Proceedings of the 42nd IEEE
Conference on Decision and Control
Maui, Hawaii USA, December 2003

Disturbance Attenuation Properties for Discrete-Time Uncertain Switched Linear Systems

Hai Lin
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, USA

Panos J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, USA

***Abstract*— In this paper, we study a class of discrete-time switched linear systems affected by both parameter variations and additive l^∞ bounded disturbances. The problem of determining upper bounds on the l^∞ norm of the output is investigated. First, we study the condition under which the switched systems have finite l^∞ induced gain. Secondly, the l^∞ induced gain is calculated for such switched system under arbitrary switching signals. Thirdly, we answer the question whether there exists proper switching mechanism for such switched system to achieve a given disturbance attenuation level. The techniques are based on positive invariant set theory.**

I. INTRODUCTION

Recently, there has been increasing interest in the stability analysis and switching control design of switched systems (see, e.g., [9], [7], [2], [8], [12] and the references cited therein). However, the literature on the robust control of hybrid/switched systems is relatively sparse. There are some related works in the literature on analyzing the induced gain in switched systems. In [11], the $\mathcal{L}_2$ gain of continuous-time switched linear systems was studied by an average dwell time approach incorporated with a piecewise quadratic Lyapunov function, and the results were extended to discrete-time case in [12]. In [8], the root-mean-square (RMS) gain of a continuous-time switched linear system was computed in terms of the solution to a differential Riccati equation when the interval between consecutive switchings was large. However, these robust performance problems considered are both in the signal's energy sense, and assume that the disturbances are constrained to have finite energy, i.e. bounded $\mathcal{L}_2$ norm. In practice, there are disturbances that do not satisfy this condition and act more or less continuously over time. Such disturbances are called persistent [6], and can not be treated in the above framework. Therefore, in this paper we consider l^∞ induced gain to deal with the robust performance problems in the signal's magnitude sense, i.e. time domain specifications. Moreover, [8], [11], [12] did not explicitly consider dynamic uncertainty in the model. Dynamics uncertainty in the plant model is one of the main challenges in control theory, and it is of practical importance to deal with systems with uncertain parameters.

In our recent work, the l^∞ disturbance attenuation properties of a class of uncertain hybrid systems were investigated. However, the termination of the procedures, which were developed for general hybrid systems, in finite steps was not guaranteed. This is mainly because of the fact that the reachability problem is undecidable for general hybrid systems [1]. Hence, an important question is to specify the decidable class for the robust performance analysis problem. Two kinds of simplifications may be employed to make the procedures decidable. One way to obtain such decidable class is to simplify the continuous variable dynamics, see for example [1]. However, this approach may not be attractive to control applications, where simple continuous variable dynamics may not be adequate to capture the system's dynamics. Alternatively, one may restrict the discrete event dynamics of the uncertain linear hybrid systems. In this paper, we will follow the second route and specify a subclass of uncertain hybrid systems, namely uncertain switched linear systems, for which the determination of an nonconservative upper bounds on the l^∞ induced gain can terminate in finite number of steps.

This paper is organized as follows. In Section II, a mathematical model for the discrete-time uncertain switched linear system affected by both parameter variations and persistent disturbances is described, and the robust disturbance attenuation performance problems are formulated. Three problems are investigated here. First, we study the condition under witch the switched system has a finite l^∞ induced gain from the disturbance to the controlled output. Secondly, non-conservative upper bounds of the l^∞ induced gain is determined for such switched system under arbitrary switching signals in Section III. Thirdly, Section IV studies whether there exists proper switching mechanism for such switched system to achieve a given disturbance attenuation level. Finally, concluding remarks are presented.

The letters $\mathcal{E}, \mathcal{P}, \mathcal{S} \cdots$ denote sets, $\partial\mathcal{P}$ the boundary of set $\mathcal{P}$, and $int\{\mathcal{P}\}$ its interior. A polyhedral set $\mathcal{P}$ will be presented either by a set of linear inequalities $\mathcal{P} = \{x : F_i x \leq g_i,\ i = 1, \cdots, s\}$, and compactly by $\mathcal{P} = \{x : Fx \leq g\}$, or by the dual representation in terms of its vertex set $\{x_j\}$, denoted by $vert\{\mathcal{P}\}$. For $x \in \mathbb{R}^n$, the l^1 and l^∞ norms are defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_i |x_i|$ respectively. l^∞ denotes the space of bounded vector sequences $h = \{h(k) \in \mathbb{R}^n\}$ equipped with the norm $\|h\|_{l^\infty} = \sup_i \|h_i(k)\|_\infty < \infty$, where $\|h_i(k)\|_\infty = \sup_{k\geq 0} |h_i(k)|$. Finally, we denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$ while $dist(x, \mathcal{P})$ denotes the distance of a point x from a set $\mathcal{P}$, defined as $dist(x, \mathcal{P}) = \inf_{y\in\mathcal{P}} \|x - y\|$.

II. PROBLEM FORMULATION

In this paper, we consider a family of discrete-time uncertain linear systems described by the following difference equations.

$$x(t+1) = A_q(w)x(t) + Ed(t), \quad t \in \mathbb{Z}^+ \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, and the disturbance input $d(t)$ is contained in $\mathcal{D} \subset \mathbb{R}^r$, the l^∞ unit ball, i.e. $\mathcal{D} = \{d : \|d\|_{l^\infty} \leq 1\}$. $A_q(w) \in \mathbb{R}^{n\times n}$ and $E \in \mathbb{R}^{n\times r}$ are state matrices indexed by $q \in Q$, where the finite set $Q = \{q_1, q_2, \cdots, q_n\}$ is called the set of *modes*.

Combine the family of discrete-time uncertain linear systems (1) with a class of piecewise constant functions of time $\sigma : \mathbb{Z}^+ \to Q$. Then we can define the following time-varying system as a discrete-time switched linear system

$$x(t+1) = A_{\sigma(t)}(w)x(t) + Ed(t), \quad t \in \mathbb{Z}^+ \tag{2}$$

The signal $\sigma(t)$ is called a *switching signal*. Let us denote the collection of all possible switching signals as Σ_a, the set of *arbitrary switching signals* [9].

Associated with the switched system (2), a controlled output $z(t)$ is considered.

$$z(t) = C(w)x(t) \tag{3}$$

where $C(w) \in \mathbb{R}^{p\times n}$ and $z(t) \in \mathbb{R}^p$.

Assumption 1. The entries of $A_\sigma(w)$ and $C(w)$ are continuous function of $w \in \mathcal{W}$, where $\mathcal{W} \subset \mathbb{R}^v$ is an assigned compact set.

Assumption 2. For all $q \in Q$ there exists $w_q \in \mathcal{W}$ such that the triplet $[A_q(w_q), E, C(w_q)]$ is reachable and observable.

For this switched system (2)-(3), and for a given class of switching sequences $\Sigma \subseteq \Sigma_a$, we are interested in determining a non-conservative bound for the l^∞ induced norm from $d(t)$ to $z(t)$, which is defined as

$$\mu_{inf}^{(\Sigma)} = \inf\{\mu : \|z\|_{l^\infty} \leq \mu, \quad \forall \sigma \in \Sigma, w \in \mathcal{W}, d \in \mathcal{D}\}.$$

The first problem we consider in this paper is to specify the condition under which such $\mu_{inf}^{(\Sigma)}$ is finite for the switched system (2)-(3) under the given class of switching sequences Σ.

Problem 1: Consider a class of switching signals Σ, determine the condition such that the l^∞ induced norm from $d(t)$ to $z(t)$ for the switched system (2)-(3) is finite.

It can be shown [4] that Problem 1 has finite solution $\mu_{inf}^{(\Sigma)}$ if and only if the discrete-time autonomous switched system $x(t+1) = A_\sigma(w)x(t)$ is asymptotically stable under the class of switching signals Σ. Therefore, Problem 1 is transformed into a stability analysis problem for autonomous switched system under some specified class of switching signals (maybe arbitrary switching), which has been studied (for the deterministic dynamics) in the literature over decades [9], [7], [10]. In the sequel, we limit our attention to asymptotically stable switched systems and have the following assumption.

Assumption 3. For all $\sigma \in \Sigma \subseteq \Sigma_a$, $x(t+1) = A_\sigma(w)x(t)$ is asymptotically stable.

From the definition of $\mu_{inf}^{(\Sigma)}$, we can derive the following relationship

$$\Sigma_1 \subseteq \Sigma_2 \Rightarrow \mu_{inf}^{(\Sigma_1)} \leq \mu_{inf}^{(\Sigma_2)}$$

Thus $\mu_{inf}^{(\Sigma)} \leq \mu_{inf}^{(\Sigma_a)}$ because for all switching signals Σ is contained in Σ_a. Let us denote $\mu_{inf}^{(\Sigma_a)}$ simply as μ_{inf}. An interesting case for Problem 1 is when $\Sigma = \Sigma_a$, namely arbitrarily switching signals. First, a necessary condition for Problem 1 under this case is that every subsystem (1)-(3) has finite $\mu_{inf}^q < +\infty$, or, equivalently, every autonomous subsystem $x(t+1) = A_q(w)x(t)$ is asymptotically stable. For this case, a necessary and sufficient condition for Problem 1 to have finite solution μ_{inf} is that the discrete-time autonomous switched system $x(t+1) = A_\sigma(w)x(t)$ is asymptotically stable under arbitrarily switching. How to check the asymptotical stability of the autonomous switched system $x(t+1) = A_\sigma(w)x(t)$ under arbitrary switching is an important problem. Unfortunately, a satisfactory answer for it is still lacking, see for example [9], [7] and the references therein.

After the above brief discussion on the condition for $\mu_{inf} < +\infty$, we will now calculate a non-conservative bound of μ_{inf} for the switched system (2)-(3) under arbitrary switching sequences $\sigma \in \Sigma_a$. This leads to the next problem studied in this paper.

Problem 2: Determine a non-conservative bound of the l^∞ induced norm from $d(t)$ to $z(t)$ for the switched system (2)-(3) under arbitrary switching signals $\sigma \in \Sigma_a$.

It is easy to conclude that $\mu_{inf} \geq \max_{q\in Q}\{\mu_{inf}^q\}$, which is because of the fact that $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \mu_{inf}^{(\Sigma_1)} \leq \mu_{inf}^{(\Sigma_2)}$ and $\Sigma = \{\sigma(t) = q, \ t \geq 0\}$ is contained in Σ_a. In other words, the disturbance attenuation performance level of switched system (2)-(3) under arbitrary switching signals $\sigma \in \Sigma_a$ is worse than its subsystems'. This deterioration comes from the careless improper switching between the subsystems.

A motivation for the study of switched systems is that a multi-modal controller can achieve better performance level than a single-modal controller does. Therefore, it is natural to ask whether the switched system (2)-(3) can achieve better disturbance attenuation performance level than its subsystems do, namely achieve a $\mu \leq \min_{q\in Q}\{\mu_{inf}^q\}$ by carefully designing switching signals. If it is possible to achieve a better disturbance attenuation level $\mu \leq \min_{q\in Q}\{\mu_{inf}^q\}$, then the next question is what is the lowest (optimal) disturbance attenuation level that can be achieved by designing the switching mechanism for the switched system (2)-(3). This is in fact a control synthesis problem, which gives a bound of the optimal l^1 norm which can be achieved by designing the switching mechanism. Let us denote the value of the optimal l^1 norm as μ_{inf}^*. This is the last problem we addressed in this paper, which can be formulated as follows:

Problem 3: Find a non-conservative bound of the optimal l^1 norm which can be achieved by designing the switching mechanism for the switched system (2)-(3).

We will mainly focus on Problem 2 in Section III, and solve Problem 3 in Section IV. The techniques for analyzing these disturbance attenuation problems are based on positive invariant set theory [5].

III. Performance Level Under Arbitrary Switching

In this section, we will focus on the disturbance attenuation performance that the switched system (2)-(3) can preserve under arbitrary switchings. We will calculate a non-conservative bound of μ_{inf} for the switched system (2)-(3) under arbitrary switching signals $\sigma \in \Sigma_a$.

For such purpose, we first introduce the definition of *positive disturbance invariant set* for the switched system (2) under a class of switching signals $\sigma \in \Sigma \subseteq \Sigma_a$.

Definition 1: Consider a class of switching signals $\sigma \in \Sigma \subseteq \Sigma_a$ for the switched system (2). A set $\mathcal{S}$ in the state space is said to be *positive disturbance invariant* for this switched system with the switching signal class Σ if for every initial condition $x(0) \in \mathcal{S}$ we have that $x(t) \in \mathcal{S}$, $t \geq 0$, for every possible switching signal $\sigma(t) \in \Sigma$, every admissible disturbance $d(t) \in \mathcal{D}$ and parameter variation $w(t) \in \mathcal{W}$.

From this definition, we can derive the following relationship for the positive disturbance invariance with respect to different classes of switching signals.

Proposition 1: If a set $\mathcal{S}$ is positive disturbance invariant for the switched system (2) with a switching signal class Σ, then $\mathcal{S}$ remains its positive disturbance invariance for all $\Sigma' \subseteq \Sigma$.

We then formalize the definition of limit set.

Definition 2: The limit set $\mathcal{L}^{(\Sigma)}$ for the switched system (2) with a switching signal class Σ is the set of states x for which there exist a switching sequence $\sigma(t) \in \Sigma$, admissible sequence $w(t)$ and $d(t)$ and a non-decreasing time sequence t_k such that

$$\lim_{k \to +\infty} \Phi(0, t_k, \sigma(\cdot), w(\cdot), d(\cdot)) = x$$

where $\lim_{k \to +\infty} t_k = +\infty$ and $\Phi(0, t_k, \sigma(\cdot), w(\cdot), d(\cdot))$ denotes the value at the instant t_k of the solution of (2) originating at $x_0 = 0$ and corresponding to σ, w and d.

The limit set $\mathcal{L}^{(\Sigma)}$ for the switched system (2) with a switching signal class Σ has the following property.

Lemma 1: Under Assumption 3, the limit set $\mathcal{L}^{(\Sigma)}$ is non-empty and the state evolution of the switched system (2), for every initial condition $x(0)$ and admissible sequence $\sigma(t) \in \Sigma$, $w(t) \in \mathcal{W}$ and $d(t) \in \mathcal{D}$, converges to $\mathcal{L}^{(\Sigma)}$. Moreover, $\mathcal{L}^{(\Sigma)}$ is bounded and it is positive disturbance invariant for the switched system (2) with respect to the switching signal class Σ.

The boundedness and convergence of the limit set come from the asymptotic stability of the switched systems under the switching signal class Σ (Assumption 3). The invariance can be easily shown by contradiction. The detailed proof is omitted here for space limit [1].

In this section, we will focus on arbitrary switching class Σ_a. Define the performance level μ set as

$$X_0(\mu) = \{x : \|C(w)x\|_\infty \leq \mu\} \tag{4}$$

A value $\mu < +\infty$ is said to be admissible for arbitrary switching signals Σ_a if $\mu > \mu_{inf}$. Clearly, given $\mu > 0$, the response of the switched system satisfies $\|z(t)\|_{l^\infty} \leq \mu$ for all $\sigma(t) \in \Sigma_a$, $w(t) \in \mathcal{W}$ and $\|d(t)\|_{l^\infty} \leq 1$ if and only if the switched system (2) admits a positive disturbance invariant set $\mathcal{P}$ under arbitrary switching such that $0 \in \mathcal{P} \subseteq X_0(\mu)$.

In the following, we will provide a procedure to compute such a positive disturbance invariant set, for arbitrary switching signals Σ_a, containing in $X_0(\mu)$. This is accomplished to find the maximal positive disturbance invariant set for the switched system (2) under arbitrary switching, i.e. a set contains any other positive disturbance invariant set under arbitrary switching in $X_0(\mu)$.

For such purpose, we first give the predecessor operator for the q-th subsystem [4]. Given a compact set $\mathcal{S} \subseteq \mathbb{R}^n$, we can define its predecessor set for the q-th subsystem, $pre_q(\mathcal{S})$, as the set of all states x that are mapped into $\mathcal{S}$ by the transformation $A_q(w)x + Ed$, for all admissible $d \in \mathcal{D}$ and $w \in \mathcal{W}$. If $\mathcal{S}$ is a polyhedral with matrix representation of the form $S = \{x : Fx \leq g\}$. Then $pre_q(\mathcal{S})$ can be represented by

$$pre_q(\mathcal{S}) = \{x : F(A_q(w)x + Ed) \leq g, \ \forall d \in \mathcal{D}, w \in \mathcal{W}\}$$

In practice uncertainties in the system model often enter linearly and they are linearly constrained [5]. We will focus on the linear constrained case and consider the class of polyhedral sets. Their main advantage is that they are suitable for computation. In the sequel, we assume polytopic uncertainty, i.e. $A_q(w) = \sum_{k=1}^r w_k A_q^k$, $w_k \geq 0$, $\sum_{k=1}^r w_k = 1$, which provides a classical description of model uncertainty. Notice that the coefficients w_k are unknown and possibly time varying. Then, the above predecessor set for the q-th subsystem, $pre_q(\mathcal{S})$, can be written as

$$F_i \sum_{k=1}^r w_k A_q^k v_j \leq g_i - \delta_i, \quad \forall v_j \in vert\{\mathcal{S}\}, \tag{5}$$

where $\delta_i = \max_{d \in \mathcal{D}}(F_i E d)$ and w_k goes through all possible convex combination coefficients. Because of linearity and convexity, it is equivalent to only considering the vertices of $A_q(w)$, i.e.

$$F_i A_q^k v_j \leq g_i - \delta_i \quad \forall v_j \in vert\{\mathcal{S}\}, \tag{6}$$

for all $i = 1, \cdots, s$ and $k = 1, \cdots, r$. For brevity, we write

$$F A_q^k v_j \leq g - \delta, \quad \forall v_j \in vert\{\mathcal{S}\}, \forall k = 1, \cdots, r \tag{7}$$

[1]Similar concepts and lemma were previously given in [4] for uncertain time-varying linear systems. The results developed here are extensions to the uncertain switched systems.

where δ has components as δ_i.

The predecessor set for the switched system (2) under arbitrary switching, $\underline{pre}(\mathcal{S})$, is defined as the set of all states x that are mapped into $\mathcal{S}$ by the transformation $A_q(w)x+Ed$, for all possible $q \in Q$, all admissible $d \in \mathcal{D}$ and $w \in \mathcal{W}$. Therefore, the predecessor set for the switched system (2) under arbitrary switching can be calculated as

$$\underline{pre}(\mathcal{S}) = \bigcap_{q \in Q} pre_q(\mathcal{S}) \tag{8}$$

As discussed above, a given scalar $\mu > 0$ is admissible for the switched system under arbitrary switching if and only if the switched system (2) admits a positive disturbance invariant set $\mathcal{P}$ under arbitrary switching such that $0 \in \mathcal{P} \subseteq X_0(\mu)$. For case of polytopic uncertain $C(w) = \sum_{i=1}^{N} w_i C_i$, where $0 \leq w_i \leq 1$ and $\sum_{i=1}^{N} w_i = 1$, then

$$\begin{aligned} X_0(\mu) &= \{x : \|C(w)x\|_\infty \leq \mu\} \\ &= \{x : \|\sum_{i=1}^{N} w_i C_i x\|_\infty \leq \mu\} \\ &= \bigcap_{i=1}^{N} \{x : \|C_i x\|_\infty \leq \mu\} \\ &= \bigcap_{i=1}^{N} \{x : \begin{bmatrix} C_i \\ -C_i \end{bmatrix} x \leq \begin{bmatrix} \bar{\mu} \\ \bar{\mu} \end{bmatrix}\} \end{aligned}$$

where $\bar{\mu}$ stands for a column vector with μ as its elements. $X_0(\mu)$ is finite intersection of polytopes containing the origin in their interior. Therefore, $X_0(\mu)$ is a polytope containing the origin in its interior.

To determine a positive disturbance invariant set $\mathcal{P}$ in $X_0(\mu)$ for the switched systems under arbitrary switching, we recursively define the sets $P^{(k)}$, $k = 0, 1, \cdots$ as

$$P^{(0)} = X_0(\mu), \; P^{(k)} = P^{(k-1)} \bigcap \underline{pre}(P^{(k-1)}) \tag{9}$$

it can be shown [3] that $P^{(\infty)}$ is the maximal positive disturbance invariant set under arbitrary switching in $X_0(\mu)$. With these assumptions and notations, we may adopt the techniques and results developed in [4] to the switched systems under arbitrary switching and get the following lemma. The proof of the lemma is not difficult by using the technique of Theorem 3.1 in [4], and thus is omitted here.

Proposition 2: Under Assumption 3, if $\mathcal{L}^{(\Sigma_a)} \subset int\{X_0(\mu)\}$ for some $\mu > 0$, then there exists $\mathbf{k}$ such that $P^{(\infty)} = P^{(\mathbf{k})}$ and this $\mathbf{k}$ can be selected as the smallest integer such that $P^{(\mathbf{k+1})} = P^{(\mathbf{k})}$.

In order to check whether a given performance level $\mu > 0$ is admissible for the switched system under arbitrary switching, one may compute the maximal positive disturbance invariant set $P^{(\infty)}$ in $X_0(\mu)$ and check whether or not $P^{(\infty)}$ contains the origin. If yes, then $\mu > \mu_{inf}$, otherwise $\mu < \mu_{inf}$. Note that in both cases we get an answer in a finite number of steps. In the first case, this is due to the above proposition. In the second case, this comes from the fact that the sequence of closed sets $P^{(k)}$ is ordered by inclusion and $P^{(\infty)}$ is their intersection. Thus $0 \notin P^{(\infty)}$ if and only if $0 \notin P^{(k)}$ for some k. Thus checking whether $\mu > \mu_{inf}$ can be obtained by starting from the initial set $X_0(\mu)$ and computing the sequence of sets $P^{(k)}$ until some appropriate stopping criterion is met. In addition, we have another stop criterion.

Proposition 3: If the set $P^{(k)} \subset int\{X_0(\mu)\}$ for some k, then the switched system (2) does not admit a positive disturbance invariant set under arbitrary switching in $X_0(\mu)$. In other words, $\mu < \mu_{inf}$.

Proof : Suppose that there exists k such that $P^{(k)} \subset int\{X_0(\mu)\}$ and the switched system (2) admits a positive disturbance invariant set in $X_0(\mu)$ under arbitrary switching, and hence $P^{(\infty)} \subset int\{X_0(\mu)\}$. Define ν as $\nu = \inf_{x \notin X_0(\mu)} dist(x, P^{(\infty)})$. For every initial condition $x_0 \notin P^{(\infty)}$ there exist sequence $\hat{\sigma}$, $\hat{w}$ and $\hat{d}$ such that the corresponding trajectory escapes from $X_0(\mu)$, i.e. $x(\bar{k}) \notin X_0(\mu)$ for some $\bar{k}$. Let $\hat{x}(t)$ and $x(t)$ denote two system trajectories, corresponding to the same sequences $\hat{\sigma}$, $\hat{w}$ and $\hat{d}$ but with different initial conditions. The updating equation for the difference $e(t) = \hat{x}(t) - x(t)$ is

$$e(t+1) = A_\sigma(\hat{w}(t))e(t) \tag{10}$$

which is stable by Assumption 3. Thus for arbitrary $0 < \varepsilon < \nu$ there exists $\delta > 0$ such that, for $\|\hat{x}(0) - x(0)\| < \delta$, we have $\|e(t)\| = \|\hat{x}(t) - x(t)\| < \varepsilon$ for $t \geq 0$. On the other hand, we may choose $\hat{x}(0) \in P^{(\infty)}$ and $x(0) \notin P^{(\infty)}$ such that $\|\hat{x}(0) - x(0)\| < \delta$. Now we have $\hat{x}(\bar{k}) \in P^{(\infty)}$ and $x(\bar{k}) \notin X_0(\mu)$. This implies that $\|e(\bar{k})\| \geq \nu$ and leads to a contradiction. □

These results suggest the following constructive procedure for finding a robust performance bound.

Procedure 1. Checking whether $\mu > \mu_{inf}$

1) Initialization: Set $k = 1$ and set $P^{(0)} = X_0(\mu)$.
2) Compute the set $P^{(k)} = P^{(k-1)} \bigcap \underline{pre}(P^{(k-1)})$.
3) If $0 \notin P^{(k+1)}$ or $P^{(k)} \subset int\{X_0(\mu)\}$ then stop, the procedure has failed. thus, the output does not robustly meet the performance level μ.
4) If the $P^{(k+1)} = P^{(k)}$, then stop, and set $P^{(\infty)} = P^{(k)}$.
5) Set $k = k + 1$ and go to step 1.

This procedure can then be used together with a bisection method on μ to approximate arbitrary close the optimal value μ_{inf}, which solves the Problem 2. In fact, if the procedure stops at step 3. we conclude that $\mu < \mu_{inf}$ and we can increase the value of the output bound μ. Else, if the procedure stops at step 4, we have determined an admissible bound for the output, say $\mu > \mu_{inf}$, that can be decreased.

IV. Improve Performance By Proper Switching

In the previous section, we determined a bound of the l^∞ induced gain μ_{inf} from $d(t)$ to $z(t)$ for the switched system (2)-(3) under arbitrary switching. Note that $\mu_{inf} \geq \max_{q\in Q}\{\mu^q_{inf}\}$, where μ^q_{inf} is the l^∞ induced norm from $d(t)$ to $z(t)$ for the q-th sub-system (1)-(3). In other words, the disturbance attenuation performance level of the switched system (2)-(3) under arbitrary switching signals $\sigma \in \Sigma_a$ is worse than its subsystems'. As we discussed in Section II, it is possible for the switched system (2)-(3) to achieve a better disturbance attenuation performance level than its subsystems', that is $\mu \leq \min_{q\in Q}\{\mu^q_{inf}\}$, by designing proper switching mechanism. The question left unclear is what is the lowest performance level μ that can be achieved by properly switching. Let us denoted such value as μ^*_{inf}. The existence of μ^*_{inf} is immediate from the fact that $0 < \mu^*_{inf} \leq \min_{q\in Q}\{\mu^q_{inf}\} < +\infty$.

In this section, we will determine a non-conservative bound for the optimal l^1 norm μ^*_{inf} which can be achieved by designing the switching mechanism for the switched system (2)-(3), namely Problem 3 described in Section II. This is basically a control synthesis problem, which gives a bound of the optimal l^1 norm which can be achieved by designing the switching mechanism for the switched system (2)-(3).

First we define the limit set for the switched system (2) with all stable switching signals as

$$\mathcal{L} = \bigcap_{\sigma\in\Sigma_s} \mathcal{L}^{(\{\sigma\})},$$

where switching signal class $\Sigma_s \subseteq \Sigma_a$ is the collection of all switching signals that the autonomous switched system $x(t+1) = A_\sigma(w)x(t)$ is asymptotically stable for all $\sigma \in \Sigma_s$. It can be shown that $\mathcal{L}$ has the property as follows.

Proposition 4: The set $\mathcal{L}$ is bounded and non-empty. For every initial condition $x(0)$, admissible $w(t) \in \mathcal{W}$ and $d(t) \in \mathcal{D}$, there exists an admissible switching sequence $\sigma(t) \in \Sigma_s$ such that the state evolution of the switched system (2) converges to $\mathcal{L}$. In addition, for all the states contained in $\mathcal{L}$, then there exists proper switching signal such that the state evolution of the switched system (2) remains in $\mathcal{L}$, despite uncertainty and disturbance.

Proof : The boundedness and non-emptiness of $\mathcal{L}$ comes from the fact that $\mathcal{L}^{(\{\sigma\})}$ is nonempty and bounded for all $\sigma \in \Sigma_s$, and $0 \in \mathcal{L}^{(\{\sigma\})}$. For any $x(0)$, there exists proper switching signal $\sigma_1 \in \Sigma_s$ and finite t_1 such that $x(t) \in \mathcal{L}^{(\{\sigma_1\})}$ for $t \geq t_1$ (from the definition of limit set $\mathcal{L}^{(\{\sigma_1\})}$). If $x(t_1) \notin \mathcal{L} = \bigcap_{\sigma\in\Sigma_s} \mathcal{L}^{(\{\sigma\})}$, then there exists at least one limit set, say $\mathcal{L}^{(\{\sigma_2\})}$, such that $x(t_1) \notin \mathcal{L}^{(\{\sigma_2\})}$. From the definition of $\mathcal{L}^{(\{\sigma_2\})}$, we know that there exists finite $t_2 > t_1$ such that $x(t) \in \mathcal{L}^{(\{\sigma_2\})}$ for $t \geq t_2$ under switch signal σ_2. If $x(t_2) \notin \mathcal{L}$, then the arguments goes on until finally $x(t) \in \mathcal{L}$. We claim that with finite number of steps $x(t) \in \mathcal{L}$. This claim and invariance of the set $\mathcal{L}$ can be easily shown by contradiction. □

It should be pointed out that the introduction of the limit set $\mathcal{L}^{(\Sigma)}$ and $\mathcal{L}$ is for the purpose of proving the termination of the procedures in finite number of steps. It is not necessary to calculate the limit set $\mathcal{L}^{(\Sigma)}$ or $\mathcal{L}$ to implement the procedures for the determination of induced gains.

Similar to the previous section, define the predecessor set for the switched system (2), $\overline{pre}(\mathcal{S})$, as the set of states from which there exist a subsystem (switching signal σ) driving the states to $\mathcal{S}$ in one step for all allowable disturbances and dynamic uncertainties. By definition, the predecessor set, $\overline{pre}(\mathcal{S})$, can be expressed as

$$\overline{pre}(\mathcal{S}) = \bigcup_{q\in Q} pre_q(\mathcal{S}) \tag{11}$$

Following the recursive definition of $P^{(k)}$ in the previous section, we recursively define the sets $\bar{P}^{(k)}$, $k = 0, 1, \cdots$ as

$$\bar{P}^{(0)} = X_0(\mu),\ \bar{P}^{(k)} = \bar{P}^{(k-1)} \bigcap \overline{pre}(\bar{P}^{(k-1)}) \tag{12}$$

By construction, $\bar{P}^{(\infty)}$ has the property that there exists a switching signal $\sigma(t)$ with respect to which $\bar{P}^{(\infty)}$ is positive disturbance invariant for the switched system (2), which is called controlled invariance. Also it can be shown that $\bar{P}^{(\infty)}$ is the maximal controlled invariant subset contained in $X_0(\mu)$. Then, given $\mu > 0$, there exists a switching signal $\sigma(t)$ such that the response of the switched system satisfies $\|z(t)\|_{l^\infty} \leq \mu$ for all $w(t) \in \mathcal{W}$ and $\|d(t)\|_{l^\infty} \leq 1$ if and only if the maximal controlled invariant subset contained in $X_0(\mu)$, $\bar{P}^{(\infty)}$, is nonempty and $0 \in \bar{P}^{(\infty)} \subseteq X_0(\mu)$.

We now give a proposition which guarantees that $\bar{P}^{(\infty)}$ can be finitely determined.

Proposition 5: Under Assumption 3, if $\mathcal{L} \subset int\{X_0(\mu)\}$ for some $\mu > 0$, then there exists $\mathbf{k}$ such that $\bar{P}^{(\infty)} = \bar{P}^{(\mathbf{k})}$ and this $\mathbf{k}$ can be selected as the smallest integer such that $\bar{P}^{(\mathbf{k+1})} = \bar{P}^{(\mathbf{k})}$.

Proof : According to the above proposition, there exists $\mathbf{k}$ such that for all $x(0) \in X_0(\mu)$, $x(t) \in \mathcal{L} \subset int\{X_0(\mu)\}$ $(\forall t \geq \mathbf{k})$ for some proper switching signals. By construction the set $\bar{P}^{(\mathbf{k})}$ has the property that $\exists\sigma(t) \in \Sigma_a$ such that $x(t) \in X_0(\mu)$, $t = 0, 1, \cdots, \mathbf{k}$, for all possible $d(t) \in \mathcal{D}$ and $w(t) \in \mathcal{W}$ if and only if $x(0) \in \bar{P}^{(\mathbf{k})}$. This implies that $\bar{P}^{(\mathbf{k})} = \bar{P}^{(\mathbf{k+1})}$. Otherwise, $\bar{P}^{(\mathbf{k})} \supset \bar{P}^{(\mathbf{k+1})}$, and there exists $x(0) \in \bar{P}^{(\mathbf{k})}$ but $x(0) \notin \bar{P}^{(\mathbf{k+1})}$, then for all possible $\sigma(t) \in \Sigma_a$, $\exists d(t) \in \mathcal{D}$ and $\exists w(t) \in \mathcal{W}$ such that $x(\mathbf{k}+1) \notin X_0(\mu)$. This leads to a contradiction. Therefore, $\bar{P}^{(\mathbf{k})} = \bar{P}^{(\mathbf{k+1})}$, and this implies that $\bar{P}^{(\mathbf{k})} = \bar{P}^{(\mathbf{k}+m)}$, for $m \geq 0$. Thus $\bar{P}^{(\infty)} = \bar{P}^{(\mathbf{k})}$. □

Problem 3 can now be solved by determining the maximal controlled invariant set $\bar{P}^{(\infty)}$ in $X_0(\mu)$ for several values of μ and checking whether or not it contains the origin. Note that in both cases we get an answer in a finite number of steps as we discussed in the previous section. Thus the solution to the Problem 3 can be obtained by starting from the initial set $X_0(\mu)$ and computing the sequence of sets $\bar{P}^{(k)}$ until some appropriate stopping criterion is met.

These results suggest the following bisection algorithm to approximate arbitrary close the optimal value μ^*_{inf}, which solves the Problem 3. The initial interval $[\mu_1^*, \mu_2^*]$ such that $\mu_1^* \le \mu_{inf}^* < \mu_2^*$ may be chosen as $\mu_1^* = \epsilon$ and $\mu_2^* = \min_{q\in Q}\{\mu_{inf}^q\} + \epsilon$.

Algorithm 1. Algorithm for Calculating μ_{inf}^*

Initialization: $\epsilon > 0$, $\mu_1^* = \epsilon$ and $\mu_2^* = \min_{q\in Q}\{\mu_{inf}^q\} + \epsilon$.
while $(\mu_2^* - \mu_1^*) > \epsilon$
  $\mu_3^* = \frac{\mu_1^*+\mu_2^*}{2}$;
  $k = 1$, $\bar{P}^{(0)} = X_0(\mu_3^*)$;
  while $\bar{P}^{(k)} \neq \bar{P}^{(k-1)}$
    $\bar{P}^{(k)} = \bar{P}^{(k-1)} \bigcap \overline{pre}(\bar{P}^{(k-1)})$
    if $0 \notin \bar{P}^{(k)}$
      $\mu_1^* = \mu_3^*$; **break**
    end if
    $k = k + 1$
  end while
end while
Output: $\mu_{inf}^* = \frac{\mu_1^*+\mu_2^*}{2}$.

Finally, let us illustrate the results here through an simple numerical example.

Example 1: Consider the following discrete-time uncertain linear hybrid systems:

$$\begin{aligned} x(t+1) &= \begin{cases} A_0(w)x(t) + Ed(t), & \sigma(t) = q_0 \\ A_1(w)x(t) + Ed(t), & \sigma(t) = q_1 \end{cases} \\ z(t) &= C(w)x(t) \end{aligned}$$

In this example the mode set $Q = \{q_0, q_1\}$, and the corresponding state matrices for each subsystem are given as

$$\begin{aligned} A_0(w) &= \begin{bmatrix} 0.1+w & 0.7 \\ -0.7 & 0.1+w \end{bmatrix}, \; E = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \\ A_1(w) &= \begin{bmatrix} 0.1+w & 1 \\ 0 & 0.5-w \end{bmatrix}, \\ C(w) &= \begin{bmatrix} 1 & 2+w \end{bmatrix}. \end{aligned}$$

We assume that the time varying uncertain parameter w is subjected to the constraint $-0.2 \le w \le 0.2$, and the continuous variable disturbance $d(t)$ is bounded by $d \in \mathcal{D} = \{d : \|d\|_{l^\infty} \le 1\} = \{d : -1 \le d \le 1\}$.

First, we calculate the l^1 induced gain μ_{inf}^q for each subsystem. Using the bisection method (with error tolerance $\epsilon = 0.01$), we get $\mu_{inf}^{q_0} = 0.145$ and $\mu_{inf}^{q_1} = 0.136$.

Then, we calculate a non-conservative bound of μ_{inf} for the switched system under arbitrary switching sequences $\sigma \in \Sigma_a$. It can be determined that $\mu_{inf} = 0.331$. Note $\mu_{inf} \ge \max_{q\in Q}\{\mu_{inf}^q\}$. In other words, the disturbance attenuation performance level of switched system under arbitrary switching signals $\sigma \in \Sigma_a$ is worse than its subsystems'.

Finally, we approximate the optimal l^1 norm μ_{inf}^* which can be achieved by designing the switching mechanism for the switched system. Using bisection method (with error tolerance $\epsilon = 0.01$) we compute the μ_{inf}^* is approximately $\mu_{inf}^* = 0.095$. Note $\mu_{inf}^* \le \min_{q\in Q}\{\mu_{inf}^q\}$ as expected.

V. CONCLUSIONS

In this paper, we investigated the problem of determining of upper bounds on the l^∞ norm of the output of discrete-time switched linear systems affected by both parameter variations and additive l^∞ bounded disturbances. Two cases were considered. First, the disturbance attenuation performance level μ_{inf} that could be preserved for the switched system (2)-(3) under arbitrary switching signals was studied. Secondly, we determined a non-conservative bound for the optimal l^1 norm μ_{inf}^* which could be achieved by designing the switching mechanism for the switched system (2)-(3). The techniques for analyzing the disturbance attenuation problem were based on positive invariant set theory.

VI. ACKNOWLEDGEMENTS

The partial support of the National Science Foundation (NSF ECS99-12458 & CCR01-13131) is gratefully acknowledged. The first author appreciates the support from Center of Applied Mathematics Fellowship (2003-04), University of Notre Dame.

VII. REFERENCES

[1] R. Alur, T. Henzinger, G. Lafferriere, and G. J. Pappas, "Discrete abstractions of hybrid systems," *Proccedings of the IEEE*, vol. 88, no. 7, pp. 971-984, July 2000.
[2] A. Bemporad, and M. Morari, "Control of systems integrating logic, dynamics, and constraints," *Automatica*, vol. 35, no. 3, pp. 407-427, 1999.
[3] F. Blanchini, "Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 39, no. 2, pp. 428-433, 1994.
[4] F. Blanchini, S. Miani, and M. Sznaier, "Robust performance with fixed and worst case signals for uncertain time-varying systems," *Automatica*, vol. 33, no. 12, pp. 2183-2189, 1997.
[5] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747-1767, 1999.
[6] M. A. Dahleh, and I. Diaz-Bobillo, *Control of Uncertain Systems: A Linear Programming Approach*, Prentice Hall, 1994.
[7] R. A. Decarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proccedings of the IEEE*, vol. 88, no. 7, pp. 1069-1082, 2000.
[8] J. P. Hespanha, "Computation of Root-Mean-Square gains of switched linear systems," *Hybrid Systems: Computation and Control*, HSCC 2002, pp. 239-252, 2002.
[9] D. Liberzon, and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine* vol. 19, no. 15, pp. 59-70, 1999.
[10] H. Ye, A. N. Michel, and L. Hou, "Stability theory for hybrid dynamical systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 461-474, 1998.
[11] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, "Disturbance attenuation properties of time-controlled switched systems," *Journal of the Franklin Institute*, vol. 338, pp. 765-779, 2001.
[12] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, "Qualitative analysis of discrete-time switched systems," in *Proceedings of 2002 American Control Conference*, vol. 3, pp. 1880-1885, 2002.