Xuping Xu, Panos J.Antsaklis, "Quadratic Optimal Control Problems for Hybrid Linear Autonomous Systems with State Jumps," Proceedings of the 2003 American Control Conference, pp. 3393-3398, Denver, Colorado, June 4-6, 2003.

Quadratic Optimal Control Problems for Hybrid Linear Autonomous Systems with State Jumps[1]

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Abstract

In this paper, results for quadratic optimal control of hybrid linear autonomous systems with state jumps are reported. In particular, we focus on problems in which a prespecified sequence of active subsystems is given. In a recent paper, we have proposed an approach for optimal control problems of hybrid autonomous systems with state jumps. However, in applying that approach, some computational issues must be addressed for the derivation of some necessary parameters. We in this paper, by taking advantage of the special structure of linear subsystems and quadratic costs, develop a more efficient method which computes these necessary parameters and hence leads to reduced computational effort. As an application, we apply our quadratic optimal control results to address the important reachability problems. Examples illustrate the results.

1 Introduction

A hybrid system is a dynamic system that involves both continuous and discrete event dynamics. The continuous dynamics is usually described by subsystem differential/difference equations and the discrete event dynamics is described by switching laws. Discontinuous jumps of continuous states may occur when the system switches from one subsystem to another. Examples of hybrid systems can be found in chemical processes, automotive systems, and electrical circuit systems, etc.

Optimal control problems are among the many important classes of problems for hybrid systems, which have recently been under intensive investigations by many researchers (see, e.g., [2, 4, 5, 6, 7, 8, 9]). In a recent paper [10], we reported an approach to optimal control problems of a class of hybrid systems in which each subsystem is autonomous (i.e., with no continuous input) and state jumps are present at the switching instants. In particular, we focus on problems in which a prespecified sequence of active subsystems is given. Such problems arise naturally in multimodal control and in logic-based control systems whose controllers are switched among several given controllers. Nonlinear autonomous subsystems and performance costs which are not necessarily quadratic are considered in [10].

However, in applying the approach in [10], some computational issues must be addressed for the derivation of some necessary parameters. In this paper, we focus on quadratic optimal control problems for hybrid linear autonomous systems with state jumps. By taking advantage of the special structure of linear subsystems and quadratic costs, we develop a method which computes these parameters more efficiently and hence leads to reduced computational effort. The method is more efficient and easier to apply than the computational method proposed in [10]. As an application, we then apply our quadratic optimal control results to address the important reachability problems.

The structure of the paper is as follows. In Section 2, we review some results from [10]. In Section 3, we focus on quadratic optimal control problems of hybrid linear autonomous systems with state jumps and propose a method for computing the necessary parameters. In Section 4, we apply the quadratic optimal control results to address reachability problems. Examples are given in Section 5. Section 6 concludes the paper.

2 Review of Previous Results

In this section, we review some previous results from [10].

2.1 Problem Formulation

A *hybrid autonomous system with state jumps* is defined as follows. The hybrid system consists of autonomous subsystems (i.e., without continuous input)

$$\dot{x} = f_i(x), \quad f_i : \mathbb{R}^n \to \mathbb{R}^n, \quad i \in I = \{1, 2, \cdots, M\}. \tag{2.1}$$

and whenever the system dynamics switches from subsystem i_k to subsystem i_{k+1}, a discontinuous jump of the state x will occur, which is described by a function

$$x(t_k^+) = \gamma^{i_k, i_{k+1}}\left(x(t_k^-)\right) \tag{2.2}$$

where $x(t_k^+)$ and $x(t_k^-)$ are the righthand and lefthand limit of the state x at t_k, respectively.

For such a hybrid system, one can control its state trajectory evolution by choosing appropriate switching sequences. Here a *switching sequence* σ in $[t_0, t_f]$ is defined as

$$\sigma = \left((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K)\right), \tag{2.3}$$

with $0 \le K < \infty$, $t_0 \le t_1 \le \cdots \le t_K \le t_f$, and $i_k \in I$, $k = 0, 1, \cdots, K$. σ indicates that subsystem i_k is active in $[t_k, t_{k+1})$ (subsystem i_K in $[t_K, t_f]$).

[1]The partial support of the National Science Foundation (NSF ECS99-12458 & CCR01-13131), and of the DARPA/ITO-NEST Program (AF-F30602-01-2-0526) is gratefully acknowledged.

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In the sequel, we assume without loss of generality that a prespecified (untimed) sequence of active subsystems is given as $(1,2,\cdots,K,K+1)$. Under such an assumption, we can simply denote the state jump function at the k-th switching as γ^k. We now consider the following optimal control problem.

Problem 2.1 (Optimal Control Problem) *Consider a hybrid autonomous system with state jumps, which consists of subsystems $f_i(x)$, $i \in I$. Assume that a prespecified sequence of active subsystems $(1,2,\cdots,K,K+1)$ is given. Find optimal switching instants $t_1,\cdots,t_K (t_0 \le t_1 \le \cdots \le t_K \le t_f)$ such that the corresponding continuous state trajectory x departs from a given initial state $x(t_0)=x_0$ and the cost*

$$J(t_1,\cdots,t_K) = \psi\big(x(t_f)\big) + \int_{t_0}^{t_f} L(x)\, dt + \sum_{k=1}^{K} \psi^k\big(x(t_k^-)\big) \qquad (2.4)$$

is minimized. Here t_0, t_f are given. □

Problem 2.1 is an optimal control problem in Bolza form. Unlike conventional optimal control problems, here the cost J includes the costs ψ^k's for discontinuous jumps at t_k's. In the sequel, we assume that f_k's, L, ψ, ψ^k's, and γ^k's are smooth enough.

2.2 An Approach

The following algorithm was proposed in [10] for solving Problem 2.1.

Algorithm 2.1

(1). Set the iteration index $j=0$. Choose initial $t_1^j,\cdots,t_K^j$.
(2). Find $J(t_1^j,\cdots,t_K^j)$, $\frac{\partial J}{\partial t_k}(t_1^j,\cdots,t_K^j)$'s and $\frac{\partial^2 J}{\partial t_k t_l}(t_1^j,\cdots,t_K^j)$'s for $1 \le k,l \le K$.
(3). Use the gradient projection method or the constrained Newton's method [1] to update t_k^j to be $t_k^{j+1} = t_k^j + \alpha^j dt_k^j$, $1 \le k \le K$. Set the iteration index $j=j+1$.
(4). Repeat steps (2), (3) and (4), until a prespecified termination condition is satisfied (e.g. $\|[\frac{\partial J}{\partial t_1},\cdots,\frac{\partial J}{\partial t_K}]\|_2 < \varepsilon$ where ε is a given small number). □

In order to apply the above algorithm, one needs to find the values of the derivatives $\frac{\partial J}{\partial t_k}$'s and $\frac{\partial^2 J}{\partial t_k t_l}$'s at each iteration. To do this, we first assume that we are given nominal values of $t_1,\cdots,t_K$ and the corresponding nominal trajectory $x(t)$. Next we define the value function at the k-th switching instant to be

$$\begin{aligned} J^k\big(x(t_k^+),t_k,\cdots,t_K\big) &\triangleq \psi\big(x(t_f)\big) + \textstyle\int_{t_k^+}^{t_{k+1}^-} L(x)\, dt \\ &+ \cdots + \textstyle\int_{t_K^+}^{t_f} L(x)\, dt + \sum_{j=k+1}^{K} \psi^j\big(x(t_j^-)\big). \end{aligned} \qquad (2.5)$$

Now the following theorem from [10] provides us with the expressions of $\frac{\partial J}{\partial t_k}$'s and $\frac{\partial^2 J}{\partial t_k t_l}$'s.

Theorem 2.1 *The cost J in Problem 2.1 satisfies*

$$\begin{aligned} &J(t_1+dt_1, t_2+dt_2,\cdots,t_K+dt_K) \\ &= J(t_1,t_2,\cdots,t_K) + \textstyle\sum_{k=1}^K J_{t_k} dt_k + \frac{1}{2}\sum_{k=1}^K J_{t_k t_k} dt_k^2 \\ &+ \textstyle\sum_{1\le k<l\le K} J_{t_k t_l} dt_k dt_l + H.O.T \end{aligned} \qquad (2.6)$$

where

$$J_{t_k} = L^{k-} - L^{k+} + \psi_x^{k-} f^{k-} + J_x^{k+}(\gamma_x^{k-} f^{k-} - f^{k+}), \qquad (2.7)$$

$$\begin{aligned} J_{t_k t_k} &= (L_x^{k-} - L_x^{k+}\gamma_x^{k-}) f^{k-} + \psi_x^{k-} f_x^{k-} f^{k-} \\ &+ (f^{k-})^T \psi_{xx}^{k-} f^{k-} + J_x^{k+}(\gamma_x^{k-} f_x^{k-} + \xi^{k-} - f_x^{k+}\gamma_x^{k-}) f^{k-} \\ &- (J_x^{k+} f_x^{k+} + L_x^{k+})(\gamma_x^{k-} f^{k-} - f^{k+}) \\ &+ (\gamma_x^{k-} f^{k-} - f^{k+})^T J_{xx}^{k+} (\gamma_x^{k-} f^{k-} - f^{k+}), \end{aligned} \qquad (2.8)$$

for any $k=1,\cdots,K$, and

$$\begin{aligned} J_{t_k t_l} &= (L_x^{l-} - L_x^{l+}\gamma_x^{l-} + \psi_x^{l-} f_x^{l-} + (f^{l-})^T \psi_{xx}^{l-} \\ &+ J_x^{l+}(\gamma_x^{l-} f_x^{l-} + \xi^{l-} - f_x^{l+}\gamma_x^{l-}) + (\gamma_x^{l-} f^{l-} \\ &- f^{l+})^T J_{xx}^{l+}\gamma_x^{l-}) H(t_l^-, t_k^+)(\gamma_x^{k-} f^{k-} - f^{k+}), \end{aligned} \qquad (2.9)$$

for any $1 \le k < l \le K$. Here $H(t_l^-, t_k^+)$ is the state transition matrix under state jumps

$$\begin{aligned} &H(t_l^-, t_k^+) \\ &= A(t_l^-, t_{l-1}^+)\gamma_x^{(l-1)-} A(t_{l-1}^-, t_{l-2}^+) \bullet \cdots \bullet \gamma_x^{(k+1)-} A(t_{k+1}^-, t_k^+) \end{aligned} \qquad (2.10)$$

where $A(t_{j+1}^-, t_j^+)$, $k \le j \le l-1$ is the state transition matrix for the time interval $[t_j^+, t_{j+1}^-]$ for the variational equation

$$\dot{y}(t) = \frac{\partial f_{j+1}\big(x(t)\big)}{\partial x} y(t). \qquad (2.11)$$

Also

$$\xi^{k-} \triangleq \begin{bmatrix} (f^{k-})^T \frac{\partial \gamma_{(1)}^k\left(x(t_k^-)\right)}{\partial x} \\ \vdots \\ (f^{k-})^T \frac{\partial \gamma_{(n)}^k\left(x(t_k^-)\right)}{\partial x} \end{bmatrix}, \ k=1,\cdots,K, \qquad (2.12)$$

where $\gamma_{(j)}^k$ refers to the j-th element of the vector-valued function γ^k. □

Remark 2.1 Note that in the above theorem, we adopt the following notational convention. We write f and f_x with a superscript $k-$ (resp. $k+$) whenever the corresponding active vector field at t_k- (resp. t_k+) is used for evaluation at $x(t_k^-)$ (resp. $x(t_k^+)$). Examples of this convention are $f^{k-} \triangleq f_k\big(x(t_k^-)\big)$, $f^{k+} \triangleq f_{k+1}\big(x(t_k^+)\big)$, $f_x^{k-} \triangleq \frac{\partial f_k}{\partial x}\big(x(t_k^-)\big)$, $f_x^{k+} \triangleq \frac{\partial f_{k+1}}{\partial x}\big(x(t_k^+)\big)$. Also, we simply write a function's name with a superscript $k-$ (resp. $k+$) whenever it is evaluated at $x(t_k^-)$ (resp. $x(t_k^+)$). Examples are $J^{k+} \triangleq J^k\big(x(t_k^+),t_k,\cdots,t_K\big)$, $J_x^{k+} \triangleq \frac{\partial J^k}{\partial x}\big(x(t_k^+),t_k,\cdots,t_K\big)$, $L^{k-} \triangleq L\big(x(t_k^-)\big)$, $L^{k+} \triangleq L\big(x(t_k^+)\big)$, $L_x^{k-} \triangleq \frac{\partial L}{\partial x}\big(x(t_k^-)\big)$, $\psi^{k-} \triangleq \psi^k\big(x(t_k^-)\big)$, etc. Also in the above theorem, we regard J_x^{k+} as a row vector, J_{xx}^{k+} as an $n \times n$ matrix and so on. □

Remark 2.2 In Theorem 2.1, in order to compute J_{t_k}, $J_{t_k t_k}$ and $J_{t_k t_l}$, we need to know the values of $H(t_l^-, t_k^+)$, J_x^{k+} and J_{xx}^{k+}. However, given nominal $t_1,\cdots,t_K$ and $x(t)$, these parameters are not readily available. In general, numerical methods need to be used to compute their values. A numerical method based on solving additional initial value ordinary differential equations (ODEs) with jumps was developed in [10]. $\frac{K(K+1)}{2}$ sets of initial value ODEs with jumps need to be solved in order to obtain the values of $H(t_l^-, t_k^+)$, J_x^{k+} and J_{xx}^{k+} for all k and $k<l$. □

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3 Results for Hybrid Linear Autonomous Systems with State Jumps

In this section, we apply the approach in Section 2.2 to a special class of problems — quadratic optimal control problems for hybrid linear autonomous systems with state jumps. In particular, we show that due to the special structure of the problem, an efficient method for the computation of the parameters $H(t_l^-,t_k^+)$, J_x^{k+} and J_{xx}^{k+} can be obtained.

We consider the following optimal control problem.

Problem 3.1 *Consider a hybrid autonomous system with linear subsystems $\dot{x} = A_i x, i \in I$. Assume a prespecified sequence of active subsystems $(1,2,\cdots,K,K+1)$ is given. Also assume that when the system switches from subsystem k to $k+1$ ($k=1,\cdots,K$), there is a discontinuous jump of the continuous state which has the linear relationship*

$$x(t_k^+) = \gamma^k\big(x(t_k^-)\big) = \Theta_k x(t_k^-) + \Gamma_k \tag{3.1}$$

where Θ_k, Γ_k are matrices of appropriate dimensions. Find optimal switching instants $t_1,\cdots,t_K$ ($t_0 \le t_1 \le \cdots \le t_K \le t_f$) such that the cost in general quadratic form

$$J(t_1,\cdots,t_K) = \psi\big(x(t_f)\big) + \int_{t_0}^{t_f} L(x)\, dt + \sum_{k=1}^{K} \psi^k\big(x(t_k^-)\big) \tag{3.2}$$

where

$$\psi\big(x(t_f)\big) = \frac{1}{2}\big(x(t_f)\big)^T Q_f x(t_f) + M_f x(t_f) + W_f, \tag{3.3}$$

$$L(x) = \frac{1}{2}\big(x(t)\big)^T Q x(t) + M x(t) + W, \tag{3.4}$$

$$\psi^k\big(x(t_k^-)\big) = \frac{1}{2}\big(x(t_k^-)\big)^T Q_k x(t_k^-) + M_k x(t_k^-) + W_k, \tag{3.5}$$

is minimized. Here t_0, t_f and $x(t_0) = x_0$ are given; Q_f, M_f, W_f, Q, M, W are matrices of appropriate dimensions with $Q_f \ge 0$, $Q \ge 0$. Q_k, M_k, W_k, ($k=1,\cdots,K$), are matrices of appropriate dimensions which form the quadratic terms for the cost of discontinuous jumps from subsystem k to $k+1$ and $Q_k \ge 0$. □

In view of the special structure of Problem 3.1, we can readily observe that

$$A(t_{k+1}^-,t_k^+) = e^{A_{k+1}(t_{k+1}-t_k)} \tag{3.6}$$

for any $k=1,\cdots,K$. Moreover,

$$\begin{aligned} & H(t_l^-,t_k^+) \\ &= e^{A_l(t_l-t_{l-1})}\gamma_x^{(l-1)-} e^{A_{l-1}(t_{l-1}-t_{l-2})} \bullet \cdots \\ &\quad \bullet \gamma_x^{(k+1)-} e^{A_{k+1}(t_{k+1}-t_k)} \\ &= e^{A_l(t_l-t_{l-1})}\Theta_{l-1} e^{A_{l-1}(t_{l-1}-t_{l-2})} \bullet \cdots \\ &\quad \bullet \Theta_{k+1} e^{A_{k+1}(t_{k+1}-t_k)}. \end{aligned} \tag{3.7}$$

The computation of J_x^{k+} and J_{xx}^{k+} is discussed next. Assume nominal $t_1,\cdots,t_K$ are given. If for any $x \in \mathbb{R}^n$ and any $t \in [t_0,t_f]$ we denote by $\tilde{J}(x,t)$ the cost incurred if the system starts from the state x at time instant t and evolves according to the portion of the switching sequence $\big((0,1),(t_1,2),(t_2,3),\cdots,(t_K,K+1)\big)$ in $[t,t_f]$. In other words,

$$\begin{aligned} \tilde{J}(x,t) &= \psi\big(x(t_f)\big) + \textstyle\int_t^{t_f} L\big(x(\tau)\big)\, d\tau \\ &\quad + \textstyle\sum_{k \text{ with } t_k \in [t,t_f]} \psi^k\big(x(t_k^-)\big) \end{aligned} \tag{3.8}$$

where $x(t) = x$. Now we note that the following dynamic programming equation holds for $\tilde{J}(x,t)$

$$\begin{cases} \tilde{J}_t = -\tilde{J}_x f^{j+1}(x) - L(x), \ t_j^+ \le t \le t_{j+1}^- \\ \qquad \text{for } j = 0,1,\cdots,K, \\ \tilde{J}(x,t_j^-) = \tilde{J}(\gamma^j(x),t_j^+) + \psi^j(x) \\ \qquad \text{for } j = 1,2,\cdots,K. \end{cases} \tag{3.9}$$

Note that (3.9) can be derived similarly to the HJB equation. However, the difference between it and the HJB equation is that (3.9) holds for any trajectory that is not necessarily optimal (for more details see [3]).

If we assume that

$$\tilde{J}(x,t) = \frac{1}{2}x^T P(t) x + S(t) x + T(t) \tag{3.10}$$

where $P(t)$, $S(t)$, $T(t)$ are matrices of appropriate dimensions, and $P(t) = P^T(t)$, then from (3.9), $P(t)$, $S(t)$, $T(t)$ must obey the following differential equations with jumps

$$\begin{cases} -\dot{P} = PA_{j+1} + A_{j+1}^T P + Q, \ t_j^+ \le t \le t_{j+1}^- \\ \qquad \text{for } j = 0,1,\cdots,K, \\ P(t_j^-) = \Theta_j^T P(t_j^+)\Theta_j + Q_j \\ \qquad \text{for } j = 1,2,\cdots,K, \end{cases} \tag{3.11}$$

$$\begin{cases} -\dot{S} = SA_{j+1} + M, \ t_j^+ \le t \le t_{j+1}^- \\ \qquad \text{for } j = 0,1,\cdots,K, \\ S(t_j^-) = \Gamma_j^T P(t_j^+)\Theta_j + S(t_j^+)\Theta_j + M_j \\ \qquad \text{for } j = 1,2,\cdots,K, \end{cases} \tag{3.12}$$

$$\begin{cases} -\dot{T} = W, \ t_j^+ \le t \le t_{j+1}^- \\ \qquad \text{for } j = 0,1,\cdots,K, \\ T(t_j^-) = \frac{1}{2}\Gamma_j^T P(t_j^+)\Gamma_j + S(t_j^+)\Gamma_j + T(t_j^+) \\ \qquad +W_j \quad \text{for } j = 1,2,\cdots,K, \end{cases} \tag{3.13}$$

along with initial conditions

$$P(t_f) = Q_f, \ S(t_f) = M_f, \ T(t_f) = W_f. \tag{3.14}$$

From the definitions of the functions $\tilde{J}$ and J^k, if $t_1,\cdots,t_K$ are fixed, we have

$$J^k\big(x(t_k^+),t_k,\cdots,t_K\big) = \tilde{J}\big(x(t_k^+),t_k^+\big), \tag{3.15}$$

$$J_x^k\big(x(t_k^+),t_k,\cdots,t_K\big) = \tilde{J}_x\big(x(t_k^+),t_k^+\big), \tag{3.16}$$

$$J_{xx}^k\big(x(t_k^+),t_k,\cdots,t_K\big) = \tilde{J}_{xx}\big(x(t_k^+),t_k^+\big). \tag{3.17}$$

Therefore the values of J_x^{k+} and J_{xx}^{k+} can be obtained as

$$J_x^{k+} = \tilde{J}_x\big(x(t_k^+),t_k^+\big) = \big(x(t_k^+)\big)^T P(t_k^+) + S(t_k^+), \tag{3.18}$$

$$J_{xx}^{k+} = \tilde{J}_{xx}\big(x(t_k^+),t_k^+\big) = P(t_k^+). \tag{3.19}$$

Remark 3.1 (Computational Cost) The computation of $H(t_l^-,t_k^+)$'s using (3.7) is straightforward and do not resort to an ODE solver. The computation of J_x^{k+} and J_{xx}^{k+} using (3.18) and (3.19) relies on the values of $P(t_k^+)$'s and $S(t_k^+)$'s which are easy to obtain by solving one set of initial value ODEs with jumps (3.11)-(3.14) backward in time only once. Therefore, the computation is much easier to implement and the computational effort for Problem 3.1 is greatly reduced as opposed to the method proposed in [10] which requires the solutions of $\frac{K(K+1)}{2}$ sets of initial value ODEs with jumps. □

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4 Reachability Problems

The quadratic optimal control problems discussed in Section 3 can also be applied to address the following important class of reachability problems. Here we still focus on hybrid linear autonomous systems with linear state jumps due to the easy implementation of Algorithm 2.1, though the discussions below can be easily extended to general hybrid autonomous systems.

Problem 4.1 (Reachability Problem) *Given a hybrid linear autonomous system with linear state jumps (as that described in Problem 3.1), does there exist a switching sequence such that the state trajectory x departs from $x(t_0)=x_0$ and meets x_f at some t_f? Here t_0, x_0, x_f are given; t_f is not given.* □

Note that x_f is reachable from x_0, if and only if the following quadratic optimal control problem achieves its minimum at $J=0$. The problem is having a free final time t_f and seeks to minimize the quadratic cost

$$J = \frac{1}{2}\|x(t_f)-x_f\|_2^2. \tag{4.1}$$

Here t_0, x_0, x_f are given. In general, the optimal control problem is difficult to solve due to the large number of possible patterns of switching sequences. But if we assume that a prespecified sequence of active subsystems is given, the problem can be handled by using optimal control methodologies. For example, we can assume subsystem k being active in $[t_{k-1}, t_k)$ (subsystem $K+1$ in $[t_K, t_f]$). In this case, we can minimize J with respect to the switching instants and the final time t_f. In other words, the reachability problem can be formulated as an optimal control problem which seeks optimal values of $t_1, \cdots, t_K, t_f$ such that

$$J(t_1,\cdots,t_K,t_f) = \frac{1}{2}\|x(t_f)-x_f\|_2^2 \tag{4.2}$$

is minimized. In this case, ideally the minimum cost should be 0 if x_f is reachable from x_0 by the given order of active subsystems. In practice, if the optimal value of J is found to be smaller than a predefined small tolerance $\varepsilon > 0$, then we regard x_f as reachable from x_0 and regard the corresponding optimal $t_1, \cdots, t_K, t_f$ as the reachability switching instants.

Remark 4.1 Note that since our approach for optimal control finds local optimal solutions, an optimal value of J greater than ε does not necessarily imply that x_f is not reachable from x_0. In the case that x_f is reachable from x_0, another trial of initial guess of switching instants may lead to the global optimal solution with $J<\varepsilon$. Therefore the optimal control approach can only be used as a sufficient condition for determining reachability. However, whenever x_f is determined to be reachable from x_0, our approach also provides an explicit sequence $(t_1,\cdots,t_K,t_f)$ that achieves it. This is the strength of the approach. □

To minimize $J(t_1,\cdots,t_K,t_f)$ with respect to $(t_1,\cdots,t_K,t_f)$, we can use Algorithm 2.1. To apply the algorithm, the derivatives of J first need to be computed. The derivative values J_{t_k}, $J_{t_kt_k}$ and $J_{t_kt_l}$ can be obtained using the expressions stated in Theorem 2.1. However, since t_f is free here, we also need to derive J_{t_f}, $J_{t_ft_f}$ and $J_{t_kt_f}$. To derive these derivatives, we define

$$J^f\big(x(t_f)\big) \triangleq \frac{1}{2}\|x(t_f)-x_f\|_2^2, \tag{4.3}$$

and consider the Taylor expansion of

$$J(t_1,\cdots,t_K,t_f) = J^f\big(x(t_f)\big). \tag{4.4}$$

By fixing $t_1,\cdots,t_K$, we can expand J into second order expansions with respect to t_f and obtain

$$\begin{aligned} J_{t_f} &= J_x^f f^f, & (4.5)\\ J_{t_ft_f} &= J_x^f f_x^f f^f + (f^f)^T J_{xx}^f f^f, & (4.6)\end{aligned}$$

Also, similarly to the derivations in [10] for $J_{t_kt_l}$, we can derive

$$J_{t_kt_f} = (J_x^f f_x^f + (f^f)^T J_{xx}^f) H(t_f, t_k^+)(\gamma_x^{k-} f^{k-} - f^{k+}). \tag{4.7}$$

In the above expressions, we have

$$\begin{aligned} J^f &= \frac{1}{2}\|x(t_f)-x_f\|_2^2, & (4.8)\\ J_x^f &= \big(x(t_f)-x_f\big)^T, & (4.9)\\ J_{xx}^f &= I_{n\times n}, & (4.10)\\ f^f &= f_{K+1}\big(x(t_f)\big) = A_{K+1}x(t_f), & (4.11)\\ f_x^f &= \frac{\partial f_{K+1}\big(x(t_f)\big)}{\partial x} = A_{K+1}, & (4.12)\end{aligned}$$

Using (4.8)-(4.12), we can simplify (4.5)-(4.7) as

$$\begin{aligned} J_{t_f} &= \big(x(t_f)-x_f\big)^T A_{K+1}x(t_f), & (4.13)\\ J_{t_ft_f} &= \big(x(t_f)-x_f\big)^T A_{K+1}^2 x(t_f) \\ &\quad + \big(x(t_f)\big)^T A_{K+1}^T A_{K+1} x(t_f), & (4.14)\\ J_{t_kt_f} &= \Big(\big(x(t_f)-x_f\big)^T A_{K+1} \\ &\quad + \big(x(t_f)\big)^T A_{K+1}^T\Big) H(t_f,t_k^+)\big(\Theta_k A_k x(t_k^-) \\ &\quad - A_{k+1}x(t_k^+)\big), & (4.15)\end{aligned}$$

where

$$\begin{aligned} &H(t_f,t_k^+) \\ &= e^{A_{K+1}(t_f-t_K)}\Theta_K e^{A_K(t_K-t_{K-1})} \bullet \cdots \bullet \Theta_{k+1} e^{A_{k+1}(t_{k+1}-t_k)}. \end{aligned} \tag{4.16}$$

5 Examples

In this section, we present two examples to illustrate the effectiveness of the method developed in this paper.

Example 5.1 Consider a hybrid linear autonomous system consisting of

$$\begin{aligned} \text{subsystem 1: } \dot{x} &= A_1 x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x, & (5.1)\\ \text{subsystem 2: } \dot{x} &= A_2 x = \begin{bmatrix} 0.1 & -0.5 \\ 0.5 & 0.1 \end{bmatrix} x, & (5.2)\\ \text{subsystem 3: } \dot{x} &= A_3 x = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x. & (5.3)\end{aligned}$$

Assume that $t_0=0$, $t_f=3$ and the system switches at $t=t_1$ from subsystem 1 to 2 and at $t=t_2$ from subsystem 2 to 3 $(0\le t_1\le t_2\le 3)$. Also assume that the system has the state jump

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$$x(t_1^+) = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.9 \end{bmatrix} x(t_1^-) + \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix} \tag{5.4}$$

when switching from subsystem 1 to 2 and

$$x(t_2^+) = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix} x(t_2^-) + \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} \tag{5.5}$$

when switching from subsystem 2 to 3. We want to find optimal switching instants t_1, t_2 such that the cost

$$\begin{aligned} J = & \tfrac{1}{2}x_1^2(3) + \tfrac{1}{2}x_2^2(3) + \tfrac{1}{2}\int_0^3 \left(x_1^2(t) + x_2^2(t)\right) dt \\ & + \textstyle\sum_{k=1}^2 \left(\tfrac{1}{2}x_1^2(t_k^-) + \tfrac{1}{2}x_2^2(t_k^-)\right) \end{aligned} \tag{5.6}$$

is minimized. Here $x_1(0) = -1$ and $x_2(0) = 3$.

For this problem, we choose initial nominal $t_1 = 1$, $t_2 = 2$. We derive the derivatives of J using the result in Theorem 2.1 and the computational method in Section 3. By using the Algorithm 2.1 with the constrained Newton's method, after 8 iterations we find that the optimal switching instants are $t_1 = 0.6167$, $t_2 = 1.6724$ and the corresponding optimal cost is 16.6518. The corresponding state trajectory is shown in Figure 1. Figure 2 shows the plot of the cost function for different $0 \le t_1 \le t_2 \le 3$. By comparing the J value for different t_1 and t_2, we verify that the solution we obtain is the global optimal (although it is difficult to tell from the cost surface, our computation shows us so). □

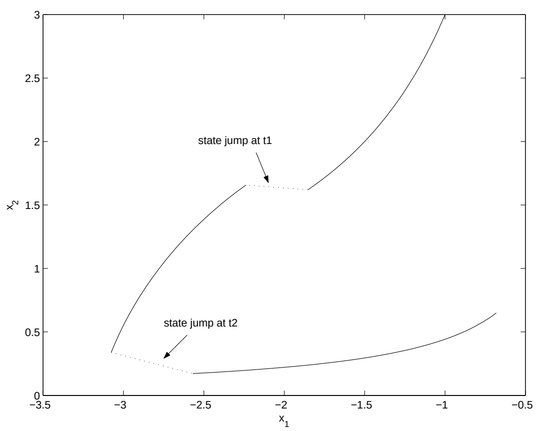

Figure 1: The state trajectory for Example 5.1.

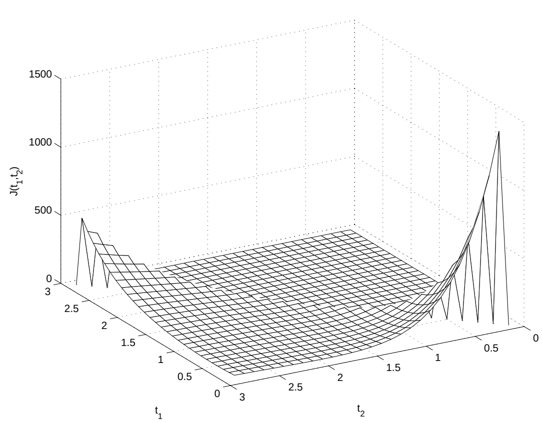

Figure 2: The cost for Example 5.1 for different (t_1, t_2)'s ($0 \le t_1 \le t_2 \le 3$).

Example 5.2 (A Reachability Problem) Consider a hybrid linear autonomous system consisting of

$$\text{subsystem 1: } \dot{x} = A_1 x = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x, \tag{5.7}$$

$$\text{subsystem 2: } \dot{x} = A_2 x = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x. \tag{5.8}$$

Assume that at $t_0 = 0$, the system state departs from the initial condition $x_1(0) = 1$ and $x_2(0) = 1$ and evolves following the dynamics of subsystem 1. Assume that the system switches once at t_1 from subsystem 1 to 2. Also assume that the system has the state jump

$$x(t_1^+) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x(t_1^-) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5.9}$$

when switching from subsystem 1 to 2. We want to find a t_1 and a t_f ($0 \le t_1 \le t_f$) such that the system state arrives at $[2e^3 + e^2,\ e^3 + e]^T$ at t_f.

This reachability problem can be posed as a quadratic optimal control problem with unknown t_f and cost $J = \frac{1}{2}\left((x_1(t_f) - (2e^3 + e^2))^2 + (x_2(t_f) - (e^3 + e))^2\right)$. We choose initial nominal $t_1 = 0.8$, $t_f = 1.8$. The values of J_{t_1}, J_{t_f}, $J_{t_1 t_1}$, $J_{t_f t_f}$ and $J_{t_1 t_f}$ can be derived by using the formulae (2.7)-(2.9), (4.13)-(4.15), and the computational method in Section 3. We use Algorithm 2.1 with the constrained Newton's method to search for an optimal solution. After 8 iterations we find that the optimal switching instants are $t_1 = 1.0000$, $t_2 = 2.0000$ and the corresponding optimal cost is 5.0487×10^{-29}. The corresponding state trajectory is shown in Figure 3. Figure 4 shows the plot of the cost function for different $0 \le t_1 \le t_f$. By comparing the J value for different t_1 and t_f, we verify that the solution we obtain is the global optimal (although it is difficult to tell from the cost surface, our computation shows us so).

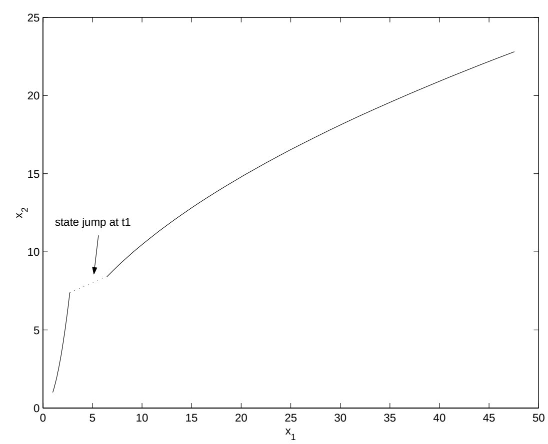

Figure 3: The state trajectory for Example 5.2.

It is worth noting that for this example we can verify the correctness of (4.13)-(4.15). For example, the expression of $J_{t_1 t_f}$ can be derived from (4.15) as (here $k = 1, K = 1$)

$$\begin{aligned} J_{t_1 t_f} = & \left(\left(x(t_f) - x_f\right)^T A_2 + \left(x(t_f)\right)^T A_2^T\right) H(t_f, t_1^+) \\ & \cdot \left(\Theta_1 A_1 x(t_1^-) - A_2 x(t_1^+)\right). \end{aligned} \tag{5.10}$$

We can substitute $x(t_1^-) = [e^{t_1},\ e^{2t_1}]^T$, $x(t_1^+) = [2e^{t_1} + 1,\ e^{2t_1} + 1]^T$, $x(t_f) = [2e^{2t_f - t_1} + e^{2t_f - 2t_1},\ e^{t_f + t_1} + e^{t_f - t_1}]^T$,

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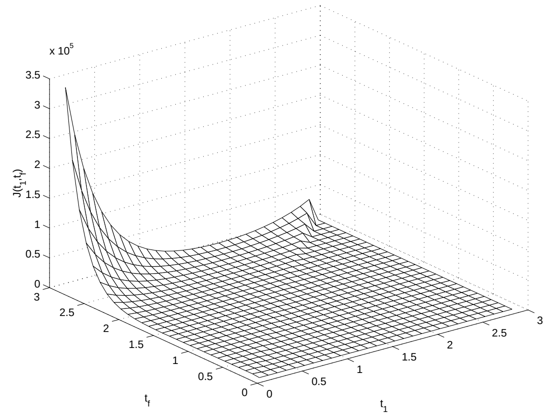

Figure 4: The cost for Example 5.2 for different (t_1,t_f)'s $(0 \le t_1 \le t_f)$.

$x_f^T = [2e^3+e^2, \ \ e^3+e]$, $H(t_f,t_1^+) = e^{A_2(t_f-t_1)}$, $\Theta_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, A_1, and A_2 into (5.10) and obtain

$$\begin{aligned} J_{t_1t_f} &= (8e^{2t_f-t_1}+4e^{2t_f-2t_1}-4e^3-2e^2)(-2e^{2t_f-t_1} \\ &\quad -2e^{2t_f-2t_1})+(2e^{t_f+t_1}+2e^{t_f-t_1} \\ &\quad -e^3-e)(e^{t_f+t_1}-e^{t_f-t_1}). \qquad (5.11) \end{aligned}$$

The correctness of (5.11) can be verified by directly differentiating the expression of J

$$\begin{aligned} J &= \frac{1}{2}\big((2e^{2t_f-t_1}+e^{2t_f-2t_1}-2e^3-e^2)^2+(e^{t_f+t_1} \\ &\quad +e^{t_f-t_1}-e^3-e)^2\big), \qquad (5.12) \\ \frac{\partial J}{\partial t_1} &= (2e^{2t_f-t_1}+e^{2t_f-2t_1}-2e^3-e^2)(-2e^{2t_f-t_1} \\ &\quad -2e^{2t_f-2t_1})+(e^{t_f+t_1}+e^{t_f-t_1} \\ &\quad -e^3-e)(e^{t_f+t_1}-e^{t_f-t_1}), \qquad (5.13) \\ \frac{\partial^2 J}{\partial t_1 \partial t_f} &= (8e^{2t_f-t_1}+4e^{2t_f-2t_1}-4e^3-2e^2)(-2e^{2t_f-t_1} \\ &\quad -2e^{2t_f-2t_1})+(2e^{t_f+t_1}+2e^{t_f-t_1} \\ &\quad -e^3-e)(e^{t_f+t_1}-e^{t_f-t_1}). \qquad (5.14) \end{aligned}$$

Similarly, we can also verify the correctness of the expressions of J_{t_1}, J_{t_f}, $J_{t_1t_1}$, $J_{t_ft_f}$ by direct differentiations of J. □

6 Conclusion

In this paper, results for quadratic optimal control of hybrid linear autonomous systems with state jumps are reported. Based on an approach proposed in [10] for optimal control problems of hybrid autonomous systems, we propose a method which leads to more efficient computation of the necessary parameters in applying the approach. The method takes advantage of the special structure of linear subsystems and quadratic costs. Application of quadratic optimal control results to reachability problems is also reported. A more detailed version of this paper can be found in [11]. A further research topic is the development of methods for searching for optimal switching sequences when the sequence of active subsystems are not prespecified.

References

[1] D.P. Bertsekas. *Nonlinear Programming, Second Edition*. Athena Scientific, 1999.

[2] M.S. Branicky, V.S. Borkar, and S.K. Mitter. A unified framework for hybrid control: model and optimal control theory. *IEEE Transactions on Automatic Control*, 43(1):31–45, January 1998.

[3] P. Dyer and S.R. McReynolds. *The Computation and Theory of Optimal Control*. Academic Press, 1970.

[4] A. Giua, C. Seatzu, and C. Van Der Mee. Optimal control of switched autonomous linear systems. In *Proc. of the 40th IEEE CDC*, pages 2472–2477, 2001.

[5] K. Gokbayrak and C.G. Cassandras. A hierarchical decomposition method for optimal control of hybrid systems. In *Proc. of the 39th IEEE CDC*, pages 2472–2477, 2000.

[6] S. Hedlund and A. Rantzer. Convex dynamic programming for hybrid systems. *IEEE Transactions on Automatic Control*, 47(9):1536–1540, September 2002.

[7] B. Piccoli. Hybrid systems and optimal control. In *Proc. of the 37th IEEE CDC*, pages 13–18, 1998.

[8] H.J. Sussmann. A maximum principle for hybrid optimal control problems. In *Proc. of the 38th IEEE CDC*, pages 425–430, 1999.

[9] X. Xu and P.J. Antsaklis. Optimal control of switched systems: new results and open problems. In *Proc. of the 2000 ACC*, pages 2683–2687, 2000.

[10] X. Xu and P.J. Antsaklis. Optimal control of hybrid autonomous systems with state jumps. In *Proc. of the 2003 ACC*, 2003. in this proceedings.

[11] X. Xu and P.J. Antsaklis. Optimal control of hybrid autonomous systems with state jumps. ISIS Technical Report ISIS-2003-001, University of Notre Dame, March 2003. (`http://www.nd.edu/~isis/tech.html`).