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Results and Perspectives on Computational Methods for Optimal Control of Switched Systems*

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Abstract. This paper surveys some of the recent progresses on computational methods for optimal control of switched systems. A general model of switched system that allows externally forced switching (EFS) and internally forced switching (IFS) is first introduced and two important classes of optimal control problems are formulated. After a brief review of some relevant theoretical results, we present the idea of two stage optimization. Based on the theoretical results and two stage optimization, we then survey computational methods based on discretization, and computational methods not based on discretization. Comments are made on the merits and restrictions of each method.

1 Introduction

Switched systems are a particular class of hybrid systems consisting of several subsystems and a switching law specifying the active subsystem at each time instant. Examples of switched systems can be found in chemical processes, automotive systems, and electrical circuit systems, to name a few.

Recently, optimal control problems of hybrid and switched systems have attracted researchers from various fields in science and engineering, due to the problems' significance in theory and application. The available results on such problems include theoretical and computational ones. The available theoretical results usually extend the classical maximum principle or dynamic programming to such problems. As to the computational results, researchers have taken advantage of efficient nonlinear optimization techniques and high-speed computers to develop efficient numerical methods for such problems.

This paper surveys some of the recent progresses on computational methods for optimal control problems of switched systems. Such problems are difficult to

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solve, due to the involvement of switchings of subsystem dynamics. The recent decade has seen some breakthrough in the development of efficient computational methods for such problems. However, the literature results are often based on different models and differ in problem formulations and approaches. Therefore, it is necessary to call for an overview of these results under a unified framework.

In this paper, we first propose a general model of switched system that includes externally forced switching (EFS) and internally forced switching (IFS) and formulate EFS and IFS optimal control problems. Such formulations then serve as unified frameworks for our survey. After a brief review of some relevant theoretical results, we present the idea of two stage optimization for EFS problems where stage 1 seeks optimal continuous input and stage 2 seeks optimal switching sequence. Extensions of the two stage optimization to IFS problems are also mentioned. Based on the theoretical results and two stage optimization, we survey two classes of computational methods — those based on discretization and those not based on discretization. Several methods based on discretization and methods for discrete-time problems are overviewed and their merits and restrictions are indicated. We then report recent developments of computational methods not based on discretization for stage 1 optimization in which a prespecified sequence of active subsystems is given. We point out that a stage 1 problem can further be decomposed into stage 1(a), which is a conventional optimal control problem that seeks the optimal cost given a switching sequence, and stage 1(b), which is a nonlinear optimization problem that seeks the optimal switching instants. Stage 1(b) poses difficulties because it is hard to obtain the information of the derivatives of the stage 1(a) optimal cost with respect to the switching instants. To address these difficulties, we finally overview two methods which can find approximations and accurate values of the derivative values, respectively.

2 Problem Statement

2.1 Switched Systems

A switched system consists of several subsystems and a switching law. A switching usually takes place when a certain event signal is received. An event signal may be an external signal (generated exogenously) or an internal signal generated when an internal condition for the states, inputs and/or time evolution is satisfied. In the sequel, we call a switching triggered by an external event an externally forced switching (EFS) and a switching triggered by an internal event an internally forced switching (IFS).

Definition 1. *A* **switched system** *is a 3-tuple* $\mathcal{S} = (\mathcal{D}, \mathcal{F}, \mathcal{L})$ *where*
• $\mathcal{D} = (I, E)$ *is a directed graph indicating the discrete mode structure of the system.* $I = \{1, 2, \cdots, M\}$ *is the set of indices for subsystems.* E *is a subset of* $I \times I - \{(i,i) | i \in I\}$ *which contains the valid events. If an event* $e = (i_1, i_2)$ *takes place, the system switches from subsystem* i_1 *to* i_2*. Furthermore* $E = E_E \cup E_I$ *(*E_E *and* E_I *may not be disjoint) where* E_E *is the external event set and* E_I *is the internal event set.*

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• $\mathcal{F} = \{f_i : X_i \times U_i \to \mathbb{R}^n | i \in I\}$ *where* f_i *describes the vector field for the* i*-th subsystem* $\dot{x} = f_i(x,u)$. *Here* $X_i \subseteq \mathbb{R}^n$ *and* $U_i \subseteq \mathbb{R}^m$ *are respectively the state and control constraint sets for the* i*-th subsystem.*
• $\mathcal{L} = \mathcal{L}_E \cup \mathcal{L}_I$ *provides logic constraints that relate the continuous state and mode switchings. Here* $\mathcal{L}_E = \{\Lambda_e | \Lambda_e \subseteq \mathbb{R}^n, \emptyset \neq \Lambda_e \subseteq X_{i_1} \cap X_{i_2}, e = (i_1, i_2) \in E_E\}$ *corresponds to the external events; only when* $x \in \Lambda_e$ *for* $e = (i_1, i_2)$, *an EFS from subsystem* i_1 *to* i_2 *is possible. Also here* $\mathcal{L}_I = \{\Gamma_e | \Gamma_e \subseteq \mathbb{R}^n, \emptyset \neq \Gamma_e \subseteq X_{i_1} \cap X_{i_2}, e = (i_1, i_2) \in E_I\}$ *corresponds to the internal events; when subsystem* i_1 *is active and the state trajectory intersects* Γ_e *for* $e = (i_1, i_2)$, *the event* $e = (i_1, i_2)$ *must be triggered and the system is forced to switch to subsystem* i_2. □

For a switched system, the presence of switchings makes the behavior of the system more complicated than that of conventional systems. In particular, the evolution of the continuous and discrete states will leave us with a timed sequence of active subsystems that is defined as a switching sequence as follows.

Definition 2. *A* **switching sequence** σ *in* $[t_0, t_f]$ *is a timed sequence* $\sigma = \big((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K)\big)$, *where* $0 \leq K < \infty$, $t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f$, *and* $i_k \in I$ *for* $0 \leq k \leq K$. □

A switching sequence σ defined above indicates that the system starts from subsystem i_0 at t_0, and switches to subsystem i_k from subsystem i_{k-1} at t_k for $1 \leq k \leq K$. Subsystem i_k will remain active in $[t_k, t_{k+1})$. For a switched system to be well-behaved, we generally exclude the undesirable *Zeno phenomenon*, i.e., infinitely many switchings in finite amount of time. The pairs (t_k, i_k)'s in σ can be classified into two categories — those corresponding to EFS denoted by (t_k^E, i_k^E), and those corresponding to IFS denoted by (t_k^I, i_k^I). By distinguishing EFS and IFS, we can define the EFS sequence $\sigma_E = \big((t_0, i_0), (t_1^E, i_1^E), \cdots, (t_{K_1}^E, i_{K_1}^E)\big)$ and the IFS sequence $\sigma_I = \big((t_0, i_0), (t_1^I, i_1^I), \cdots, (t_{K_2}^I, i_{K_2}^I)\big)$. The combination of σ_E and σ_I gives us the overall switching sequence $\sigma \stackrel{\triangle}{=} \sigma_E \cup \sigma_I$ (here $K_1 + K_2 = K$).

Given a switched system, the overall exogenous control input is a pair (σ_E, u). Along with the evolution of $x(t)$, an IFS sequence σ_I will be generated implicitly. σ_E and σ_I then lead to the overall σ. Given initial pair $\big(x(t_0), i_0\big)$, an exogenous input pair (σ_E, u) is said to be *valid* if the evolution of the system under it generates a nonblocking state trajectory $x(t)$ and a nonZeno σ.

Remark 1. For switched systems in Definition 1, the continuous state does not exhibit jumps at switching instants. We propose this framework due to two reasons. First, in many applications such as some chemical processes, there are no state jumps at switchings. Second, analysis and design of switched systems without jumps usually require simpler notations and are more amenable to rigorous study as opposed to systems with jumps. Therefore, we mainly focus on systems without jumps in this paper. However, we note that many methods reported in the paper can actually be extended to problems with jumps. □

Remark 2. If we let all subsystems be discrete-time systems, we can similarly define discrete-time switched systems. □

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2.2 Optimal Control Problems

Although general optimal control problems can be formulated for switched systems with both EFS and IFS, notations would be complicated and results would be difficult to obtain. Hence, we choose to study two important classes of problems, i.e., optimal control problems for systems with EFS only (EFS Problems), and problems for systems with IFS only (IFS Problems). In doing so, the objective of overviewing available results can be fulfilled, because most of the available literature results are on computational methods for one of these two classes of problems. In the following, we call a valid (σ_E, u) (or u for IFS problems) *admissible* if the corresponding $x(t)$ meets the terminal manifold.

Problem 1 (EFS Problem). Consider a switched system $\mathcal{S}$ with EFS only. Find an admissible control pair (σ_E, u) (u is piecewise continuous) such that x departs from a given initial state $x(t_0) = x_0$ at the given initial time t_0 and meets an $(n - l_f)$-dimensional smooth manifold S_f at t_f (t_f is free) and

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))\ dt + \sum_{1 \leq k \leq K} \delta(x(t_k), i_{k-1}, i_k)$$

is minimized (here K is the number of switchings in σ_E). □

Problem 2 (IFS Problem). Consider a switched system $\mathcal{S}$ with IFS only. Find an admissible $u(t)$ (u is piecewise continuous) such that x departs from a given initial state $x(t_0) = x_0$ at the given initial time t_0 and meets an $(n - l_f)$-dimensional smooth manifold S_f at t_f (t_f is free) and

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))\ dt + \sum_{1 \leq k \leq K} \delta(x(t_k), i_{k-1}, i_k)$$

is minimized (here K is the number of switchings in σ_I). □

Remark 3. Problems 1 and 2 are formulated as general Bolza problems with terminal cost ψ, running cost $\int_{t_0}^{t_f} L\ dt$, and switching cost δ. In the sequel, we assume that f_i, ψ, L, and δ are smooth enough (e.g., twice continuously differentiable). In the results we will overview, various additional assumptions may be imposed. For example, for the convenience of developing numerical methods, problems with fixed t_f are sometimes studied. In fact, a free-final-time problem can always be transcribed into a fixed-final-time one by introducing additional state variables (see page 101 in [30]). □

Remark 4. Problems 1 and 2 are different due to the different exogenous inputs. Such difference makes the two problems different in many aspects when we develop computational methods. In fact, the implicit generation of σ_I makes the IFS problem more difficult. □

3 An Overview of Some Theoretical Results

In this section, we briefly overview some basic theoretical results that can help understand Problems 1 and 2 and can serve as foundations for the development of various computational methods.

The Maximum Principle

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[28] is an early paper on continuous-time optimal control problems for a class of hybrid systems in which transitions of discrete state are triggered by the continuous state (an IFS problem). The main results in [28] include a version of MP. In particular, it is proved that the costate satisfies some jump conditions at the switching instants. Another early paper [25] studies problems akin to IFS Problem 2 and reports conditions for the existence of optimal solutions for problems with two subsystems.

More general optimal control problems for hybrid systems with switchings and jumps have recently been reported in [20, 26]. [20] introduces a general hybrid system model similar to Definition 1 except for the jumps. Due to the general hybrid systems model, even analysis of the existence of solutions is quite involved (and additional assumptions must be made). Hence [20] then focuses on two special classes of problems and proves a version of MP. In [26], optimal control problems for hybrid systems with a similar formulation to that of [20] are studied. The optimal control problems studied there are more like EFS problems. Several versions of MP are then given there, including a nonsmooth version.

More recent results on EFS problems have been reported in [23, 24]. A version of MP is proposed for problems with running cost only. The MP is specifically applied to time optimal control problems and linear quadratic problems.

The MPs proposed in the above papers usually provide necessary conditions with no specific sequence of active subsystems assumed. This introduces a problem formed by a continuous-time mixed differential algebraic equation (DAE) and an integer programming problem, which is not amenable for numerical computations. However, in Section 4.2, after introducing the two stage optimization method, we will state a version of MP for a given sequence of active subsystems that only involves DAE and can be used for numerical methods.

The Dynamic Programming

Similar to conventional optimal control, DP can be used to derive the HJB equation for Problems 1 and 2. Detailed derivations of HJB equations for various formulations of optimal control problems for switched systems and systems with impulse effect and costs for switchings can be found in [37]. Following the ideas in [37], a version of HJB for EFS Problem 1 without cost for switchings is derived in [29], under additional assumptions $X_i = \mathbb{R}^n$, $U_i = \mathbb{R}^m$, $\Lambda_e = \mathbb{R}^n$. To briefly reiterate the result, we define

$$V^{*i}(x,t) \triangleq \min_{\text{admissible } (\sigma_E,u)'s} \left\{\psi(x(t_f)) + \int_t^{t_f} L(x(\tau),u(\tau))\, d\tau\right\}$$

where $x(\tau)$ is equal to x at time instant $\tau = t$. If $V^{*i}(x,t) \in C^1[t_0,t_f]$, then we have the following HJB equation

$$\min\Big\{ \min_{j\in\{i'|(i,i')\in E_E\}} \{V^{*j}(x,t)\} - V^{*i}(x,t),\ V_t^{*i}(x,t) + \min_{u\in\mathbb{R}^m}\{L(x,u) + V_x^{*i}(x,t) f_i(x,u)\} \Big\} = 0 \qquad (1)$$

The DP method and HJB equation can be useful in developing methods utilizing value functions. (1) is very difficult to solve analytically. One way to solve it is to discretize the continuous time and state spaces and then apply backward searching methods to solve the discretized problem. However, combinatorial explosion problems may arise as the discretization level becomes finer.

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A result closely related to DP is the derivation of the generalized quasi-variational inequalities (GQVI's) reported in [5]. The hybrid DP developed in [12] is also relevant in this regard.

4 Two Stage Optimization

4.1 Two Stage Optimization Formulation

Problem 1 requires the solution of a valid optimal pair (σ_E^*, u^*) such that

$$J(\sigma_E^*, u^*) = \min_{\text{admissible } (\sigma_E,u)'s} J(\sigma_E, u). \tag{2}$$

In [29], a lemma is given which proposes a way to formulate (2) into a two stage optimization problem. With only slight modifications, we can rewrite the lemma to be applicable to Problem 1 as follows.

Lemma 1. *For Problem 1, if*

(a). an admissible optimal solution (σ_E^*, u^*) exists and
(b). for any given EFS sequence σ_E for which at least one continuous input u exists so that (σ_E, u) is admissible, there exists a corresponding admissible $u^* = u^*_{\sigma_E}$ such that the function $J_{\sigma_E}(u) \triangleq J(\sigma_E, u)$ is minimized,

then the following equation holds

$$\min_{ad.\ (\sigma_E,u)\text{'s}} J(\sigma_E, u) = \min_{\sigma \in \{\sigma_E | \exists u, (\sigma_E,u) \ is\ ad.\}} \min_{u \in \{u | (\sigma_E,u)\ is\ ad.\}} J(\sigma_E, u). \tag{3}$$

Here 'ad.' stands for 'admissible'. □

The right hand side of (3) needs twice the minimization processes. This implies that the following two stage optimization method can be applied.

A Two Stage Optimization Method

Stage 1. Fixing σ, solve the inner minimization problem (or claim that σ_E is invalid, i.e., no u exists for the given σ_E such that (σ_E, u) is admissible).
Stage 2. Regarding the optimal cost for each valid σ_E as a function

$$J_1 = J_1(\sigma_E) = \min_{u \in \{u | (\sigma_E,u) \text{ is admissible}\}} J(\sigma_E, u),$$

minimize J_1 with respect to all valid σ_E's. □

In [29], under the assumptions $X_i = \mathbb{R}^n$, $U_i = \mathbb{R}^m$, $\Lambda_e = \mathbb{R}^n$, and t_f being given, we implement the above method by the following more detailed algorithm.

Algorithm 1 (A Two Stage Algorithm).

Stage 1. (a). Fix the total number of switchings to be K and the sequence of active subsystems and let the minimum value of J with respect to u be a function of the K switching instants, i.e., $J_1 = J_1(t_1, \cdots, t_K)$ for $K \geq 0$ $(t_0 \leq t_1 \leq \cdots t_K \leq t_f)$. Find J_1. ($J_1 = \infty$ when no admissible pair (σ_E, u) can be found.)

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(b). Minimize J_1 with respect to $t_1, \cdots, t_K$.

Stage 2. (a). Vary the sequence of active subsystems to find an optimal solution under K switchings.

(b). Vary the value of K to find an optimal solution for Problem 1. □

In the following, when we mention stage 1 optimization, we actually refer to stage 1 in Algorithm 1. In practice, many problems only require the solutions of optimal continuous inputs and optimal switching instants for stage 1 optimization in which a prespecified sequence of active subsystems is given. We will focus on such stage 1 optimization in the rest of this section.

IFS problems are more difficult due to the additional constraint that x must be in the set $\Gamma_{(i_1,i_2)}$ when the system switches from subsystem i_1 to i_2. Moreover, the switching instants can depend on the continuous input in a complicated way. In [31], an extension of the algorithm for EFS problems to stage 1 optimization of IFS problems is proposed (in the case that the switching set is a hypersurface).

Method 1 (A Method for IFS Problems)

1. Denote in a redundant fashion that an optimal solution to the IFS problem contains both an optimal continuous input and an optimal switching sequence (starting at subsystem i_0), i.e., regard an IFS problem as an EFS problem with additional state constraints at the switching instants. Solve the corresponding EFS problem.
2. Verify the validity of the solution for the IFS problem (i.e., if the system under the continuous input can evolve validly and generate the corresponding switching sequence). □

The decomposition of the problem into two stages and the conceptual Algorithm 1 are still applicable to step 1 in the above method. Such an extension must address the additional requirement that the system's state is restricted to a switching hypersurface at each switching instant.

4.2 More on Stage 1 Optimization

Now we concentrate on stage 1 optimization. Note that many real world problems are in fact stage 1 optimization problems. For example, the speeding-up of a power train only requires switchings from gear 1 to 2 to 3 to 4. As can be seen from Algorithm 1, stage 1 can be further decomposed into two sub-steps (a) and (b). Stage 1(a) is in essence a conventional optimal control problem which seeks the minimum of J with respect to u under a given switching sequence $\sigma = ((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K))$. We denote the corresponding optimal cost as a function $J_1(\hat{t})$, where $\hat{t} \triangleq (t_1, t_2, \cdots, t_K)^T$. Stage 1(b) is in essence a constrained nonlinear optimization problem

$$\begin{array}{c} \min_{\hat{t}} J_1(\hat{t}) \\ \text{subject to } \hat{t} \in T \end{array} \tag{4}$$

where $T \triangleq \{\hat{t} = (t_1, \cdots, t_K)^T | t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f\}$.

Stage 1(a)

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For stage 1(a) where a switching sequence $\sigma = ((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K))$ is given, finding $J_1(\hat{t})$ for the corresponding $\hat{t}$ is a conventional optimal control problem. In stage 1(a), we need to find an optimal continuous input u and the corresponding minimum J. In order to find solutions for stage 1(a) problems, computational methods must be adopted in most cases. Most of the available numerical methods for unconstrained conventional optimal control problems with fixed end-time can be used. See [16, 21] for surveys of computational methods. It is not difficult to use the calculus of variations techniques (see e.g. [13]) to prove the following necessary conditions.

Theorem 1 (Necessary Conditions for Stage 1(a)). *Consider the stage 1(a) problem for Problem 1. Assume subsystem k is active in $[t_{k-1}, t_k)$ for $1 \leq k \leq K$ and subsystem $K+1$ in $[t_K, t_f]$. Let u be a piecewise continuous input such that x departs from a given initial state $x(t_0) = x_0$ and meets $S_f = \{x | \phi_f(x) = 0, \phi_f : \mathbb{R}^n \to \mathbb{R}^{l_f}\}$ at t_f. In order that u be optimal, it is necessary that there exists a vector function $p(t) = [p_1(t), \cdots, p_n(t)]^T$, $t \in [t_0, t_f]$, such that*

(a). For almost any $t \in [t_0, t_f]$ the state equation $\frac{dx(t)}{dt} = \left(\frac{\partial H}{\partial p}(x(t), p(t), u(t))\right)^T$ and costate equation $\frac{dp(t)}{dt} = -\left(\frac{\partial H}{\partial x}(x(t), p(t), u(t))\right)^T$ hold. Here $H(x, p, u) \triangleq L(x, u) + p^T f_k(x, u)$, if $t \in [t_{k-1}, t_k)$ ($k = K + 1$ if $t \in [t_K, t_f]$).

(b). For almost any $t \in [t_0, t_f]$, the stationarity condition $0 = \left(\frac{\partial H}{\partial u}(x(t), p(t), u(t))\right)^T$ holds.

(c). At t_f, the function p satisfies $p(t_f) = \left(\frac{\partial \psi}{\partial x}(x(t_f))\right)^T + \left(\frac{\partial \phi_f}{\partial x}(x(t_f))\right)^T \lambda$, where λ is an l_f-dimensional vector.

(d). At any t_k, $k = 1, 2, \cdots, K$, we have $p(t_k-) = p(t_k+)$. □

In general, it is difficult or even impossible to find an analytical expression of $J_1(\hat{t})$ using the above conditions. The reason is that conditions (a)-(d) present a two point boundary value differential algebraic equation (DAE) that, in most cases, cannot be solved analytically. However, the above DAE can be solved efficiently using many numerical methods (e.g., shooting methods and collocation methods). Note that Theorem 1 can also be extended to IFS problems if a prespecified sequence of active subsystems is given and each set Γ_e is a hypersurface. The only difference in the IFS case is that the costate will have discontinuous jumps at the switching instants (see [28, 31] for more details).

Stage 1(b)

In stage 1(b), we need to solve the constrained nonlinear optimization problem (4) with simple constraints. Computational methods for the solution of such problems are abundant in the nonlinear optimization literature. For example, feasible direction methods and penalty function methods are two commonly used classes of methods. These methods use the information of first-order derivative $\frac{\partial J_1}{\partial \hat{t}}$ and even second-order derivative $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ (see [4, 17] for details).

Finally in this section, we should point out that [9, 10] independently propose a conceptual method of hierarchical decomposition similar to stage 1(a) and 1(b) for a class of hybrid systems optimal control problems under the assumption of

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a given sequence of active subsystems. The problems are motivated by previous works in manufacturing systems [6, 18, 19] but the formulation is more general. Both IFS and EFS problems can be formulated. The method decomposes the solution of such problems into two levels. The lower level is a conventional optimal control problem seeking an optimal continuous input and the higher level is a nonlinear optimization problem seeking initial and final states (since they are not prespecified in the problem formulations in [9, 10]) and the optimal time durations.

5 Computational Methods Based on Discretization

Now we will look at specific computational methods for Problems 1 and 2. The first class of computational methods are based on discretization of the original problem, or discretization of some continuous conditions. The benefits of using discretization-based methods are two fold. First, such methods are usually not restricted to stage 1. Second, since such methods are usually directly built on nonlinear programming methods, general constraints can be dealt with and both EFS and IFS problems can be handled within a unified framework.

5.1 Methods Based on Discretization

A discretization-based method in line with the two stage optimization in Section 4 is reported in [27]. An EFS problem with running cost and switching cost is considered, assuming no state, continuous input, and switching set constraints. The subsystem dynamics and the cost functional are discretized to obtain a discrete-time problem. The cost functional becomes

$$J\big(i(0), \cdots, i(N-1), u(0), \cdots, u(N-1)\big) = \sum_{j=0}^{N} \Big(L\big(x(j), u(j)\big) + \delta(j)\Big)$$

where the switching cost δ is 0 if no switching occurs during the time interval $[t(j), t(j+1))$. A discrete-time version of the two stage optimization then is

$$J_{opt} = \inf_{i(0),\cdots,i(N-1)\in I} \; \inf_{u(0),\cdots,u(N-1)\in\mathbb{R}^m} J\big(i(0), \cdots, i(N-1), u(0), \cdots, u(N-1)\big). \quad (5)$$

It can be seen that (5) can be solved in two stages. For a given discrete sequence of active subsystems $\big(i(0), \cdots, i(N-1)\big)$ (denoted as i in the following), denote

$$J_1(i, x(0)) = \inf_{u(0),\cdots,u(N-1)\in\mathbb{R}^m} J\big(i(0), \cdots, i(N-1), u(0), \cdots, u(N-1)\big) \quad (6)$$

subject to the discretized system equation. (6) is a classic discrete-time optimal control problem. The optimal hybrid control is achieved by

$$J_{opt} = \inf_i J_1\big(i, x(0)\big). \quad (7)$$

One disadvantage of solving (7) to find the optimal i is that it is enumerative.

Another approach based on discretization is reported in [11, 12], in which free-final-time EFS problems with running and switching costs are studied. Instead of directly discretizing the optimal control problem, the authors first introduces a set of piecewise C^1 functions V_i's and forms inequalities of Bellman type. It is

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then proved that for every given initial condition (x_0, i_0), $V_{i_0}(x_0)$ gives a lower bound on the cost for optimally bringing the system from (x_0, i_0) to the given (x_f, i_f). To simplify the notation, let us assume that all $X_i = X$ and $U_i = U$. The inequalities can then be written as

$$\begin{aligned}
&0 \leq \frac{\partial V_i(x)}{\partial x} f_i(x,u) + L(x,u), \ \forall x \in X, u \in U, i \in I \\
&0 \leq V_{i_1}(x) - V_{i_2}(x) + \delta(x, i_1, i_2), \ \forall x \in \Lambda_{(i_1,i_2)}, \ i_1, i_2 \in I, \ i_1 \neq i_2 \\
&0 = V_{i_f}(x_f)
\end{aligned}$$

For the above equations, a set of value functions $V_i(x)$ is involved, with i being the index of the initial active subsystem. Note that the result of a maximization of $V_i(x)$ is always identical to the optimal cost for the corresponding initial (x, i). In order to utilize a computer to solve the above inequalities and maximize $V_{i_0}(x_0)$, a straightforward approach is to grid the state space and inequalities to be met at a set of uniformly distributed points. Such a discretization method leads to a lower bound V_i which is a good approximation of the optimal cost. A suboptimal feedback control law can then be obtained as $u(x,i)=\mathrm{argmin}_{u\in U}\{\frac{\partial V_i}{\partial x} f_i(x,u)+L(x,u)\}$ and $i(t)=\mathrm{argmin}_{(i(t^-),i)\in E_E | x\in \Lambda_{(i(t^-),t)}}\{V_i(x)$ $+\delta(x, i(t^-), i)\}$.

A closely related paper which also utilizes Bellman type inequalities is [22]. The paper focuses on piecewise linear quadratic optimal control and uses linear matrix inequalities (LMIs) to solve the maximization problems.

As can be seen from the above discussions, discretization-based methods are capable of dealing with problems with constraints. Today's efficient nonlinear optimization solvers can even provide us with solutions of global optima or solutions close to them. However, the solutions thus obtained may not be accurate enough in terms of their continuous counterparts.

5.2 Discrete-time Problems

Since discrete-time problems are closely related to discretization methods, here we briefly mention some results related to our discussions.

[14] utilizes a discrete-time DP approach for quadratic full information EFS optimal control problems for systems with stochastic linear subsystems. In the discrete-time setting, the optimal value function can be expressed as a quadratic function of the continuous state and the optimal continuous input can be expressed in state feedback form. The optimal switching sequence is found by backward iteration. The main result in [14] is a method for efficient pruning of the backward search tree to avoid combinatoric explosion. The idea of the algorithm is to make sure that pruned sequences would have resulted in a higher cost than those remaining after the pruning. Although this result is proposed in stochastic settings, it may be extended to deterministic cases.

An early result on discrete-time IFS problems is reported in [15]. In [15], an IFS optimal control problem for a discrete-time switched system is studied. The switching of the subsystems depends on the continuous state as well as the current active subsystem. For each i, it is assumed that $\Gamma_{(i,j)}$, $j \in I$ do not overlap (expect for the shared boundaries) and cover the state space $\mathbb{R}^n$ (Here

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$\Gamma_{(i,i)}$ is understood as the set where the system stays at subsystem i). An iterative algorithm using constrained differential dynamic programming, which is similar to that for IFS problems proposed in Section 4, is proposed. Assume an initial $(x(0), i(0))$ is given. An initial guess of $u(k), k = 0, \cdots, N-1$, is chosen and then the corresponding state trajectory $x(k)$ and the active subsystem sequence $i(k)$ can be computed. Now regard the sequence of active subsystems as a constraint and solve a constrained optimal control problem. Accept the result if no resultant state $x(k)$ lies on the boundary of some switching set. Otherwise, according to certain termination rules, either accept the result or perturb the active subsystem sequence constraint and repeat the constrained optimization.

One of the very nice modeling frameworks for hybrid systems is the mixed logical dynamical (MLD) systems [1], which describes discrete-time switched and hybrid systems as follows

$$x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) \tag{8}$$

$$E_2\delta(k) + E_3z(k) \leq E_1u(k) + E_4x(k) + E_5 \tag{9}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_l}$ is a vector of continuous and binary states, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_l}$ are the continuous and binary inputs, $\delta \in \{0,1\}^{r_l}$ and $z \in \mathbb{R}^{r_c}$ represent auxiliary binary and continuous variables respectively, which are introduced when transforming logic relations into mixed-integer linear inequalities.

Based on that modeling framework, [2] studies optimal control problems which require the minimization of a weighted l_1/∞-norm of the tracking error and the input trajectory over a finite horizon. The problem is motivated by the requirement of designing a stabilizing model predictive controller. The overall optimal control problem subject to constraints (8)-(9) can then be regarded as a multiparametric mixed-integer linear programming (MILP) problem. If the optimal control problem uses performance criteria based on quadratic norms, a recent result in [3] shows that the optimal control for the finite time optimal control problem is a time-varying piecewise affine state feedback control law. Such optimal control law can be computed by means of dynamic programming and multiparametric quadratic programming. Note that the benefit of using an MLD framework is that there is no need to distinguish the optimization of continuous and discrete inputs (i.e., stages 1 and 2). However, such result can only be obtained for a class of discrete-time problems with linear subsystems.

6 Computational Methods Not Based on Discretization

As mentioned in Section 5, solutions obtained by discretization-based methods may not be accurate enough in terms of their continuous counterparts. Recently, some results that do not rely on discretization have been reported. In particular, many of these results are developed for stage 1 of Algorithm 1. The ideas of these results are based on the observation that stages 1(a) and 1(b) can be solved separately and iteratively. The following conceptual algorithm describes such an iterative method and provides a formal framework for the optimization methods in the sequel.

Algorithm 2 (A Conceptual Algorithm for Stage 1 Optimization).

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(1). Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.
(2). By solving an optimal control problem (i.e., stage (a)), find $J_1(\hat{t}^j)$.
(3). Find $\frac{\partial J_1}{\partial \hat{t}}(\hat{t}^j)$ (and $\frac{\partial^2 J_1}{\partial \hat{t}^2}(\hat{t}^j)$ if second-order method is to be used).
(4). Use some feasible direction method to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$ (here $d\hat{t}^j = -\big(\frac{\partial J_1}{\partial \hat{t}}(\hat{t}^j)\big)^T$ or $d\hat{t}^j = -\big(\frac{\partial^2 J_1}{\partial \hat{t}^2}(\hat{t}^j)\big)^{-1}\big(\frac{\partial J_1}{\partial \hat{t}}(\hat{t}^j)\big)^T$ and the stepsize α^j is chosen using the Armijo's rule [4]). Set the iteration index $j = j + 1$.
(5). Repeat Steps (2), (3), (4) and (5), until a prespecified termination condition is satisfied (e.g. the norm of the projection of $\frac{\partial J_1}{\partial \hat{t}}(\hat{t}^j)$ on any feasible direction is smaller than a given small number ϵ). □

Note that following the similar two stage decomposition idea, [9, 10] also independently come up with iterative solution methods similar to Algorithm 2. It should be noted that the key difficulty of Algorithm 2 lies in the computation of $\frac{\partial J_1}{\partial \hat{t}}(\hat{t}^j)$ (and $\frac{\partial^2 J_1}{\partial \hat{t}^2}(\hat{t}^j)$) in step (3). In [9], it is proposed that the lower level problems be solved analytically first and then the result is substituted into high level to seek the optimal switching instants. However, we should point out that it is not always possible to derive analytical solutions to the lower level optimal control problems. This is evident from the fact that only few classes of conventional optimal control problems possess closed form solutions. Even for the case of linear quadratic (LQ) problems, we do not have the closed form solutions [35]. Therefore it is necessary to devise optimization methods that do not require the explicit expression of J_1 as a function of t_k's. In [32, 34, 35], the authors noticed that stage 1(b) (i.e., higher level) is a nonlinear optimization problem that can be optimized if we know the derivatives of the cost with respect to the switching instants. Instead of seeking closed form solutions to stage 1(a), we only need accurate values of the derivatives in order to carry out stage 1(b) optimization. Here we will survey two methods of computing these derivatives. In the following, we assume that $X = \mathbb{R}^n$, $U_i = \mathbb{R}^m$, $\Lambda_e = \mathbb{R}^n$, $S_f = \mathbb{R}^n$, and there is no switching cost in the problem formulation.

6.1 Method Based on Direct Differentiations of Value Functions

The first method was reported in [34] that approximates the derivatives by direct differentiations of value functions. Consider stage 1 optimization where the number of switchings is K and the order of active subsystems is $1, 2, \cdots, K, K+1$. We need to find an optimal switching instant vector $\hat{t} = (t_1, \cdots, t_K)^T$ and an optimal control input u.

Given a nominal $\hat{t}$ and a nominal u, we denote the corresponding cost as a value function (which is not necessarily optimal)

$$V^0(x(t_0), t_0, t_1, \cdots, t_K) = \psi(x(t_f)) + \int_{t_0}^{t_1} L(x,u)\, dt + \cdots + \int_{t_K}^{t_f} L(x,u)\, dt$$

where the superscript 0 is to indicate that the starting time for evaluation is t_0. Similarly, we can define the value function at the k-th switching instant as

$$V^k(x(t_k), t_k, t_{k+1}, \cdots, t_K) = \psi(x(t_f)) + \int_{t_k}^{t_{k+1}} L(x,u)\, dt + \cdots + \int_{t_K}^{t_f} L(x,u)\, dt.$$

The idea of the method is to approximate $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ by $\frac{\partial V^0}{\partial \hat{t}}$ and $\frac{\partial^2 V^0}{\partial \hat{t}^2}$, respectively.

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The explicit expressions of $\frac{\partial V^0}{\partial \hat{t}}$ and $\frac{\partial^2 V^0}{\partial \hat{t}^2}$ are derived in [34]. They are

$$\begin{aligned} V^0_{t_k} &= L^{k-} - L^{k+} + V_x^{k+}(f^{k-} - f^{k+}), \\ V^0_{t_k t_k} &= (f^{k-} - f^{k+})^T V_{xx}^{k+}(f^{k-} - f^{k+}) - (V_x^{k+} f_x^{k+} + L_x^{k+})(f^{k-} - f^{k+}) + \big(V_x^{k+}(f_x^{k-} \\ &\quad - f_x^{k+}) + L_x^{k-} - L_x^{k+}\big) f^{k-} + (V_x^{k+} f_u^{k-} + L_u^{k-})\dot{u}^{k-} - (V_x^{k+} f_u^{k+} + L_u^{k+})\dot{u}^{k+}, \\ V^0_{t_k t_l} &= \big(V_x^{l+}(f_x^{l-} - f_x^{l+}) + (f^{l-} - f^{l+})^T V_{xx}^{l+} + L_x^{l-} - L_x^{l+}\big) A(t_l, t_k)(f^{k-} - f^{k+}). \end{aligned}$$

where we write a function with a superscript $k-$ (resp. $k+$) whenever it is evaluated at t_k and the nominal values $x(t_k)$, $u(t_k-)$ (resp. t_k and the nominal values $x(t_k)$, $u(t_k+)$) (see [34] for details). Note here $A(t_l, t_k)$ is the state transition matrix for the variational time-varying equation $\dot{y}(t) = \frac{\partial f\left(x(t), u(t)\right)}{\partial x} y(t)$ for $y(t), t \in [t_k, t_l]$; here f corresponds to the active subsystem vector field at each time instant and u, x are the current nominal input and state. Also note here $\dot{u}^{k-} \triangleq \frac{du(t_k-)}{dt}$ and $\dot{u}^{k+} \triangleq \frac{du(t_k+)}{dt}$.

Furthermore, a modified and more efficient version of the method is reported in [34] for EFS problems with linear subsystems and general quadratic costs.

6.2 Method Based on Parameterization of the Switching Instants

As opposed to approximations in [34], a method is proposed in [32, 35] to obtain accurate values of the derivatives of J_1. The method is based on solving boundary value differential algebraic equations (DAEs). Such DAEs are similar to those for conventional optimal control problems, except for more equations due to the differentiations with respect to the switching instants (regarded as parameters).

To illustrate the method and make notation clear, let us concentrate on the case of two subsystems where subsystem 1 is active in the interval $t \in [t_0, t_1)$ and subsystem 2 is active in the interval $t \in [t_1, t_f]$ (t_1 is the switching instant to be determined). We also assume that $S_f = \mathbb{R}^n$ (for general S_f, we can introduce Lagrange multipliers as in Theorem 1 and develop similar method). It was shown in [32, 35] that the stage 1 problem can be transcribed into the following equivalent problem.

Problem 3 (An Equivalent Problem). For a system with dynamics

$$\frac{dx(\tau)}{d\tau} = \begin{cases} (x_{n+1} - t_0) f_1(x, u), & \text{for } \tau \in [0, 1), \\ (t_f - x_{n+1}) f_2(x, u), & \text{for } \tau \in [1, 2], \end{cases}$$

find optimal parameter x_{n+1} and $u(\tau)$, $\tau \in [0, 2]$ such that the cost functional

$$J = \psi\big(x(2)\big) + \int_0^1 (x_{n+1} - t_0) L(x, u)\, d\tau + \int_1^2 (t_f - x_{n+1}) L(x, u)\, d\tau$$

is minimized. Here t_0, t_f and $x(0) = x_0$ are given. x_{n+1} is a parameter which corresponds to the switching instant t_1. The independent time variable τ has the following relationship to the original time variable t

$$t = \begin{cases} t_0 + (x_{n+1} - t_0)\tau, & 0 \le \tau < 1, \\ x_{n+1} + (t_f - x_{n+1})(\tau - 1), & 1 \le \tau \le 2. \end{cases} \qquad \square$$

Based on the equivalent Problem 3, we now develop a method for deriving accurate numerical value of $\frac{dJ_1}{dt_1}$. First note that for Problem 3, the optimal x and u will be functions in x_{n+1}. Also note that

$$J_1(x_{n+1}) = \psi\big(x(2, x_{n+1})\big) + \int_0^1 (x_{n+1} - t_0) L(x, u)\, d\tau + \int_1^2 (t_f - x_{n+1}) L(x, u)\, d\tau.$$

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Differentiating J_1 with respect to x_{n+1} provides us with

$$\frac{dJ_1}{dx_{n+1}} = \frac{\partial\psi\big(x(2,x_{n+1})\big)}{\partial x}\frac{\partial x(2,x_{n+1})}{\partial x_{n+1}} + \int_0^1 \big(L(x,u) + (x_{n+1} - t_0)(\tfrac{\partial L}{\partial x}\tfrac{\partial x}{\partial x_{n+1}} + \tfrac{\partial L}{\partial u}\tfrac{\partial u}{\partial x_{n+1}})\big)\,d\tau + \int_1^2 \big(-L(x,u) + (t_f - x_{n+1})(\tfrac{\partial L}{\partial x}\tfrac{\partial x}{\partial x_{n+1}} + \tfrac{\partial L}{\partial u}\tfrac{\partial u}{\partial x_{n+1}})\big)\,d\tau.$$

Therefore, what we need to know are the values of x, u, $\frac{\partial x}{\partial x_{n+1}}$, $\frac{\partial u}{\partial x_{n+1}}$. In [32,35], it was shown that their values for $\tau \in [0,2]$ can be obtained by solving a two point boundary value DAE formed by the parameterized state, costate, stationarity equations (in order to make exposition brief, we do not list them here but refer the readers to Theorem 1), the boundary and continuity conditions

$$x(0,x_{n+1}) = x_0,$$
$$p(2,x_{n+1}) = \Big(\frac{\partial\psi}{\partial x}(x(2,x_{n+1}))\Big)^T.$$

for Problem 3 and their derivatives with respect to the parameter x_{n+1}.

For each given x_{n+1}, the DAE can be solved using numerical methods. Assume we have solved the above DAE and obtained the optimal $x(\tau,x_{n+1})$, $p(\tau,x_{n+1})$ and $u(\tau,x_{n+1})$, we can then obtain the value of the derivative $\frac{dJ_1}{dx_{n+1}}$. Using this method, we can address general problems involving nonlinear systems and cost functionals. This method is also applicable to the case of more than one switchings and to the computation of higher order derivatives. Moreover, for EFS problems with linear subsystems and general quadratic costs, the burden of solving DAE can be relieved and one only needs to solve a set of ODEs formed by the Riccati equations and their differentiations with respect to the switching instants. A preliminary result of applying this method to IFS problem based on Method 1 discussed at the end of Section 4.1 is also reported in [31,35].

6.3 Results for Switched Autonomous Systems

When each subsystem is autonomous (i.e., with no continuous input), stage 1 problem becomes a nonlinear optimization problem and J becomes a function of the switching instants. The method in Section 6.1 can still be applied to similar stage 1 problems where a prespecified sequence of active subsystems is given. Because u is absent, it is shown in [33] that accurate values of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ can be obtained. The result has been extended to hybrid autonomous systems with state jumps [36]. Closely related papers are [7,8] that present closed-loop solutions to a special class of problems, i.e., infinite horizon problems for switched linear autonomous systems. However, there are some differences between the results in [7,8] and [33,36]. First, [33,36] deal with finite horizon problems with nonlinear subsystems, and with costs which are not necessarily quadratic, while [7,8] deal with infinite horizon problems with linear subsystems and quadratic costs. Moreover, the result in [33,36] can be applied to reachability problems, while the result in [7,8] fits better for stability problems.

7 Conclusion

In this paper, we have surveyed some recent results on computational methods for optimal control of switched systems. In particular, we formally formulate

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the problems and discussed in detail the two stage optimization. Computational methods based on discretization and computational methods not based on discretization are surveyed. We have aimed at providing an overview of general results and ideas. For technical details, the reader may consult the reference listed below. Despite the results reported here, it can be seen that the subject is still largely open and we are far from total understanding and complete solution of such problems. Each result we presented usually imposes several additional assumptions so that a specific method can be developed. There are many future research directions which we can pursue. For example, even stage 1 optimization for problems with state and input constraints has not been solved yet. By solving this problem, applications to real-world processes can be greatly expanded. From the paper, it can also be seen that very few results are available for efficient optimization of the number and the order of active subsystems. This can be a very challenging problem that our continuous methods may no longer be capable of handling. Moreover, extensions of the surveyed methods to general hybrid systems can also provide us with many new results. In all, the subject of computational methods for optimal control problems of switched and hybrid systems is an exciting open area that deserves more attention and can stimulate the development of hybrid system theory and application.

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