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Stability and persistent disturbance attenuation properties for a class of networked control systems: switched system approach

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In this paper, both the asymptotic stability and l^∞ persistent disturbance attenuation issues are investigated for a class of networked control systems (NCSs) under bounded uncertain access delay and packet dropout effects. The basic idea is to formulate such NCSs as discrete-time switched systems with arbitrary switching. Then the NCSs' stability and performance problems can be reduced to the corresponding problems of such switched systems. The asymptotic stability problem is considered first, and a necessary and sufficient condition is derived for the NCSs' asymptotic stability based on robust stability techniques. Secondly, the NCSs' l^∞ persistent disturbance attenuation properties are studied and an algorithm is introduced to calculate the l^∞ induced gain of the NCSs. The decidability issue of the algorithm is also discussed. A network controlled integrator system is used throughout the paper for illustration.

1. Introduction

By networked control systems (NCSs), we mean feedback control systems where networks, typically digital band-limited serial communication channels, are used for the connections between spatially distributed system components like sensors and actuators to controllers. These channels may be shared by other feedback control loops. In traditional feedback control systems these connections are established by point-to-point cables. Compared with point-to-point cables, the introduction of serial communication networks has several advantages, such as high system testability and resource utilization, as well as low weight, space, power and wiring requirements (Zhang *et al*. 2001, Ishii and Francis 2002). These advantages have made the networks connecting sensors/actuators to controllers increasingly popular in many applications including traffic control, satellite clusters, mobile robotics, etc. Recently, modelling, analysis and control of networked control systems with limited communication capability has emerged as a topic of significant interest to the control community, see for example Wong and Brockett (1999), Brockett and Liberzon (2000), Elia and Mitter (2001), Zhang *et al*. (2001), Ishii and Francis (2002), and recent special issue edited by Antsaklis and Baillieul (2004).

Time delay typically has negative effects on the NCSs' stability and performance. There are several situations where time delay may arise. First, transmission delay is caused by the limited bit rate of the communication channels. Secondly, the channel in NCSs is usually shared by multiple sources of data, and the channel is usually multiplexed by a time-division method. Therefore, there are delays caused by a node waiting to send out a message through a busy channel, which is usually called accessing delay and serves as the main source of delays in NCSs. There are also some delays caused by processing and propagation which are usually negligible for NCSs. Another interesting problem in NCSs is the packet dropout phenomenon. Because of the uncertainties and noise in the communication channel there may exist unavoidable errors or losses in the transmitted packet even when an error control coding and/or Automatic Repeat reQuest (ARQ) mechanisms are employed. If this happens, the corrupted packet is

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dropped and the receiver (controller or actuator) uses the packet that it received most recently. In addition, packet dropout may occur when one packet, say sampled values from the sensor, reaches the destination later than its successors. In this situation, the old packet is dropped and its successive packet is used instead. There is another important issue in NCSs, namely the quantization effect. With the finite bit-rate constraints, quantization has to be taken into consideration in NCSs. Therefore, quantization and limited bit rate issues have attracted many researchers' attention with the aim to identify the minimum bit rate required to stabilize a NCS, see for example Delchamps (1990), Brockett and Liberzon (2000), Elia and Mitter (2001), Nair *et al.* (2004), Tatikonda and Mitter (2004). In this paper we will focus on packet exchange networks, in which the minimum unit of data transmission is the packet which typically is with the size of several hundred bits. Therefore, sending a single bit or several hundred of bits does not make a significant difference in the network resource usage. Hence, we will omit the quantization effects here and focus our attention on the effects of network induced delay and packet dropout on NCSs' stability and performance.

The effects of network induced delay on the NCSs' stability have been studied in the literature. In Branicky *et al.* (2000), the delay was assumed to be constant and then the NCSs' could be transformed into a time-invariant discrete-time system. Therefore, the NCSs' stability could be checked by the Schurness of certain augmented state matrix. Since most network protocols introduce delays that can vary from packet to packet, Zhang *et al.* (2001) extended the results to non-constant delay case. They employed Lyapunov methods, in particular a common quadratic Lyapunov function, to study bounds on the maximum delay allowed by the NCSs. However, the choice of a common quadratic Lyapunov function could make the conclusion for maximum allowed delay conservative in some cases. The packet dropouts have also been studied and there are two typical ways to model packet dropouts in the literature. The first approach assumes that the packet dropouts follow certain probability distributions and describes NCSs with packet dropouts via stochastic models, such as Markovian jump linear systems. The second approach is deterministic, and specifies the dropouts in the time average sense or in terms of bounds on maximum allowed consecutive dropouts. For example, Hassibi *et al.* (1999) modelled a class of NCSs with package dropouts as asynchronous dynamical systems, and derived a sufficient condition on packet dropouts in the time-average sense for the NCSs' stability based on common Lyapunov function approach. Note that most of the results obtained so far are for the NCSs' stability problem and the delay and packet dropouts are usually dealt with separately. Here, we will consider both network induced delay and packet dropouts in a unified switched system model. In addition, the disturbance attenuation issues for NCSs are investigated as well as stability problems.

In this paper, the asymptotic stability and l^∞ persistent disturbance attenuation properties for a class of NCSs under bounded uncertain access delay and packet dropout effects are investigated. The basic idea is to formulate such NCSs as discrete-time switched systems with arbitrary switching signals. Then the NCSs' stability and performance problems can be studied in the switched system framework. The strength of this approach comes from the solid theoretic results existing in the literature of switched systems. By a switched system, we mean a hybrid dynamical system consisting of a finite number of subsystems described by differential or difference equations and a logical rule that orchestrates switching between these subsystems. Properties of this type of model have been studied for the past fifty years to consider engineering systems that contain relays and/or hysteresis. Recently, there has been increasing interest in the stability analysis and switching control design of switched systems (see, e.g., Liberzon and Morse (1999), Decarlo *et al.* (2000) and the references cited therein).

The paper is organized as follows. The assumptions on the network link layer of the NCSs are described in § 2, and the NCSs with bounded uncertain access delay and packet dropout effects are modelled as a class of discrete-time switched linear systems with arbitrary switching in § 3. The stability for such NCSs is studied in § 4, and a necessary and sufficient matrix norm condition is derived for the NCSs' global asymptotic stability. This result also improves the sufficient only conditions found in the literature of asymptotic stability for arbitrarily switching systems. The persistent disturbance attenuation properties for such NCSs are studied in § 5, and a non-conservative bound of the l^∞ induced gain for the NCS is calculated. The techniques are based on the recent progress on robust performance of switched systems (Lin and Antsaklis 2003). A networked controlled integrator with disturbances is used throughout the paper for illustration. Finally, concluding remarks are presented.

Notation: The letters $\mathcal{E}, \mathcal{P}, \mathcal{S} \ldots$ denote sets, $\partial\mathcal{P}$ the boundary of set $\mathcal{P}$, and $int\{\mathcal{P}\}$ its interior. A bounded polyhedral set $\mathcal{P}$ will be presented either by a set of linear inequalities $\mathcal{P} = \{x: \ F_i x \le g_i, \ i = 1, \ldots, s\}$, and compactly by $\mathcal{P} = \{x: \ Fx \le g\}$, or by the dual representation in terms of the convex hull of its vertex set $vert\{\mathcal{P}\} = \{x_j\}$, denoted by $Conv\{x_j\}$. For $x \in \mathbb{R}^n$, the l^1 and l^∞ norms are defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_i |x_i|$ respectively. l^∞ denotes the space of bounded vector sequences $h = \{h(k) \in \mathbb{R}^n\}$ equipped with the norm $\|h\|_{l^\infty} = \sup_i \|h_i(k)\|_\infty < \infty$, where $\|h_i(k)\|_\infty = \sup_{k\geq 0} |h_i(k)|$.

2. The access delay and packet dropout

For the network link layer we assume that the delays caused by processing and propagation are ignored, and we only consider the access delay which serves as the main source of delays in NCSs. Dependent on the data traffic, the communication bus is either busy or idle (available). If the link is available the communication between sender and receiver is assumed to be instantaneous. Errors may occur during the communication and destroy the packet and this is considered as a packet dropout.

The model of the NCS used in this paper is shown in figure 1. For simplicity, but without loss of generality, we may combine all the time delay and packet dropout effects into the sensor to controller path and assume that the controller and the actuator communicate ideally.

We assume that the plant can be modelled as a continuous-time linear time-invariant system described by

$$\begin{cases} \dot{x}(t) = A^c x(t) + B^c u(t) + E^c d(t) \\ z(t) = C^c x(t) \end{cases}, \quad t \in \mathbb{R}^+, \qquad (1)$$

where $\mathbb{R}^+$ stands for non-negative real numbers, $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is control input, and $z(t) \in \mathbb{R}^p$ is the controlled output. The disturbance input $d(t)$ is contained in $\mathcal{D} \subset \mathbb{R}^r$. $A^c \in \mathbb{R}^{n\times n}$, $B^c \in \mathbb{R}^{n\times m}$ and $E^c \in \mathbb{R}^{n\times r}$ are constant matrices related to the system state, and $C^c \in \mathbb{R}^{p\times n}$ is the output matrix.

For the above NCS, it is assumed that the plant output node (sensor) is time driven. In other words, after each clock cycle (sampling time T_s), the output node attempts to send a packet containing the most recent state (output) samples. If the communication bus is idle, then the packet will be transmitted to the controller. Otherwise, if the bus is busy, then the output node will wait for some time, say $\varpi < T_s$, and try again. After several attempts or when newer sampled data become available, if the transmission still cannot be completed, then the packet is discarded, which is also considered as a packet dropout. On the other hand, the controller and actuator are event driven and work in a simpler way. The controller, as a receiver, has a receiving buffer which contains the most recently received data packet from the sensors (the overflow of the buffer may be dealt with as packet dropouts). The controller reads the buffer periodically at a higher frequency than the sampling frequency, say every T_s/N for some integer N large enough. Whenever there are new data in the buffer the controller will calculate the new control signal and transmit it to the actuator. Upon the arrival of the new control signal, the actuator updates the output of the Zero-Order-Hold (ZOH) to the new value.

Based on the above assumptions a typical time delay and packet dropout pattern is shown in figure 2. In this

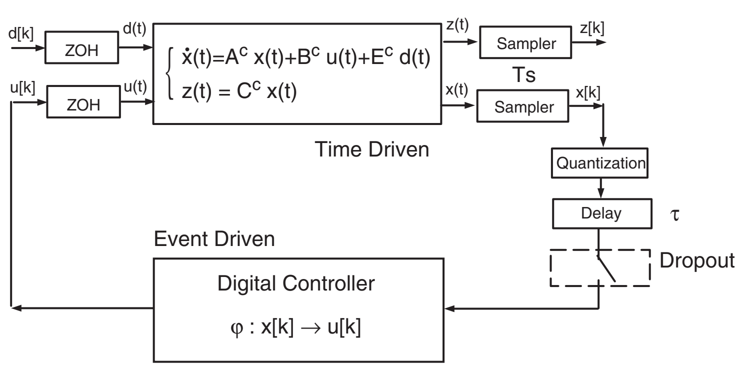

Figure 1. The networked control systems' model.

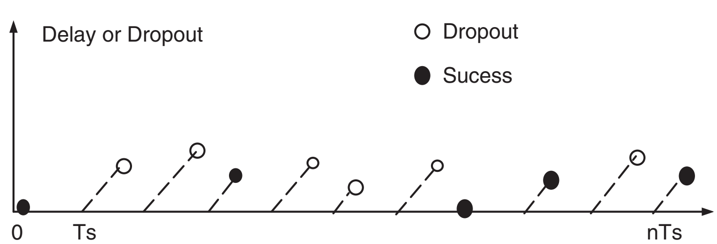

Figure 2. The illustration of uncertain time delay and packet dropout of networked control systems.

figure the small bullet • stands for the packet being transmitted successfully from the sensor to the controller's receiving buffer, maybe with some delay, and being read by the controller, at some time $t = kT_s + h(T_s/N)$ (k and h are integers). The new control signal is sent to the actuator and the actuator holds this new value until the next update control signal comes. The symbol ∘ denotes the packet being dropped due to error, bus being busy, conflict or buffer overflow etc.

3. A switched system model for NCSs

In this section, we will consider the sampled-data model of the plant. Because we do not assume time synchronization between the sampler and the digital controller, the control signal is no longer of constant value within a sampling period. Therefore the control signal within a sampling period has to be divided into subintervals corresponding to the controller's reading buffer period, $T = T_s/N$. Within each subinterval, the control signal is constant under the assumptions of the previous section. Hence the continuous-time plant may be discretized and approximated by the following sampled-data model

$$x[k+1] = Ax[k] + \underbrace{[B \quad B \quad \cdots \quad B]}_{N} \begin{bmatrix} u^1[k] \\ u^2[k] \\ \vdots \\ u^N[k] \end{bmatrix} + Ed[k] \tag{2}$$

where $A = e^{A^c T_s}$, $B = \int_0^{T_s/N} e^{A^c \eta} B^c d\eta$ and $E = \int_0^{T_s} e^{A^c \eta} E^c d\eta$. The controlled output $z[k]$ is given by

$$z[k] = Cx[k] \tag{3}$$

where $C = C^c$. Note that for a linear time-invariant plant and constant-periodic sampling, the matrices A, B, C and E are constant.

3.1. *Modelling uncertain access delay*

During each sampling period, there are several different cases that may arise.

First, if there is no delay, namely $\tau = 0$, $u^1[k] = u^2[k] = \cdots = u^N[k] = u[k]$, then the state transition equation (2) for this case can be written as

$$\begin{aligned} x[k+1] &= Ax[k] + [B \quad B \quad \cdots \quad B] \begin{bmatrix} u[k] \\ u[k] \\ \vdots \\ u[k] \end{bmatrix} + Ed[k] \\ &= Ax[k] + N \cdot Bu[k] + Ed[k]. \end{aligned}$$

Secondly, if the delay $\tau = h \times T$, where $T = T_s/N$, and $h = 1, 2, \ldots, d_{\max}$ (the value of $d_{\max}$ is determined as the least integer greater than the positive scalar $\tau_{\max}/T$, where $\tau_{\max}$ stands for the maximum access delay), then $u^1[k] = u^2[k] = \cdots = u^h[k] = u[k-1]$, $u^{h+1}[k] = u^{h+2}[k] = \cdots = u^N[k] = u[k]$, and (2) can be written as

$$\begin{aligned} x[k+1] &= Ax[k] + [B \quad B \quad \cdots \quad B] \begin{bmatrix} u[k-1] \\ \vdots \\ u[k-1] \\ u[k] \\ \vdots \\ u[k] \end{bmatrix} + Ed[k] \\ &= Ax[k] + h \cdot Bu[k-1] \\ &\quad + (N-h) \cdot Bu[k] + Ed[k]. \end{aligned}$$

Note that $h = 0$ implies $\tau = 0$, which corresponds to the previous "no delay" case.

Finally, if a packet-dropout happens, which may be due to a corrupted packet or sending it out with delay greater than $\tau_{\max}$, then the actuator will implement the previous control signal, i.e. $u^1[k] = u^2[k] = \cdots = u^N[k] = u[k-1]$. Therefore, the state transition equation (2) for this case can be written as

$$\begin{aligned} x[k+1] &= Ax[k] + [B \quad B \quad \cdots \quad B] \begin{bmatrix} u[k-1] \\ u[k-1] \\ \vdots \\ u[k-1] \end{bmatrix} + Ed[k] \\ &= Ax[k] + N \cdot Bu[k-1] + Ed[k]. \end{aligned}$$

In the following, we will model the uncertain multiple successive packet dropouts.

3.2. *Modelling packet dropout*

Here, we assume that the maximum number of the consecutive dropped packets is bounded, by some integer $D_{\max}$. In this subsection, we will analyse the bounded uncertain packet dropout pattern and model the NCSs as switched systems with arbitrary switching.

We first consider the simplified case when the packets are dropped periodically, with period T_m. Note that T_m is integer multiple of the sampling period T_s, i.e., $T_m = mT_s$. In case of $m = T_m/T_s \geq 2$, the first $(m-1)$ packets are dropped. Then, for these first $(m-1)$ steps, the previous control signal is used. Therefore

$$
\begin{aligned}
x(kT_m + T_s) &= Ax(kT_m) + NBu(kT_m - T_s) \\
&\quad + Ed(kT_m) \\
x(kT_m + 2T_s) &= A^2 x(kT_m) + N \cdot (AB + B)u \\
&\quad \times (kT_m - T_s) + AEd(kT_m) \\
&\quad + Ed(kT_m + T_s) \\
&\;\vdots \\
x(kT_m + (m-1)T_s) &= A^{m-1} x(kT_m) \\
&\quad + N \cdot \sum_{i=0}^{m-2} A^i Bu(kT_m - T_s) \\
&\quad + [A^{m-2}E, \ldots, E] \\
&\quad \times \begin{bmatrix} d(kT_m) \\ \vdots \\ d(kT_m + (m-2)T_s) \end{bmatrix}.
\end{aligned}
$$

Note that the integer $N = T_s/T$, where T is the period of the controller reading its receiving buffers. During the period $t \in [kT_m + (m-1)T_s, (k+1)T_m)$, the new packet is transmitted successfully with some delay, say $\tau = h(T_s/N)$, where $h = 0, 1, 2, \ldots, d_{\max}$. Then

$$
\begin{aligned}
x((k+1)T_m) &= Ax(kT_m + (m-1)T_s) + hBu(kT_m - T_s) \\
&\quad + (N-h)Bu(kT_m + (m-1)T_s) \\
&\quad + Ed(kT_m + (m-1)T_s) \\
&= A^m x(kT_m) + \left[N \cdot \sum_{i=1}^{m-1} A^i + h \right] Bu(kT_m - T_s) \\
&\quad + (N-h)Bu(kT_m + (m-1)T_s) \\
&\quad + [A^{m-1}E, \ldots, E] \begin{bmatrix} d(kT_m) \\ \vdots \\ d(kT_m + (m-1)T_s) \end{bmatrix}
\end{aligned}
$$

Note that $x(kT_m + mT_s)$ equals $x((k+1)T_m)$, and $x(kT_m + (m-1)T_s) = x((k+1)T_m - T_s)$. Let us assume that the controller uses just the time-invariant linear feedback control law, $u(t) = Kx(t)$. Then, we may substitute the $u(\cdot)$ to obtain

$$
\begin{aligned}
&x((k+1)T_m - T_s) \\
&\quad = A^{m-1} x(kT_m) + N \sum_{i=0}^{m-2} A^i BKx(kT_m - T_s) \\
&\quad + \sum_{i=0}^{m-2} A^i Ed(kT_m)
\end{aligned}
$$

and

$$
\begin{aligned}
x((k+1)T_m) &= A^m x(kT_m) \\
&\quad + \left[N \sum_{i=1}^{m-1} A^i + h \right] BKx(kT_m - T_s) \\
&\quad + (N-h)BKx((k+1)T_m - T_s) \\
&\quad + \sum_{i=0}^{m-1} A^i Ed(kT_m).
\end{aligned}
$$

Substitute $x((k+1)T_m - T_s)$ into the above equation. Then

$$
\begin{aligned}
x((k+1)T_m) &= [A^m + (N-h)BKA^{m-1}]x(kT_m) \\
&\quad + \left[N \sum_{i=1}^{m-1} A^i + (N-h)NBK \sum_{i=0}^{m-2} A^i + h \right] \\
&\quad \times BKx(kT_m - T_s) + \left[(N-h)BK \sum_{i=0}^{m-2} A^i \right. \\
&\quad \left. + \sum_{i=0}^{m-1} A^i \right] Ed(kT_m).
\end{aligned}
$$

If we let $\hat{x}[k] = \begin{bmatrix} x(kT_m - T_s) \\ x(kT_m) \end{bmatrix}$ and $d[k] = d(kT_m)$, then the above equations can be written as

$$
\hat{x}[k+1] = \Phi_{(m,h)} \hat{x}[k] + E_m d[k],
$$

where

$$
\Phi_{(m,h)} = \begin{bmatrix} N \sum_{i=0}^{m-2} A^i BK & A^{m-1} \\ \left(N \sum_{i=1}^{m-1} A^i + (N-h)NBK \sum_{i=0}^{m-2} A^i + h \right) BK & A^m + (N-h)BKA^{m-1} \end{bmatrix}
$$

$$
E_m = \begin{bmatrix} \sum_{i=0}^{m-2} A^i E \\ (N-h)BK \sum_{i=0}^{m-2} A^i E + \sum_{i=0}^{m-1} A^i E \end{bmatrix}.
$$

that $d(kT_m) = d(kT_m + 1) = \cdots = d(kT_m + m - 1)$, and

In this case, $m = T_m/T_s \geq 2$, and $h = 0, 1, \ldots, d_{\max}$.

For the case of $m=1$, namely no packet dropout, the following dynamic equation is derived

$$\hat{x}[k+1] = \Phi_{(1,h)}\hat{x}[k] + E_1 d[k],$$

where

$$\Phi_{(1,h)} = \begin{bmatrix} 0 & I \\ hBK & A+(N-h)BK \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 \\ E \end{bmatrix}.$$

For the the case of aperiodic packet dropouts, one may look at the delay and packet dropout pattern (figure 2) of the NCS as a succession of ramps of various lengths $(T_{m_1}+h_1,\ T_{m_2}+h_2, \ldots,\ T_{m_k}+h_k, \ldots)$. Therefore, the NCS along with a typical aperiodic delay and packet dropout pattern can be seen as a dynamical system switching among the dynamics with different periodic delay and packet dropout pattern $\Phi_{(m,h)}$, for $m = 1, \ldots, D_{\max}$ and $h = 0, 1, 2, \ldots, d_{\max}$. This observation leads to modelling the NCS as a switched system namely as

$$\left.\begin{aligned} \hat{x}[k+1] &= \Phi_{(m,h)}\hat{x}[k] + E_m d[k] \\ z[k] &= \begin{bmatrix} C & 0 \end{bmatrix}\hat{x}[k], \end{aligned}\right\} \tag{4}$$

where

$$\Phi_{(m,h)} = \begin{cases} \begin{bmatrix} 0 & I \\ hBK & A+(N-h)BK \end{bmatrix}, & m=1 \\ \begin{bmatrix} N\sum_{i=0}^{m-2} A^i BK & A^{m-1} \\ \left(N\sum_{i=1}^{m-1} A^i + (N-h)NBK\sum_{i=0}^{m-2} A^i + h\right)BK & A^m + (N-h)BKA^{m-1} \end{bmatrix}, & m \ge 2 \end{cases}$$

$$E_m = \begin{cases} \begin{bmatrix} 0 \\ E \end{bmatrix}, & m=1 \\ \begin{bmatrix} \sum_{i=0}^{m-2} A^i E \\ (N-h)BK\sum_{i=0}^{m-2} A^i E + \sum_{i=0}^{m-1} A^i E \end{bmatrix}, & m \ge 2. \end{cases}$$

Here $\Phi_{(m,h)} \in \{\Phi_{(1,0)}, \Phi_{(1,1)}, \ldots, \Phi_{(1,D_{\max})}, \Phi_{(2,0)}, \ldots, \Phi_{(D_{\max},0)}, \ldots, \Phi_{(D_{\max},d_{\max})}\}$, where D_{max} corresponds to the maximum number of successively dropped packets, and $d_{\max}$ is the maximum access delay.

For notational simplicity, let us denote the index of all the subsystems by $q = m + h \times D_{\max}$, and call the collection $\{1, 2, \ldots, D_{\max} \times (d_{\max}+1)\}$ the mode set Q, $q \in Q$. Therefore, we rewrite (4) as

$$\left.\begin{aligned} \hat{x}[k+1] &= \Phi_q\hat{x}[k] + E_q d[k] \\ z[k] &= \begin{bmatrix} C & 0 \end{bmatrix}\hat{x}[k]. \end{aligned}\right\} \tag{5}$$

Associate (5) with a class of piecewise constant functions of time $\sigma\colon \mathbb{Z}^+ \to Q$, called switching signals. Note that each switching signal σ corresponds to a (maybe aperiodic) delay and packet dropout pattern. In order to study the effects of bounded uncertain access delay and packet dropouts on the NCSs' stability and performance, one needs to consider all possible delay and packet dropout patterns which corresponds to considering the arbitrary switching signals for (5). Therefore, the stability and performance problems of the NCS are equivalent to the stability and performance problems of the switched system (5) with arbitrary switching. To illustrate, we consider the following example.

Example 1: Consider the continuous-time integrator with disturbances as the plant

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} d(t) \\ z(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t). \end{aligned}$$

Assume that the sampling period T_s is 0.1 second. The controller reads the receiving buffer every $T = 0.01s$, i.e. $N = T_s/T = 10$. It is assumed that the sensor only tries to send the new sampled state value during the first $0.02s$ of each sampling period T_s, from which we may obtain that the maximum delay (if successfully arrived) is $\tau_{\max} = 0.02s$ and $d_{\max} = 0.02/T = 2$.

Also assume that at most three successive packet-dropouts can occur, namely $D_{\max} = 4$. Therefore, the above NCS can be modelled as an arbitrary switching system with $D_{\max} \times (d_{\max} + 1) = 12$ modes. The state matrices for each mode can be determined by substituting the following values

$$A = e^{A^c T_s} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad B = \int_0^T e^{A^c t} B^c dt$$
$$= \begin{bmatrix} 0.00005 \\ 0.01 \end{bmatrix}$$
$$E = \int_0^{T_s} e^{A^c t} E^c dt = \begin{bmatrix} 0.105 \\ 0.1 \end{bmatrix}, \quad K = \begin{bmatrix} -2 & -1 \end{bmatrix}$$

into the expressions for $\Phi_{(m,h)}$ and E_m in (4) for all possible values of $m \in \{1, 2, 3, 4\}$ and $h = \{0, 1, 2\}$. For instance, the mode corresponding to the case of two successive packet dropouts ($m = 3$) and the third packet arriving with delay $0.02s$ ($h = 2$), i.e., the eleventh mode ($2 \times D_{\max} + 3 = 11$), can be described by

$$\hat{x}[k+1] = \begin{bmatrix} -0.0220 & -0.0110 & 1.0000 & 0.2000 \\ -0.4000 & -0.2000 & 0 & 1.0000 \\ -0.1020 & -0.0510 & 0.9992 & 0.2994 \\ -0.4047 & -0.2023 & -0.1600 & 0.8880 \end{bmatrix} \hat{x}[k] + \begin{bmatrix} 0.2200 \\ 0.2000 \\ 0.2399 \\ 0.1736 \end{bmatrix} d[k]$$
$$z[k] = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \hat{x}[k]. \qquad \square$$

Remarks: Similar techniques were used in Bauer *et al.* (2001), in which a NCS with bounded packet dropout was modelled as a polytopic uncertain linear time-variant system. In Bauer *et al.* (2001), it is assumed that the plant and the controller are well synchronized, while in this paper we do not have the synchronization assumption. In addition, we also consider uncertain access delay in our switched NCS model.

Now we have modelled the NCS with uncertain access delay and packet dropout effects as a switched system (5) with arbitrary switching between its $\mathbf{N} = D_{\max} \times (d_{\max} + 1)$ modes. In the following sections we will study the asymptotic stability and disturbance attenuation properties of such NCSs within the framework of switched systems. For notational simplicity we will write $\hat{x}$ as x in the sequel.

4. Stability analysis

The effects of the uncertain access delay and packet dropouts on the persistent disturbance attenuation property, namely the l^∞ induced norm from $d[k]$ to $z[k]$ for the NCSs (5) will now be investigated. It is assumed that the disturbance $d[k]$ is contained in the l^∞ unit ball, i.e., $\mathcal{D} = \{d : \|d\|_{l^\infty} \leq 1\}$. The l^∞ induced norm from $d[k]$ to $z[k]$ is defined as

$$\mu_{inf} = \inf\{\mu : \|z[k]\|_{l^\infty} \leq \mu, \ \forall d[k], \ \|d[k]\|_{l^\infty} \leq 1\}.$$

The first problem we need to answer is

Problem 1: Under what conditions the l^∞ induced norm from $d[k]$ to $z[k]$ for the NCSs with bounded uncertain access delay and packet dropouts is finite?

The answer to Problem 1 is equivalent to the condition for the arbitrarily switching system (4) to have a finite l^∞ induced gain. In Lin and Antsaklis (2003), it is shown that a necessary and sufficient condition for an arbitrarily switching system (5) to have a finite μ_{inf} is that the corresponding autonomous switched system $x[k+1] = \Phi_\sigma x[k]$ be asymptotically stable under arbitrary switching signals. Therefore, Problem 1 is transformed into a stability analysis problem for autonomous switched systems under arbitrary switching, which has been studied extensively in the literature, and is typically being dealt with by constructing a common Lyapunov function; see the survey papers by Liberzon and Morse (1999), Decarlo *et al.* (2000), Lin an Antsaklis (2005), the recent book by Liberzon (2003) and the references cited therein. Various attempts have been made, for example in Narendra and Balakrishnan (1994), Shorten and Narendra (1998), Liberzon *et al.* (1999) and Liberzon and Tempo (2003) to find a common quadratic Lyapunov function for the family of systems, ensuring the asymptotic stability of switched systems for all switching signals. In Liberzon *et al.* (1999) and Agrachev and Liberzon (2001), Lie algebra conditions were given which imply the existence of a common quadratic Lyapunov function. It is worth pointing out that a converse Lyapunov theorem was derived in Dayawansa and Martin (1999) for the globally asymptotic stability of arbitrary switching systems. This converse Lyapunov theorem justifies the common Lyapunov method which was pursued in the literature. However, most of the work has been restricted to the case of quadratic Lyapunov function which only gives sufficient stability test criteria. In the present paper a necessary and sufficient condition for asymptotic stability of arbitrarily switching systems is given thus improving the sufficient only conditions found in the literature.

Let us first introduce a technical lemma (Bauer *et al.* 1993) for the robust stability of linear time variant systems

$$x[k+1] = \Phi(k)x[k], \tag{6}$$

where $\Phi(k) \in \mathcal{A} \hat{=} \text{Conv}\{\Phi_1, \Phi_2, \ldots, \Phi_\mathbf{N}\}$.

Lemma 1: *The polytopic uncertain linear time-variant system* (6) *is globally asymptotically stable if and only if there exists a finite* $\mathbf{n}$ *such that* $\|\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_\mathbf{n}}\| < 1$ *for all* $\mathbf{n}$*-tuple* $\Phi_{i_j} \in \text{vert}\{\mathcal{A}\} = \{\Phi_1, \Phi_2, \ldots, \Phi_\mathbf{N}\}$, *for* $j = 1, \ldots, \mathbf{n}$.

Here the norm $\|\cdot\|$ stands for either 1 norm or ∞ norm of a matrix. A necessary and sufficient condition for the asymptotic stability of the switched NCS (5) can be presented as the following proposition.

Proposition 1: *A switched linear system* $x[k+1] = \Phi_{\sigma(k)}x[k]$, *where* $\Phi_{\sigma(k)} \in \{\Phi_1, \Phi_2, \ldots, \Phi_\mathbf{N}\}$, *is globally asymptotically stable under arbitrary switching if and only if there exists a finite* $\mathbf{n}$ *such that*

$$\|\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_\mathbf{n}}\| < 1, \quad \forall \Phi_{i_j} \in \{\Phi_1, \Phi_2, \ldots, \Phi_\mathbf{N}\}, \tag{6a}$$

for $j = 1, \ldots, \mathbf{n}$.

Proof: See Appendix. □

Example 2: For the NCS example considered in the previous section (Example 1), we tested the matrix norm condition via stochastic methods and observed that

$$\|\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_{24}}\|_\infty < 1, \quad \forall \Phi_{i_j} \in \{\Phi_1, \Phi_2, \ldots, \Phi_{12}\},$$

holds for $j = 1, \ldots, 24$. Therefore, by Proposition 1, the NCS in the above example is globally asymptotically stable.

It is worth pointing out that the above matrix norm condition, although necessary and sufficient, is not easy to check. It requires a permutation of all possible matrix products of length $\mathbf{n}$. In addition, the result does not give any clue on how large the integer $\mathbf{n}$ needs to be. However, starting from the condition obtained here some equivalent conditions could be identified in the literature, which are expressed in matrix equation form and efficient numerical methods have been proposed. Interested readers may refer to Bhaya and Mota (1994), Liu and Molchanov (2002), and the references therein.

Based on Proposition 1 and Lemma 1, we may conclude the *equivalence* between the robust asymptotic stability for polytopic uncertain linear time-variant systems and the asymptotic stability for switched linear systems with arbitrary switching.

Corollary 1: The global asymptotic stability for arbitrary switching systems $x[k+1] = \Phi_{\sigma(k)}x[k]$ is equivalent to the global asymptotic stability for polytopic uncertain linear time-variant systems $x[k+1] = \Phi(k)x[k]$, where $\Phi(k) \in \text{Conv}\{\Phi_{q_1}, \Phi_{q_2}, \ldots, \Phi_{q_N}\}$, or $\Phi(k) = \sum_{i=1}^{N} w_i(k)\Phi_{q_i}$, where $w_i(k) \geq 0$ and $\sum_{i=1}^{N} w_i(k) = 1$ for all k.

Remarks: It is quite interesting that the study of robust stability of a polytopic uncertain linear time-variant system, which has infinite number of possible dynamics (modes), is equivalent to only considering a finite number of its vertex dynamics as an arbitrary switching system. In fact, it is not a surprising result since this fact has already been implied by the finite vertex stability criteria for robust stability in the literature e.g. Molchanov and Pyatnitskiy (1989), Bhaya and Mota (1994). By explicitly exploring this equivalence relationship, we may obtain some ''new'' stability criteria for switched linear systems using the existing robust stability results. Although we only discuss here the discrete-time case, this result is also true in the continuous-time case. This fact bridges two distinct research fields. Therefore, existing results in the robust stability area, which has been extensively studied for over two decades, can be directly introduced to study the arbitrarily switching systems and vice versa.

5. Disturbance attenuation property

In view of the above discussion on the conditions for μ_{inf} to be finite, we limit our attention to asymptotically stable switched systems under arbitrary switching in the sequel. We aim to calculate a non-conservative bound on μ_{inf} for the NCS with bounded uncertain access delay and packet dropouts. This leads to the second problem studied in this paper.

Problem 2: Determine the minimal l^∞ induced norm from $d[k]$ to $z[k]$ for NCSs with bounded uncertain access delay and packet dropouts.

To solve this problem, we consider the disturbance attenuation performance that the switched system (5) can preserve under arbitrary switching. We will calculate a non-conservative bound on μ_{inf} for the arbitrarily switching system (5). The basic idea is to translate the required level of performance into constraints on the controlled system. The techniques are based on the positive invariant set theory and our recent results on robust performance for switched linear systems (Lin and Antsaklis 2003).

We first introduce the definition of a *positive disturbance invariant set* for the switched system (5) under arbitrary switching signals.

Definition 1: A set $\mathcal{P}$ in the state space is said to be *positive disturbance* invariant for the switched system (3.4) with arbitrary switching if for every initial condition $x[0] \in \mathcal{P}$ we have that $x[k] \in \mathcal{P}$, $k \geq 0$, for every possible switching signal $\sigma(k)$ and every admissible disturbance $d[k] \in \mathcal{D}$.

We now formalize the definition of a limit set.

Definition 2: The limit set $\mathcal{L}$ for the switched system (5) with arbitrary switching is the set of states x for which there exist a switching sequence $\sigma(k)$, admissible sequence $d[k]$ and a non-decreasing time sequence t_k such that

$$\lim_{k\to+\infty} \Xi(0, t_k, \sigma(\cdot), d[\cdot]) = x$$

where $\lim_{k\to+\infty} t_k = +\infty$ and $\Xi(0, t_k, \sigma(\cdot), d[\cdot])$ denotes the value of the solution of (5) at the instant t_k, which originates at $x_0 = 0$ and corresponds to σ and d.

The limit set $\mathcal{L}$ has the following property.

Lemma 2: *Consider the switched system* (5) *with arbitrary switching, under the asymptotic stability assumption, the limit set $\mathcal{L}$ is non-empty and the state evolution of the switched system* (5) *converges to $\mathcal{L}$, for every initial condition $x[0]$, all switching sequences $\sigma(k)$ and all admissible disturbances $d[k] \in \mathcal{D}$. Moreover, $\mathcal{L}$ is bounded and positive disturbance invariant.*

The boundedness and convergence of the limit set come from the asymptotic stability of the switched system under arbitrary switching. The invariance can be easily shown by contradiction. The detailed proof is omitted here due to space limitations. (Similar concepts and lemma were previously given in Blanchini *et al.* (1997) for uncertain linear time-varying systems. The results developed here are direct extensions to the switched systems.)

Define now the set

$$X_0(\mu) = \{x : \|Cx\|_\infty \leq \mu\} = \left\{x : \begin{bmatrix} C \\ -C \end{bmatrix} x \leq \begin{bmatrix} \bar{\mu} \\ \bar{\mu} \end{bmatrix}\right\},$$

where $\bar{\mu}$ stands for a column vector with each of its elements equals to μ. $X_0(\mu)$ is a polytope containing the origin in its interior.

A value $\mu < +\infty$ is said to be admissible for arbitrary switching signals if $\mu > \mu_{inf}$. Clearly, given $\mu > 0$, the response of the switched system satisfies $\|z[k]\|_{l^\infty} \leq \mu$ and $\|d[k]\|_{l^\infty} \leq 1$ if and only if the switched system (5) admits a positive disturbance invariant set $\mathcal{P}$ under arbitrary switching such that $0 \in \mathcal{P} \subseteq X_0(\mu)$.

In the following, we provide a procedure to compute a positive disturbance invariant set, for arbitrary switching signals, containing in $X_0(\mu)$. This is accomplished by finding the maximal positive disturbance invariant set for the switched system (5) under arbitrary switching, i.e., a set contains any other positive disturbance invariant set under arbitrary switching in $X_0(\mu)$.

Given a compact set $\mathcal{P} \subseteq \mathbb{R}^n$, we can define its predecessor set for switched systems (5) under arbitrary switching, $pre(\mathcal{P})$, as all the states x that can reach the set $\mathcal{P}$ in the next step in spite of disturbances or switching signals. It can be calculated as

$$\underline{pre}(\mathcal{P}) = \bigcap_{q\in Q} pre_q(\mathcal{P}), \tag{7}$$

where $pre_q(\mathcal{P})$ stands for the predecessor set of the q-th subsystem, that is the set of all states x that are mapped into $\mathcal{P}$ by the transformation $\Phi_q x + E_q d$, for all admissible $d \in \mathcal{D}$. See (Blanchini 1994) for linear programming procedures to calculate the $pre_q(\mathcal{P})$ for a polyhedral set $\mathcal{P}$.

By recursively defining the sets $\mathcal{P}^{(k)}$, $k = 0, 1, \ldots$ as

$$\mathcal{P}^{(0)} = X_0(\mu),\ \mathcal{P}^{(k)} = \mathcal{P}^{(k-1)} \bigcap \underline{pre}(\mathcal{P}^{(k-1)}) \tag{8}$$

it can be shown (Blanchini 1994) that $\mathcal{P}^{(\infty)}$ is the maximal positive disturbance invariant set under arbitrary switching in $X_0(\mu)$. We now introduce a lemma guaranteeing that this set can be expressed by a finite set of linear inequalities (i.e. it is a polyhedral) and thus can be finitely determined.

Proposition 2: *Under the asymptotic stability assumption, if $\mathcal{L} \subset int\{X_0(\mu)\}$ for some $\mu > 0$, then there exists some integer* $\mathbf{k}$ *such that $\mathcal{P}^{(\infty)} = \mathcal{P}^{(\mathbf{k})}$ and this* $\mathbf{k}$ *can be selected as the smallest integer such that $\mathcal{P}^{(\mathbf{k+1})} = \mathcal{P}^{(\mathbf{k})}$.*

Proof: See Appendix. □

In order to check whether a given performance level $\mu > 0$ is admissible for the switched system under arbitrary switching, one may compute the maximal positive disturbance invariant set $\mathcal{P}^{(\infty)}$ in $X_0(\mu)$ and check whether or not $\mathcal{P}^{(\infty)}$ contains the origin. If it does, then $\mu > \mu_{inf}$, otherwise $\mu < \mu_{inf}$. Note that in both cases we obtain an answer in a finite number of steps. In the first case, this is due to the above proposition. In the second case, this stems from the fact that the sequence of closed sets $\mathcal{P}^{(k)}$ is ordered by inclusion and $\mathcal{P}^{(\infty)}$ is their intersection. Thus $0 \notin \mathcal{P}^{(\infty)}$ if and only if $0 \notin P^{(k)}$ for some k. Thus checking whether $\mu > \mu_{inf}$ can be obtained by starting from the initial set $X_0(\mu)$ and computing the sequence of sets $\mathcal{P}^{(k)}$ until some appropriate stopping criterion is met. Note that another useful stopping criterion is derived as follows.

Proposition 3: *If the set $\mathcal{P}^{(k)} \subset int\{X_0(\mu)\}$ for some k, then the switched system* (5) *does not admit a positive disturbance invariant set under arbitrary switching in $X_0(\mu)$. In other words, $\mu < \mu_{inf}$.*

Proof: See Appendix. □

These results suggest the following constructive procedure for finding a robust performance bound.

Procedure 1: Checking whether $\mu > \mu_{inf}$

1. Initialization: Set $k = 1$ and set $\mathcal{P}^{(0)} = X_0(\mu)$.
2. Compute the set $\mathcal{P}^{(k)} = \mathcal{P}^{(k-1)} \cap \underline{pre}(\mathcal{P}^{(k-1)})$.
3. If $0 \notin \mathcal{P}^{(k+1)}$ or $\mathcal{P}^{(k)} \subset int\{X_0(\mu)\}$ then stop, the procedure has failed. Thus, the output does not robustly meet the performance level μ.
4. If the $\mathcal{P}^{(k+1)} = \mathcal{P}^{(k)}$, then stop, and set $\mathcal{P}^{(\infty)} = \mathcal{P}^{(k)}$.
5. Set $k = k + 1$ and go to step 1.

This procedure can then be used together with a bisection method on μ to approximate the optimal value μ_{inf} arbitrarily close. This solves the Problem 2. In fact, if the procedure stops at step 3, we conclude that $\mu < \mu_{inf}$ and we should increase the value of the output bound μ. Otherwise, if the procedure stops at step 4, we have determined an admissible bound for the output, say $\mu > \mu_{inf}$, that can be decreased. This can be formalized as a bisection algorithm:

Algorithm 1: Algorithm for Calculating μ_{inf}

1. Initialization: Choose the initial interval $[\mu_1, \mu_2]$ such that $\mu_1 \le \mu_{inf} < \mu_2$. Choose $\epsilon > 0$, a tolerance level. A good candidate for μ_1 is $\mu_1 = \max_{q \in Q}\{\mu^q_{inf}\} - \epsilon$, where μ^q_{inf} is the l^1 norm of the q-th subsystem. If no value for μ^q_{inf} is available, μ_1 may be chosen to be $\mu_1 = \epsilon$.
2. While $(\mu_2 - \mu_1) > \epsilon$, set $\mu_3 = (\mu_1 + \mu_2)/2$, and check whether $\mu_3 > \mu_{inf}$ by the above Procedure. If $\mu_3 > \mu_{inf}$, then set $\mu_2 = \mu_3$, else set $\mu_1 = \mu_3$.
3. Output $\mu_{inf} = (\mu_1 + \mu_2)/2$.

Example 3: Consider Example. A non-conservative bound of μ_{inf} for the switched NCSs under arbitrary switching sequences is obtained as $\mu_{inf} = 0.809$, via the bisection method (with error tolerance $\epsilon = 0.01$).

6. Concluding remarks

In this paper, NCSs under bounded uncertain access delay and packet dropout were modelled as switched linear systems with arbitrary switching. The asymptotic stability and persistent disturbance attenuation properties of the NCSs were studied in the switched system framework. It was shown that the asymptotic stability of switched linear systems with arbitrary switching is equivalent to the robust stability of polytopic uncertain linear time-variant systems. Therefore, a necessary and sufficient condition, which roots from robust stability literature, was given for the NCSs' asymptotic stability. This equivalence bridges two distinct research fields and the stability study of arbitrary switching systems may benefit from the existing results in the robust stability area, which has been extensively studied for over two decades.

Assuming an absolute upper bound on the maximum number of packets dropped in a row could be conservative in certain cases. In Lin *et al.* (2003), we provided an alternative way to model NCSs as switched systems. Instead of incorporating all possible delay-dropout patterns to relax the switching signal to be arbitrary, we specified a subclass of the switching signal by restricting the occurring frequency and the number of dropped and seriously-delayed packets in the time average sense. In particular, it was shown that there exist bounds on the delay and packet dropout rate and percentage, below which the NCS stability and $\mathcal{L}_2$ disturbance attenuation properties may be preserved to a desired level. In Lin *et al.* (2003), these bounds were identified based on multiple Lyapunov functions incorporated with average dwell time scheme.

We believe that switched system approaches to NCSs are promising, as many research topics like networked continuous-controller design, controller and scheduling policy co-design could be pursued in the switched system framework. For example, in Lin *et al.* (2005), a stability and $\mathcal{L}_2$ performance preserving network bandwidth management policy was proposed based on switched systems approaches. The potential of dealing with NCSs as switched systems comes from the existence of solid theoretic results in the field of switched systems, jump linear systems etc. Interested readers may refer to Sun and Ge (2005) and Lin and Antsaklis (2005) for surveys on the most recent progress on switched linear systems.

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Appendix

Proof for Proposition 1: The sufficiency is obvious because under the conditions stated, the uncertain time variant system $x[k+1] = \Phi(k)x[k]$ is robustly stable according to the Lemma 1, where $\Phi(k) \in \mathcal{A} \hat{=} \text{Conv}\{\Phi_1, \Phi_2, \ldots, \Phi_{\mathbf{N}}\}$. This implies the globally asymptotic stability of the switched systems $x[k+1] = \Phi_{\sigma(k)}x[k]$ with arbitrary switching.

To prove the necessity, let us assume that there exists no such $\mathbf{n}$ that

$$\|\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_{\mathbf{n}}}\| < 1, \quad \forall \Phi_{i_j} \in \{\Phi_1, \Phi_2, \ldots, \Phi_{\mathbf{N}}\},$$

for $j = 1, \ldots, n$. Therefore, given any $\mathbf{n}$, we may always find a switching sequence $\sigma(k)$ such that $\|\Phi_{\sigma(0)}\Phi_{\sigma(1)}\cdots\Phi_{\sigma(\mathbf{n}-1)}\| \geq 1$. Then there exists an initial vector $x[0]$ such that $\|x[\mathbf{n}]\| \geq \|x[0]\|$, for any $\mathbf{n}$, which contradicts the condition for globally asymptotic stability. □

Proof for Proposition 2: By the asymptotic stability assumption, the switched system is asymptotically stable under arbitrary switching. Then according to Lemma 2, there exists $\mathbf{k}$ such that for all $x[0] \in X_0(\mu)$, $x[k] \in \mathcal{L} \subset int\{X_0(\mu)\}$ $(\forall k \geq \mathbf{k})$ for all possible switching signals. By construction, the set $\mathcal{P}^{(\mathbf{k})}$ has the property that $x[k] \in X_0(\mu)$, $k = 0, 1, \ldots, \mathbf{k}$, for all possible $d[k] \in \mathcal{D}$ if and only if $x[0] \in \mathcal{P}^{(\mathbf{k})}$. This implies that $\mathcal{P}^{(\mathbf{k})} = \mathcal{P}^{(\mathbf{k}+1)}$. Otherwise, $\mathcal{P}^{(\mathbf{k})} \supset \mathcal{P}^{(\mathbf{k}+1)}$, and there exists $x[0] \in \mathcal{P}^{(\mathbf{k})}$ but $x[0] \notin \mathcal{P}^{(\mathbf{k}+1)}$, then for some $\sigma(k)$ and $d[k] \in \mathcal{D}$ we have $x[\mathbf{k}+\mathbf{1}] \notin \mathbf{X_0}(\mu)$. This is a contradiction. Therefore, $\mathcal{P}^{(\mathbf{k})} = \mathcal{P}^{(\mathbf{k}+1)}$, and this implies that $\mathcal{P}^{(\mathbf{k})} = \mathcal{P}^{(\mathbf{k}+m)}$, for $m \geq 0$. Thus $\mathcal{P}^{(\infty)} = \mathcal{P}^{(\mathbf{k})}$. □

Proof for Proposition 3: Suppose that there exists k such that $\mathcal{P}^{(k)} \subset int\{X_0(\mu)\}$ and the switched system (5) admits a positive disturbance invariant set in $X_0(\mu)$ under arbitrary switching, and hence $\mathcal{P}^{(\infty)} \subset int\{X_0(\mu)\}$. Define ν as $\nu = \inf_{x \notin X_0(\mu)} dist(x, \mathcal{P}^{(\infty)})$. For every initial condition $x_0 \notin \mathcal{P}^{(\infty)}$, there exist sequences $\bar{\sigma}$ and $\bar{d}$ such that the corresponding trajectory escapes from $X_0(\mu)$, i.e. $x[\bar{k}] \notin X_0(\mu)$ for some $\bar{k}$. Let $\bar{x}[t]$ and $x[t]$ denote two system trajectories, corresponding to the same sequences $\bar{\sigma}$ and $\bar{d}$ but with different initial conditions. The updating equation for the difference $e[t] = \bar{x}[t] - x[t]$ is

$$e[t+1] = \Phi_\sigma e[t] \tag{9}$$

which is stable. Thus for arbitrary $0 < \varepsilon < \nu$ there exists $\delta > 0$ such that, for $\|\bar{x}[0] - x[0]\| < \delta$, we have $\|e[t]\| = \|\bar{x}[t] - x[t]\| < \varepsilon$ for $t \geq 0$. On the other hand, we may choose $\bar{x}[0] \in \mathcal{P}^{(\infty)}$ and $x[0] \notin \mathcal{P}^{(\infty)}$ such that $\|\bar{x}[0] - x[0]\| < \delta$. Now we have $\bar{x}[\bar{k}] \in \mathcal{P}^{(\infty)}$ and $x[\bar{k}] \notin X_0(\mu)$. This implies that $\|e[\bar{k}]\| \geq \nu$ and leads to a contradiction. □

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