponding continuous state trajectory x departs from a given initial state $x(t_0)=x_0$ and the cost*

$$J(t_1,\cdots,t_K) = \psi\big(x(t_f)\big) + \int_{t_0}^{t_f} L(x)\,dt + \sum_{k=1}^{K} \psi^k\big(x(t_k^-)\big) \qquad (2.4)$$

is minimized. Here t_0, t_f are given. □

Problem 2.1 is an optimal control problem in Bolza form. Unlike conventional optimal control problems, here J includes the costs ψ^k's for discontinuous jumps at t_k's. In the sequel, we assume that f_k's, L, ψ, ψ^k's, and γ^k's are smooth enough. Under these assumptions, it can be shown that J is a continuously differentiable function of $(t_1,\cdots,t_K)$.

2.1 An Algorithm

Note that Problem 2.1 is actually a constrained multivariable optimization problem

$$\begin{array}{c} \min_{\hat t} J(\hat t) \\ \text{subject to } \hat t \in T \end{array} \qquad (2.5)$$

where $T \triangleq \{\hat t = (t_1,t_2,\cdots,t_K)^T | t_0 \le t_1 \le t_2 \le \cdots \le t_K \le t_f\}$. We propose the following algorithm for (2.5).

Algorithm 2.1

(1). Set the iteration index $j=0$. Choose an initial $\hat t^j$.

(2). Find $J(\hat t^j)$, $\frac{\partial J}{\partial \hat t}(\hat t^j)$ and $\frac{\partial^2 J}{\partial \hat t^2}(\hat t^j)$.

(3). Use the gradient projection method or the constrained Newton's method [2] to update $\hat t^j$ to be $\hat t^{j+1} = \hat t^j + \alpha^j d\hat t^j$. Set the iteration index $j = j+1$.

(4). Repeat steps (2), (3) and (4), until a prespecified termination condition is satisfied (e.g. $\|\frac{\partial J}{\partial \hat t}(\hat t^j)\|_2 < \varepsilon$ where ε is a given small number). □

3 Differentiations of the Cost Function

In order to apply the above algorithm, one needs to find the values of the derivatives $\frac{\partial J}{\partial \hat t}$ and $\frac{\partial^2 J}{\partial \hat t^2}$ (step (2)). In this section, we propose an approach based on the direct differentiations of the cost function to finding the values of $\frac{\partial J}{\partial \hat t}$ and $\frac{\partial^2 J}{\partial \hat t^2}$. This extends the results in [8].

Given a nominal $\hat t = (t_1,\cdots,t_K)^T$ and the corresponding nominal trajectory $x(t)$, we can compute the cost J by using (2.4). Since x_0 and t_0 are fixed, J is not a function of them. Next we define the value function at the k-th switching instant as

$$\begin{aligned} J^k\big(x(t_k^+),t_k,\cdots,t_K\big) &\triangleq \psi\big(x(t_f)\big) + \textstyle\int_{t_k^+}^{t_{k+1}^-} L(x)\,dt \\ &+ \cdots + \textstyle\int_{t_K^+}^{t_f} L(x)\,dt + \sum_{j=k+1}^{K} \psi^j\big(x(t_j^-)\big). \end{aligned} \qquad (3.1)$$

Unlike J, J^k's for $k \ge 1$ are functions of t_k and the initial state $x(t_k^+)$ which depends on the trajectory before t_k. Also note that J^K does not have the state jump cost and it is $J^K\big(x(t_K^+),t_K\big) \triangleq \psi\big(x(t_f)\big) + \int_{t_K^+}^{t_f} L(x)\,dt$. In the sequel, we denote $\frac{\partial J^k}{\partial x}$ for the function J^k as a row vector J_x^k, $\frac{\partial^2 J^k}{\partial x^2}$ as an $n \times n$ matrix J_{xx}^k and so on.

3.1 Single Switching

Let us first consider the case of a single switching. Given a nominal t_1 and the corresponding nominal trajectory $x(t)$, we denote by $\hat x(t)$ the trajectory after a variation dt_1 has taken place. In the sequel, we write f and f_x with a superscript $1-$ (resp. $1+$) whenever the corresponding active vector field at t_1- (resp. t_1+) is used for evaluation at $x(t_1^-)$ (resp. $x(t_1^+)$). Examples are $f^{1-} \triangleq f_1\big(x(t_1^-)\big)$, $f^{1+} \triangleq f_2\big(x(t_1^+)\big)$, $f_x^{1-} \triangleq \frac{\partial f_1}{\partial x}\big(x(t_1^-)\big)$, $f_x^{1+} \triangleq \frac{\partial f_2}{\partial x}\big(x(t_1^+)\big)$. Also, we simply write a function's name with a superscript $1-$ (resp. $1+$) whenever it is evaluated at $x(t_1^-)$ (resp. $x(t_1^+)$). Examples are $J^{1+} \triangleq J^1\big(x(t_1^+),t_1\big)$, $J_x^{1+} \triangleq \frac{\partial J^1}{\partial x}\big(x(t_1^+),t_1\big)$, $L^{1-} \triangleq L\big(x(t_1^-)\big)$, $L^{1+} \triangleq L\big(x(t_1^+)\big)$, $L_x^{1-} \triangleq \frac{\partial L}{\partial x}\big(x(t_1^-)\big)$, $\psi^{1-} \triangleq \psi^1\big(x(t_1^-)\big), \cdots$ (be careful to distinguish the values J^{1+}, J_x^{1+}, L^{1-}, $L_x^{1-}, \cdots$ from the functions $J^1\big(x(t_1^+),t_1\big)$, $J_x^1\big(x(t_1^+),t_1\big)$, $L(x)$, $L_x(x), \cdots$). We also simply denote the lefthand (resp. righthand) limit of (t_1+dt_1) as $t_1+dt_1^-$ (resp. $t_1+dt_1^+$).

Now consider $J(t_1)$ which can be expressed as

$$J(t_1) = \int_{t_0}^{t_1^-} L(x)\,dt + \psi^1\big(x(t_1^-)\big) + J^1\big(x(t_1^+),t_1\big). \qquad (3.2)$$

For a small variation dt_1 of t_1, we have

$$\begin{aligned} J(t_1+dt_1) &= \textstyle\int_{t_0}^{t_1+dt_1^-} L(\hat x)\,dt + \psi^1\big(\hat x(t_1+dt_1^-)\big) \\ &+ J^1\big(\hat x(t_1+dt_1^+),t_1+dt_1\big). \end{aligned} \qquad (3.3)$$

There are three terms in (3.3). Let us consider the second order Taylor expansion of each term. In the following derivations we denote $dx(t_1^-) \triangleq \hat x(t_1+dt_1^-) - x(t_1^-)$ and $dx(t_1^+) \triangleq \hat x(t_1+dt_1^+) - x(t_1^+)$.

Consider the first term $\int_{t_0}^{t_1+dt_1^-} L(\hat x)\,dt$ in (3.3), for either $dt_1 \ge 0$ or $dt_1 < 0$, we have

$$\begin{aligned} \textstyle\int_{t_0}^{t_1+dt_1^-} L(\hat x)\,dt &= \textstyle\int_{t_0}^{t_1^-} L(x)\,dt + L^{1-}dt_1 \\ &+ \tfrac{1}{2} dt_1 L_x^{1-} dx(t_1^-) + \text{H.O.T.} \end{aligned} \qquad (3.4)$$

where H.O.T. stands for Higher Order Terms.

For the second term in (3.3), we have

$$\begin{aligned} \psi^1\big(\hat x(t_1+dt_1^-)\big) &= \psi^1\big(x(t_1^-)+dx(t_1^-)\big) = \psi^{1-} \\ &+ \psi_x^{1-} dx(t_1^-) + \tfrac{1}{2}\big(dx(t_1^-)\big)^T \psi_{xx}^{1-} dx(t_1^-) + \text{H.O.T.} \end{aligned} \qquad (3.5)$$

For the third term in (3.3), we have the second order expansion

$$\begin{aligned} &J^1\big(\hat x(t_1+dt_1^+),t_1+dt_1\big) = J^{1+} + J_x^{1+} dx(t_1^+) + J_{t_1}^{1+} dt_1 \\ &+ \tfrac{1}{2}\big(dx(t_1^+)\big)^T J_{xx}^{1+} dx(t_1^+) + \tfrac{1}{2} J_{t_1 t_1}^{1+} dt_1^2 + dt_1 J_{t_1 x}^{1+} dx(t_1^+) + \text{H.O.T.} \end{aligned} \qquad (3.6)$$

In order to express (3.3) into second order expansions with respect to dt_1, we need to find the second order expansions of $dx(t_1^-)$, $dx(t_1^+)$ in terms of dt_1. Note that

$$dx(t_1^-) \triangleq \hat x(t_1+dt_1^-) - x(t_1^-) = f^{1-}dt_1 + \frac{1}{2} f_x^{1-} f^{1-} dt_1^2 + o(dt_1^2). \qquad (3.7)$$

where $o(dt_1^2)$ refers to a column vector with each element being $o(dt_1^2)$. We will not explicitly mention this later since it will be clear from the context. Next we have

$$\begin{aligned} dx(t_1^+) &\triangleq \hat x(t_1+dt_1^+) - x(t_1^+) = \gamma^1\big(\hat x(t_1+dt_1^-)\big) - \gamma^1\big(x(t_1^-)\big) \\ &= \gamma_x^{1-} dx(t_1^-) + \frac{1}{2} \begin{bmatrix} \big(dx(t_1^-)\big)^T \frac{\partial^2 \gamma_{(1)}^1\left(x(t_1^-)\right)}{\partial x^2} dx(t_1^-) \\ \vdots \\ \big(dx(t_1^-)\big)^T \frac{\partial^2 \gamma_{(n)}^1\left(x(t_1^-)\right)}{\partial x^2} dx(t_1^-) \end{bmatrix} + \text{H.O.T.} \end{aligned} \qquad (3.8)$$

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where $\gamma^1_{(j)}$ refers to the j-th element of the vector-valued function γ^1. Since

$$\begin{bmatrix} \left(dx(t_1^-)\right)^T \frac{\partial^2\gamma^1_{(1)}\left(x(t_1^-)\right)}{\partial x^2} dx(t_1^-) \\ \vdots \\ \left(dx(t_1^-)\right)^T \frac{\partial^2\gamma^1_{(n)}\left(x(t_1^-)\right)}{\partial x^2} dx(t_1^-) \end{bmatrix} = \xi^{1-} f^{1-} dt_1^2 + o(dt_1^2) \quad (3.9)$$

where

$$\xi^{1-} \triangleq \begin{bmatrix} (f^{1-})^T \frac{\partial^2\gamma^1_{(1)}\left(x(t_1^-)\right)}{\partial x^2} \\ \vdots \\ (f^{1-})^T \frac{\partial^2\gamma^1_{(n)}\left(x(t_1^-)\right)}{\partial x^2} \end{bmatrix}, \quad (3.10)$$

we can then rewrite (3.8) as

$$dx(t_1^+) = \gamma_x^{1-} f^{1-} dt_1 + \frac{1}{2}(\gamma_x^{1-} f_x^{1-} + \xi^{1-}) f^{1-} dt_1^2 + o(dt_1^2) \quad (3.11)$$

Substituting (3.7) and (3.11) into (3.4), (3.5) and (3.6) and summing them, we obtain

$$\begin{aligned} J(t_1+dt_1) &= J(t_1) + L^{1-}dt_1 + \tfrac{1}{2}dt_1 L_x^{1-} dx(t_1^-) + \psi_x^{1-} dx(t_1^-) \\ &\quad + \tfrac{1}{2}\left(dx(t_1^-)\right)^T \psi_{xx}^{1-} dx(t_1^-) + J_x^{1+} dx(t_1^+) + J_{t_1}^{1+} dt_1 \\ &\quad + \tfrac{1}{2}\left(dx(t_1^+)\right)^T J_{xx}^{1+} dx(t_1^+) + \tfrac{1}{2} J_{t_1t_1}^{1+} dt_1^2 \\ &\quad + dt_1 J_{t_1x}^{1+} dx(t_1^+) + \text{H.O.T.} \\ &= J(t_1) + (L^{1-} + \psi_x^{1-} f^{1-} + J_x^{1+}\gamma_x^{1-} f^{1-} + J_{t_1}^{1+}) dt_1 \\ &\quad + \tfrac{1}{2}\Big(L_x^{1-} f^{1-} + \psi_x^{1-} f_x^{1-} f^{1-} + (f^{1-})^T \psi_{xx}^{1-} f^{1-} \\ &\quad + J_x^{1+}(\gamma_x^{1-} f_x^{1-} + \xi^{1-}) f^{1-} + (f^{1-})^T (\gamma_x^{1-})^T J_{xx}^{1+} \gamma_x^{1-} f^{1-} \\ &\quad + J_{t_1t_1}^{1+} + 2J_{t_1x}^{1+}\gamma_x^{1-} f^{1-}\Big) dt_1^2 + o(dt_1^2) \\ &\triangleq J(t_1) + J_{t_1} dt_1 + \tfrac{1}{2} J_{t_1t_1} dt_1^2 + o(dt_1^2). \end{aligned} \quad (3.12)$$

Note that the following dynamic programming equation holds for $J^1\left(x(t_1^+), t_1\right)$

$$J_{t_1}^{1+} = -J_x^{1+} f^{1+} - L^{1+}. \quad (3.13)$$

(3.13) can be derived similarly to the HJB equation. However, the difference between it and the HJB equation is that (3.13) holds for any trajectory that is not necessarily optimal (for more details see [3]).

By differentiating (3.13), we obtain

$$J_{t_1x}^{1+} = -(f^{1+})^T J_{xx}^{1+} - J_x^{1+} f_x^{1+} - L_x^{1+}, \quad (3.14)$$

$$\begin{aligned} J_{t_1t_1}^{1+} &= -J_{t_1x}^{1+} f^{1+} = (f^{1+})^T J_{xx}^{1+} f^{1+} + (J_x^{1+} f_x^{1+} \\ &\quad + L_x^{1+}) f^{1+}. \end{aligned} \quad (3.15)$$

Substituting these into (3.12) we have

$$J_{t_1} = L^{1-} - L^{1+} + \psi_x^{1-} f^{1-} + J_x^{1+}(\gamma_x^{1-} f^{1-} - f^{1+}), \quad (3.16)$$

$$\begin{aligned} J_{t_1t_1} &= (L_x^{1-} - L_x^{1+}\gamma_x^{1-}) f^{1-} + \psi_x^{1-} f_x^{1-} f^{1-} \\ &\quad + (f^{1-})^T \psi_{xx}^{1-} f^{1-} + J_x^{1+}(\gamma_x^{1-} f_x^{1-} + \xi^{1-} - f_x^{1+}\gamma_x^{1-}) f^{1-} \\ &\quad - (J_x^{1+} f_x^{1+} + L_x^{1+})(\gamma_x^{1-} f^{1-} - f^{1+}) \\ &\quad + (\gamma_x^{1-} f^{1-} - f^{1+})^T J_{xx}^{1+} (\gamma_x^{1-} f^{1-} - f^{1+}). \end{aligned} \quad (3.17)$$

3.2 Two or More Switchings

Now consider the case of two switchings. Assume that a system switches from subsystem 1 to 2 at t_1 and from subsystem 2 to 3 at t_2 ($t_0 \le t_1 \le t_2 \le t_f$). Then

$$J(t_1,t_2) = \int_{t_0}^{t_1^-} L(x)\,dt + \psi^1\left(x(t_1^-)\right) + J^1\left(x(t_1^+), t_1, t_2\right) \quad (3.18)$$

$$\begin{aligned} &= \int_{t_0}^{t_1^-} L(x)\,dt + \psi^1\left(x(t_1^-)\right) + \int_{t_1^+}^{t_2^-} L(x)\,dt \\ &\quad + \psi^2\left(x(t_2^-)\right) + J^2\left(x(t_2^+), t_2\right). \end{aligned} \quad (3.19)$$

Using (3.18), by holding t_2 fixed, J_{t_1}, $J_{t_1t_1}$ can be derived similarly to that in subsection 3.1. In the similar manner, J_{t_2}, $J_{t_2t_2}$ can be derived using (3.19). However, to derive $J_{t_1t_2}$, we need to use arguments from the calculus of variations. Let us first define the important notion of incremental change which will be used later.

Definition 3.1 (Incremental Change) *Given variations dt_1 and dt_2, we define* the incremental change $\delta x(t), \min\{t_1^+, t_1 + dt_1^+\} \le t \le \max\{t_2^-, t_2 + dt_2^-\}$ *as*

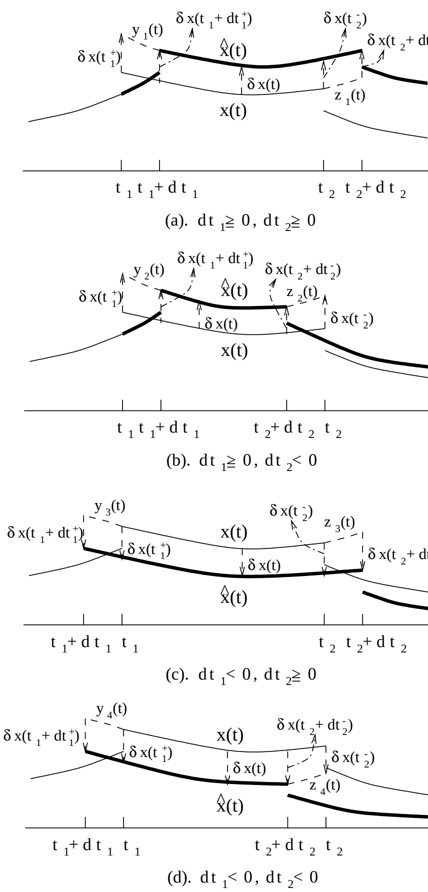

Figure 1: The incremental change $\delta x(t)$.

Case 1: $dt_1 \ge 0, dt_2 \ge 0$ *(see figure 1(a))*

In this case, $\delta x(t)$ *is defined to be*

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1 + dt_1^+, t_2^-] \\ y_1(t) - x(t), \ t \in [t_1^+, t_1 + dt_1^+] \\ \hat{x}(t) - z_1(t), \ t \in [t_2^-, t_2 + dt_2^-] \end{cases} \quad (3.20)$$

where $y_1(t)$ *the solution of*

$$\begin{cases} \dot{y}_1(t) = f_2\big(y_1(t)\big), \ t \in [t_1^+, t_1 + dt_1^+] \\ y_1(t_1 + dt_1^+) = \hat{x}(t_1 + dt_1^+) \end{cases} \quad (3.21)$$

and $z_1(t)$ *is the solution of*

$$\begin{cases} \dot{z}_1(t) = f_2\big(z_1(t)\big), \ t \in [t_2^-, t_2 + dt_2^-] \\ z_1(t_2^-) = x(t_2^-). \end{cases} \quad (3.22)$$

Case 2: $dt_1 \ge 0, dt_2 < 0$ *(see figure 1(b))*

In this case, $\delta x(t)$ *is defined to be*

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1 + dt_1^+, t_2 + dt_2^-] \\ y_2(t) - x(t), \ t \in [t_1^+, t_1 + dt_1^+] \\ z_2(t) - x(t), \ t \in [t_2 + dt_2^-, t_2^-] \end{cases} \quad (3.23)$$

where $y_2(t)$ is the solution of

$$\begin{cases} \dot{y}_2(t) = f_2(y_2(t)),\ t \in [t_1^+, t_1 + dt_1^+] \\ y_2(t_1 + dt_1^+) = \hat{x}(t_1 + dt_1^+) \end{cases} \tag{3.24}$$

and $z_2(t)$ is the solution of

$$\begin{cases} \dot{z}_2(t) = f_2(z_2(t)),\ t \in [t_2 + dt_2^-, t_2^-] \\ z_2(t_2 + dt_2^-) = \hat{x}(t_2 + dt_2^-). \end{cases} \tag{3.25}$$

Case 3: *$dt_1 < 0, dt_2 \geq 0$ (see figure 1(c))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t),\ t \in [t_1^+, t_2^-] \\ \hat{x}(t) - y_3(t),\ t \in [t_1 + dt_1^+, t_1^+] \\ \hat{x}(t) - z_3(t),\ t \in [t_2^-, t_2 + dt_2^-] \end{cases} \tag{3.26}$$

where $y_3(t)$ is the solution of

$$\begin{cases} \dot{y}_3(t) = f_2(y_3(t)),\ t \in [t_1 + dt_1^+, t_1^+] \\ y_3(t_1^+) = x(t_1^+) \end{cases} \tag{3.27}$$

and $z_3(t)$ is the solution of

$$\begin{cases} \dot{z}_3(t) = f_2(z_3(t)),\ t \in [t_2^-, t_2 + dt_2^-] \\ z_3(t_2^-) = x(t_2^-). \end{cases} \tag{3.28}$$

Case 4: *$dt_1 < 0, dt_2 < 0$ (see figure 1(d))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t),\ t \in [t_1^+, t_2 + dt_2^-] \\ \hat{x}(t) - y_4(t),\ t \in [t_1 + dt_1^+, t_1^+] \\ z_4(t) - x(t),\ t \in [t_2 + dt_2^-, t_2^-] \end{cases} \tag{3.29}$$

where $y_4(t)$ is the solution of

$$\begin{cases} \dot{y}_4(t) = f_2(y_4(t)),\ t \in [t_1 + dt_1^+, t_1^+] \\ y_4(t_1^+) = x(t_1^+) \end{cases} \tag{3.30}$$

and $z_4(t)$ is the solution of

$$\begin{cases} \dot{z}_4(t) = f_2(z_4(t)),\ t \in [t_2 + dt_2^-, t_2^-] \\ z_4(t_2 + dt_2^-) = \hat{x}(t_2 + dt_2^-). \end{cases} \tag{3.31}$$

□

Remark 3.1 In $[\min\{t_1^+, t_1 + dt_1^+\}, \max\{t_2^-, t_2 + dt_2^-\}]$, at least one of $\hat{x}(t)$ and $x(t)$ evolves along subsystem 2. $\delta x(t)$ is the difference between $\hat{x}(t)$ and $x(t)$ in this interval (by possibly extending $\hat{x}$ and x under subsystem 2 to it). □

Lemma 3.1 *The expressions of $\delta x(t_2^-)$ and $\delta x(t_2 + dt_2^-)$ are as follows*

$$\delta x(t_2^-) = A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}) dt_1 + o(dt_1), \tag{3.32}$$

$$\begin{aligned} \delta x(t_2 + dt_2^-) &= A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}) dt_1 \\ &+ f_x^{2-} A(t_2, t_1)(\gamma_x^{1-} f^{1-} - f^{1+}) dt_1 dt_2 \\ &+ (\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}), \end{aligned} \tag{3.33}$$

where $A(t_2^-, t_1^+)$ is the state transition matrix for the variational equation $\dot{y}(t) = \frac{\partial f_2(x(t))}{\partial x} y(t)$ for $y(t), t \in [t_1^+, t_2^-]$; here x is the current nominal state.

Proof: See [9]. □

In fact, from the proof of Lemma 3.1 (see [9]), we can obtain the following important *the forward decoupling principle*, which reveals some intrinsic relationship among different switching instants.

The Forward Decoupling Principle:

(a) The value of the incremental change $\delta x(t_1^+)$ at t_1^+ does not depend on dt_2.

(b) The value of the incremental change $\delta x(t_2^-)$ at t_2^- does depend on dt_1. □

The forward decoupling principle tells us that a variation of an earlier switching instant will affect the value of δx) at a later switching instant, but not vice versa.

Lemma 3.2 *The expressions of $dx(t_2^-)$ (i.e., $\hat{x}(t_2 + dt_2^-) - x(t_2^-)$) and $dx(t_2^+)$ (i.e., $\hat{x}(t_2 + dt_2^+) - x(t_2^+)$) are*

$$\begin{aligned} dx(t_2^-) &= A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}) dt_1 \\ &+ f_x^{2-} A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}) dt_1 dt_2 + f^{2-} dt_2 \\ &+ (\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}), \end{aligned} \tag{3.34}$$

$$\begin{aligned} dx(t_2^+) &= \gamma_x^{2-} A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}) dt_1 \\ &+ (\gamma_x^{2-} f_x^{2-} + \xi^{2-}) A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}) dt_1 dt_2 \\ &+ \gamma_x^{2-} f^{2-} dt_2 + (\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}) \end{aligned} \tag{3.35}$$

where ξ^{2-} is defined similarly to ξ^{1-} in (3.10) as $\xi^{2-} \triangleq \begin{bmatrix} (f^{2-})^T \frac{\partial^2 \gamma_{(1)}^2 (x(t_2^-))}{\partial x^2} \\ \vdots \\ (f^{2-})^T \frac{\partial^2 \gamma_{(n)}^2 (x(t_2^-))}{\partial x^2} \end{bmatrix}$ with $\gamma_{(j)}^2$ referring to the j-th element of the vector-valued function γ^2.

Proof: See [9]. □

Equipped with Lemmas 3.1 and 3.2, we are ready to derive the coefficient for $dt_1 dt_2$ in the expansion of

$$\begin{aligned} J(t_1 + dt_1, t_2 + dt_2) &= \int_{t_0}^{t_1 + dt_1^-} L(\hat{x}(t))\, dt + \psi^1(\hat{x}(t_1 + dt_1^-)) \\ &+ \int_{t_1 + dt_1^+}^{t_2 + dt_2^-} L(\hat{x}(t))\, dt + \psi^2(\hat{x}(t_2 + dt_2^-)) \\ &+ J^2(\hat{x}(t_2 + dt_2^+), t_2 + dt_2). \end{aligned} \tag{3.36}$$

There are five terms in (3.36). From the forward decoupling principle, we conclude that none of $\delta x(t_1^-)$, $\delta x(t_1^+)$, $dx(t_1^-)$, and $dx(t_1^+)$ depends on dt_2. Consequently the expansions of the first two terms have no terms in dt_2, dt_2^2 and $dt_1 dt_2$ and will not contribute to the coefficient of $dt_1 dt_2$. For the third term in (3.36), we have

Lemma 3.3 *The contribution of $\int_{t_0}^{t_2 + dt_2^-} L(\hat{x})\, dt$ to the coefficient of $dt_1 dt_2$ is*

$$L_x^{2-} A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}). \tag{3.37}$$

Proof: See [9]. □

The fourth term in (3.36) can be expanded as

$$\begin{aligned} &\psi^2(\hat{x}(t_2 + dt_2^-)) = \psi^2(x(t_2^-) + dx(t_2^-)) \\ &= \psi^{2-} + \psi_x^{2-} dx(t_2^-) + \tfrac{1}{2}(dx(t_2^-))^T \psi_{xx}^{2-} dx(t_2^-) + \text{H.O.T.} \end{aligned} \tag{3.38}$$

Therefore the contribution to the coefficient of $dt_1 dt_2$ by the fourth term is

$$(\psi_x^{2-} f_x^{2-} + (f^{2-})^T \psi_{xx}^{2-}) A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}). \tag{3.39}$$

For the fifth term in (3.36), similar to the single switching case, we can obtain its Taylor expansion as

$$\begin{aligned} J^2(\hat{x}(t_2 + dt_2^+), t_2 + dt_2) &= J^{2+} + J_x^{2+} dx(t_2^+) \\ &+ J_{t_2}^{2+} dt_2 + \tfrac{1}{2}(dx(t_2^+))^T J_{xx}^{2+} dx(t_2^+) + \tfrac{1}{2} J_{t_2 t_2}^{2+} dt_2^2 \\ &+ dt_2 J_{t_2 x}^{2+} dx(t_2^+) + \text{H.O.T.} \end{aligned} \tag{3.40}$$

In (3.40), the terms that will possibly contribute to the coefficient of $dt_1 dt_2$ are those containing $dx(t_2^+)$. They are $J_x^{2+} dx(t_2^+)$, $\frac{1}{2}(dx(t_2^+))^T J_{xx}^{2+} dx(t_2^+)$, $dt_2 J_{t_2 x}^{2+} dx(t_2^+)$. Substituting the expression of $dx(t_2^+)$ into these three terms and summing them, we obtain the contribution of the fifth term to the coefficient of $dt_1 dt_2$ as

$$\begin{aligned} &(J_x^{2+}(\gamma_x^{2-} f_x^{2-} + \xi^{2-}) + (f^{2-})^T (\gamma_x^{2-})^T J_{xx}^{2+} \gamma_x^{2-} \\ &+ J_{t_2 x}^{2+} \gamma_x^{2-}) A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}). \end{aligned} \tag{3.41}$$

Summing (3.37), (3.39), and (3.41) and also taking into consideration the expression of $J_{t_2 x}^{2+}$ which can be obtained similarly to the expression of $J_{t_1 x}^{1+}$ in (3.14), we conclude that

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the coefficient of $dt_1 dt_2$ (i.e., $J_{t_1t_2}$ in the expansion of $J(t_1+dt_1, t_2+dt_2)$ is

$$\begin{aligned} J_{t_1t_2} &= \big(L_x^{2-}+\psi_x^{2-}f_x^{2-}+(f^{2-})^T\psi_{xx}^{2-} \\ &+J_x^{2+}(\gamma_x^{2-}f_x^{2-}+\xi^{2-})+(f^{2-})^T(\gamma_x^{2-})^T J_{xx}^{2+}\gamma_x^{2-} \\ &+J_{t_2x}^{2+}\gamma_x^{2-}\big)A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+}) \\ &= \big(L_x^{2-}-L_x^{2+}\gamma_x^{2-}+\psi_x^{2-}f_x^{2-}+(f^{2-})^T\psi_{xx}^{2-} \\ &+J_x^{2+}(\gamma_x^{2-}f_x^{2-}+\xi^{2-}-f_x^{2+}\gamma_x^{2-})+(\gamma_x^{2-}f^{2-} \\ &-f^{2+})^T J_{xx}^{2+}\gamma_x^{2-}\big)A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+}). \end{aligned} \tag{3.42}$$

The above results can also be similarly generalized to the case of K switchings as follows.

Theorem 3.1 *The cost J in Problem 2.1 satisfies*

$$\begin{aligned} &J(t_1+dt_1,t_2+dt_2,\cdots,t_K+dt_K) \\ &= J(t_1,t_2,\cdots,t_K)+\textstyle\sum_{k=1}^K J_{t_k}dt_k+\frac{1}{2}\sum_{k=1}^K J_{t_kt_k}dt_k^2 \\ &+\textstyle\sum_{1\le k<l\le K} J_{t_kt_l}dt_k dt_l + H.O.T. \end{aligned} \tag{3.43}$$

where

$$J_{t_k} = L^{k-}-L^{k+}+\psi_x^{k-}f^{k-}+J_x^{k+}(\gamma_x^{k-}f^{k-}-f^{k+}), \tag{3.44}$$

$$\begin{aligned} J_{t_kt_k} &= (L_x^{k-}-L_x^{k+}\gamma_x^{k-})f^{k-}+\psi_x^{k-}f_x^{k-}f^{k-} \\ &+(f^{k-})^T\psi_{xx}^{k-}f^{k-}+J_x^{k+}(\gamma_x^{k-}f_x^{k-}+\xi^{k-}-f_x^{k+}\gamma_x^{k-})f^{k-} \\ &-(J_x^{k+}f_x^{k+}+L_x^{k+})(\gamma_x^{k-}f^{k-}-f^{k+}) \\ &+(\gamma_x^{k-}f^{k-}-f^{k+})^T J_{xx}^{k+}(\gamma_x^{k-}f^{k-}-f^{k+}), \end{aligned} \tag{3.45}$$

for any $k=1,\cdots,K$, and

$$\begin{aligned} J_{t_kt_l} &= \big(L_x^{l-}-L_x^{l+}\gamma_x^{l-}+\psi_x^{l-}f_x^{l-}+(f^{l-})^T\psi_{xx}^{l-} \\ &+J_x^{l+}(\gamma_x^{l-}f_x^{l-}+\xi^{l-}-f_x^{l+}\gamma_x^{l-})+(\gamma_x^{l-}f^{l-} \\ &-f^{l+})^T J_{xx}^{l+}\gamma_x^{l-}\big)H(t_l^-,t_k^+)(\gamma_x^{k-}f^{k-}-f^{k+}), \end{aligned} \tag{3.46}$$

for any $1\le k<l\le K$. Here $H(t_l^-,t_k^+)$ is the state transition matrix under state jumps

$$\begin{aligned} &H(t_l^-,t_k^+) \\ &= A(t_l^-,t_{l-1}^+)\gamma_x^{(l-1)-}A(t_{l-1}^-,t_{l-2}^+)\bullet\cdots\bullet\gamma_x^{(k+1)-}A(t_{k+1}^-,t_k^+) \end{aligned} \tag{3.47}$$

where $A(t_{j+1}^-,t_j^+)$, $k\le j\le l-1$ is the state transition matrix for the time interval $[t_j^+,t_{j+1}^-]$ for the variational equation $\dot y(t)=\frac{\partial f_{j+1}(x(t))}{\partial x}y(t)$. Also here $\xi^{k-}\triangleq\begin{bmatrix}(f^{k-})^T\frac{\partial\gamma_{(1)}^k(x(t_k^-))}{\partial x}\\ \vdots\\ (f^{k-})^T\frac{\partial\gamma_{(n)}^k(x(t_k^-))}{\partial x}\end{bmatrix}$, $k=1,\cdots,K$. □

Remark 3.2 Due to discontinuous jumps in $[t_k^+,t_l^-]$, $H(t_l^-,t_k^+)$ appears in (3.46) (instead of $A(t_l^-,t_k^+)$). In the special case when $l=k+1$, $H(t_l^-,t_k^+)$ is reduced to be $A(t_{k+1}^-,t_k^+)$. □

4 Computation of $H(t_l^-,t_k^+)$, J_x^{k+}, and J_{xx}^{k+}

In order to use Theorem 3.1 to compute J_{t_k}, $J_{t_kt_k}$ and $J_{t_kt_l}$, we need to know the values of $H(t_l^-,t_k^+)$, J_x^{k+} and J_{xx}^{k+}. In this subsection, we develop an efficient numerical method for computing them.

First note that if $l=k+1$ then $H(t_l^-,t_k^+)$ is equal to $A(t_{k+1}^-,t_k^+)$, which is the state transition matrix for $\dot y(t)=\frac{\partial f_{k+1}(x(t))}{\partial x}y(t)$. To find its value, we can first find the solution $y^{(1)}(t),\cdots,y^{(n)}(t)$ corresponding to initial conditions

$$y^{(1)}(t_k^+)=e_1,\ \cdots,\ y^{(n)}(t_k^+)=e_n \tag{4.1}$$

respectively, where e_p is the unit column vector with all 0's except that the p-th element being 1, $p=1,2,\cdots,n$. From linear systems theory, $A(t_{k+1}^-,t_k^+)$ is equal to the square matrix whose p-th column is $y^{(p)}(t_{k+1}^-)$, i.e., in this case $H(t_l^-,t_k^+)=A(t_{k+1}^-,t_k^-)=[y^{(1)}(t_{k+1}^-),\cdots,y^{(n)}(t_{k+1}^-)]$.

Now if $l>k$, the similar method can be adopted to compute $H(t_l^-,t_k^+)$. Instead of solving initial value ODEs for $y^{(p)}$'s, $y^{(p)}(t)$'s are now obtained by solve the following ODEs with jumps with initial conditions (4.1).

$$\begin{cases} \dot y(t)=\frac{\partial f_{j+1}(x(t))}{\partial x}y(t),\ \text{for } t_j^+\le t\le t_{j+1}^-, \\ y(t_j^+)=\gamma_x^{j-}y(t_j^-), k<j<l. \end{cases} \tag{4.2}$$

We then have

$$H(t_l^-,t_k^+)=[y^{(1)}(t_l^-),\cdots,y^{(n)}(t_l^-)]. \tag{4.3}$$

To obtain the value of J_x^k, note that

$$\begin{aligned} &J^k(x(t_k^+),t_k,\cdots,t_K)=\psi(x(t_f)) \\ &+\textstyle\sum_{j=k}^K\int_{t_j^+}^{t_{j+1}^-}L(x(t))\,dt+\sum_{j=k+1}^K\psi^j(x(t_j^-)). \end{aligned} \tag{4.4}$$

In (4.4), we regard t_f as t_{K+1}^- for simplicity of notation.

If $x(t_k^+)$ has a variation $\delta x(t_k^+)$, then

$$\begin{aligned} &J^k(x(t_k^+)+\delta x(t_k^+),t_k,\cdots,t_K) \\ &=\psi(x(t_f)+H(t_f,t_k^+)\delta x(t_k^+)+\text{H.O.T.}) \\ &+\textstyle\sum_{j=k}^K\int_{t_j^+}^{t_{j+1}^-}L(x(t)+H(t,t_k^+)\delta x(t_k^+)+\text{H.O.T})\,dt \\ &+\textstyle\sum_{j=k+1}^K\psi^j(x(t_j^-)+H(t_j^-,t_k^+)\delta x(t_k^+)+\text{H.O.T.}) \\ &=J^k(x(t_k^+),t_k,\cdots,t_K)+\Big(\psi_x(x(t_f))H(t_f,t_k^+) \\ &+\textstyle\sum_{j=k}^K\int_{t_j^+}^{t_{j+1}^-}L_x(x(t))H(t,t_k^+)\,dt \\ &+\textstyle\sum_{j=k+1}^K\psi_x^j(x(t_j^-))H(t_j^-,t_k^+)\Big)\delta x(t_k^+)+\text{H.O.T.} \end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned} J_x^{k+}&=\psi_x(x(t_f))H(t_f,t_k^+)+\textstyle\sum_{j=k}^K\int_{t_j^+}^{t_{j+1}^-}L_x(x(t))H(t,t_k^+)\,dt \\ &+\textstyle\sum_{j=k+1}^K\psi_x^j(x(t_j^-))H(t_j^-,t_k^+). \end{aligned} \tag{4.6}$$

Now if we apply the similar procedure by varying $x(t_k^+)$ as in (4.5) to $J_x^k(x(t_k^+),t_k,\cdots,t_K)$, we can obtain

$$\begin{aligned} J_{xx}^{k+}&=H^T(t_f,t_k^+)\psi_{xx}(x(t_f))H(t_f,t_k^+) \\ &+\textstyle\sum_{j=k}^K\int_{t_j^+}^{t_{j+1}^-}H^T(t,t_k^+)L_{xx}(x(t))H(t,t_k^+)\,dt \\ &+\textstyle\sum_{j=k+1}^K H^T(t_j^-,t_k^+)\psi_{xx}^j(x(t_j^-))H(t_j^-,t_k^+). \end{aligned} \tag{4.7}$$

From the above discussion, we find that $H(t_l^-,t_k^+)$ can be obtained by solving ODEs with jumps (4.2) along with initial conditions (4.1). $H(t_f,t_k^+)$ can be obtained in the same fashion. J_x^{k+} and J_{xx}^{k+} are in the forms (4.6) and (4.7) which can easily be rewritten as

$$J_x^{k+} = \psi_x(x(t_f))H(t_f,t_k^+)+\eta_1(t_f), \tag{4.8}$$

$$J_{xx}^{k+} = H^T(t_f,t_k^+)\psi_{xx}(x(t_f))H(t_f,t_k^+)+\eta_2(t_f) \tag{4.9}$$

with η_1 and η_2 satisfying the following ODEs with jumps

$$\begin{cases} \dot\eta_1=L_x(x(t))H(t,t_k^+),\ t_j^+\le t\le t_{j+1}^-, \\ \eta_1(t_j^+)=\eta_1(t_j^-)+\psi_x^j(x(t_j^-))H(t_j^-,t_k^+), \\ \eta_1(t_k)=0_{1\times n}, \end{cases} \tag{4.10}$$

$$\begin{cases} \dot\eta_2=H^T(t,t_k^+)L_{xx}(x(t))H(t,t_k^+),\ t_j^+\le t\le t_{j+1}^-, \\ \eta_2(t_j^+)=\eta_2(t_j^-)+H^T(t_j^-,t_k^+)\psi_{xx}^j(x(t_j^-))H(t_j^-,t_k^+), \\ \eta_2(t_k)=0_{n\times n}. \end{cases} \tag{4.11}$$

Remark 4.1 (Computational Cost) The main computational cost for J_{t_k}, $J_{t_kt_k}$, J_{t_k,t_l} occurs in the computation of $H(t_l^-,t_k^+)$, J_x^{k+}, and J_{xx}^{k+}, since all other terms in (3.44)-(3.46) are readily available. The above method we propose reduces

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the computation of $H(t_l^-, t_k^+)$ to solving initial value ODEs with jumps (4.2) for any $k < l$ and the computation of J_x^{k+} and J_{xx}^{k+} to solving initial value ODEs with jumps (4.10)-(4.11) for $k = 1, \cdots, K$. Hence we altogether need to solve $\frac{(K-1)K}{2} + K = \frac{K(K+1)}{2}$ sets of ODEs with jumps. With today's powerful ODE solvers (e.g., `ode45` function in MATLAB), these equations can be solved efficiently and accurately. □

5 An Example

In this section, we present an example to illustrate the effectiveness of the approach developed in this paper.

Example 5.1 Consider a hybrid autonomous system consisting of

$$\text{subsystem 1: } \begin{cases} \dot{x}_1 = x_1 + 0.5 \sin x_2 \\ \dot{x}_2 = -0.5 \cos x_1 - x_2 \end{cases} \quad (5.1)$$

$$\text{subsystem 2: } \begin{cases} \dot{x}_1 = 0.3 \sin x_1 + 0.5 x_2 \\ \dot{x}_2 = -0.5 x_1 + 0.3 \cos x_2 \end{cases} \quad (5.2)$$

$$\text{subsystem 3: } \begin{cases} \dot{x}_1 = -x_1 - 0.5 \cos x_2 \\ \dot{x}_2 = 0.5 \sin x_1 + x_2 \end{cases} \quad (5.3)$$

Assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 1 to 2 and at $t = t_2$ from subsystem 2 to 3 $(0 \le t_1 \le t_2 \le 3)$. Also assume that the system has the state jump

$$\begin{cases} x_1(t_1^+) = x_1(t_1^-) + 0.2 \\ x_2(t_1^+) = x_2(t_1^-) + 0.2 \end{cases} \quad (5.4)$$

when switching from subsystem 1 to 2 and

$$\begin{cases} x_1(t_2^+) = x_1(t_2^-) + 0.2 \\ x_2(t_2^+) = x_2(t_2^-) - 0.2 \end{cases} \quad (5.5)$$

when switching from subsystem 2 to 3. We want to find optimal switching instants t_1, t_2 such that the cost

$$J = \frac{1}{2}x_1^2(3) + \frac{1}{2}x_2^2(3) + \frac{1}{2}\int_0^3 \left(x_1^2(t) + x_2^2(t)\right) dt + \sum_{k=1}^{2} \left(\frac{1}{2}x_1^2(t_k^-) + \frac{1}{2}x_2^2(t_k^-)\right) \quad (5.6)$$

is minimized. Here $x_1(0) = 1$ and $x_2(0) = 3$.

For this problem, we choose initial nominal $t_1 = 1$, $t_2 = 1.5$. By using Algorithm 2.1 (using constrained Newton's method) along with Theorem 3.1, after 8 iterations we find the optimal $t_1 = 0.4847$, $t_2 = 1.9273$ and the corresponding optimal cost 18.8310. The corresponding state trajectory is shown in Figure 2. □

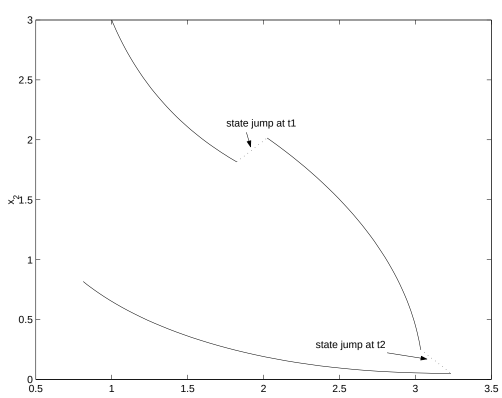

Figure 2: The state trajectory for Example 5.1.

6 Conclusion

In this paper, we propose an approach for solving optimal control problems for hybrid autonomous systems with state jumps given prespecified sequences of active subsystems. In particular, we derive the derivatives of the cost with respect to the switching instants and use nonlinear optimization techniques to locate the optimal switching instants. It is also shown how to address the computational issues in applying the approach. Note that a more detailed version of this paper can be found in [9]. A further research topic is the development of methods for searching for optimal switching sequences when the sequence of active subsystems are not prespecified.

References

[1] A. Bemporad, F. Borrelli, and M. Morari. On the optimal control law for linear discrete time hybrid systems. In *HSCC 2002, LNCS 2289*, pages 105–119. Springer, 2002.

[2] D.P. Bertsekas. *Nonlinear Programming, Second Edition*. Athena Scientific, 1999.

[3] P. Dyer and S.R. McReynolds. *The Computation and Theory of Optimal Control*. Academic Press, 1970.

[4] A. Giua, C. Seatzu, and C. Van Der Mee. Optimal control of switched autonomous linear systems. In *Proc. of the 40th IEEE CDC*, pages 2472–2477, 2001.

[5] S. Hedlund and A. Rantzer. Optimal control of hybrid system. In *Proc. 38th IEEE CDC*, pages 3972–3977, 1999.

[6] J. Lu, L. Liao, A. Nerode, and J.H. Taylor. Optimal control of systems with continuous and discrete states. In *Proc. of the 32nd IEEE CDC*, pages 2292–2297, 1993.

[7] L.Y. Wang, A. Beydoun, J. Cook, J. Sun, and I. Kolmanovsky. Optimal hybrid control with applications to automotive powertrain systems. In *Control Using Logic-Based Switching, LNCIS 222*, pages 190–200, 1997.

[8] X. Xu and P.J. Antsaklis. Optimal control of switched autonomous systems. In *Proc. of the 41st IEEE CDC*, pages 4401–4406, 2002.

[9] X. Xu and P.J. Antsaklis. Optimal control of hybrid autonomous systems with state jumps. ISIS Technical Report ISIS-2003-001, University of Notre Dame, March 2003. (`http://www.nd.edu/~isis/tech.html`).