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CONTROL DESIGN USING POLYNOMIAL MATRIX INTERPOLATION

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Introduction

The theory of polynomial and rational function interpolation is well understood and its importance in many disciplines is well known [1,2]. Recently, a theory generalizing certain interpolation results to polynomial matrices was introduced [3,4]. This generalization appears to be well suited to study and resolve a variety of multivariable system and control problems.

Many control problems can be formulated in terms of polynomial equations (e.g. the Diophantine equation $XD+YN=Q$) where solutions with specific properties are of interest (e.g. $X^{-1}Y$ exists and it is proper). It is well known that equations involving just polynomials can be solved by either equating coefficients of equal powers of the indeterminate s or equivalently, by using the values obtained when appropriate values for s are substituted in the given polynomials; in the latter case one employs results from the theory of polynomial interpolation. In multivariable control, polynomial matrix equations appear, which can be similarly studied either by equating coefficients [5,6] or by using the recently introduced polynomial matrix interpolation theory [3,4]. Using this latter method, the given matrices do not have to be reduced to any appropriate form before the method is applied (e.g. in $XD+YN=Q$, D does not have to be column reduced), equations involving rational matrices can also be directly treated (e.g. $\det|X+YP|$ desired with $P=ND^{-1}$ in eigenvalue placement problem) and equations where certain given matrices are not completely known but they belong to certain known class can be solved (e.g. $XD+YN=WR$, W arbitrary polynomial matrix). Note that polynomial matrix interpolation methods have been used in the past to solve special cases of $XD+YN=Q$; these include the eigenvalue placement problem via real output feedback and via state feedback [7,8,6]. Note however that this was done without using the theoretical polynomial matrix interpolation results which are essential in fully solving the general Diophantine equation.

A suite of interactive computer programs has been developed [9] to study the effectiveness of polynomial matrix interpolation in solving control problems and certain of the results are reported here. In addition certain basic interpolation results together with some new extensions are briefly outlined and the relation to interpolation conditions used in [10,11] is shown.

Basic Interpolation Results

The basic theorem of polynomial interpolation can be stated as follows [2]: Given n+1 distinct complex points s_j $j=1,\dots,n+1$ and n+1 complex values b_j, there exists a unique nth degree polynomial $q(s)$ for which

$$q(s_j) = b_j \qquad j=1,\dots,n+1 \tag{1}$$

A generalization of this result to polynomial matrices is as follows:
Let $S(s) = \text{blk diag}\,[1,s,\dots,s^{d_i}]^T$ where d_i $i=1,\dots,m$ are non-negative integers; let $a_j\neq 0$ and b_j denote (mx1) and (px1) complex vectors respectively and s_j complex scalars.

Theorem 1: Given $(s_j,\ a_j,\ b_j)\ j=1,\dots,\ell$ with $\ell=\sum_1^m d_i+m$ such that the $(\ell x\ell)$ matrix

$$S_\ell = [S(s_1)a_1,\dots,S(s_\ell)a_\ell] \tag{2}$$

has full rank, there exists a unique (pxm) polynomial matrix $Q(s)$ with column degrees $\partial_{ci}[Q(s)]=d_i$ for which

$$Q(s_j)a_j = b_j \qquad j=1,\dots,\ell. \tag{3}$$

When $p=m=1$ and $d_i=n$ this theorem reduces to the polynomial interpolation theorem. To see this, note that the non-zero scalars a_j $j=1,\dots,\ell=n+1$ can be taken equal to 1 in which case S_ℓ is exactly the well known Vandermonde matrix which is non-singular if and only if s_j are distinct.

In view of the above polynomial matrix interpolation result, the basic rational matrix interpolation theorem follows:

Theorem 2: Assume that (s_j,c_j,b_j) $j=1,\dots,\ell$ with $\ell = \sum_1^m d_i+m$ and an (mxm) polynomial matrix $D(s)$ with $|D(s_j)|\neq 0$ are given, such that the S_ℓ matrix in (2) with $a_j=[D(s_j)]^{-1}c_j$, $c_j\neq 0$ has full rank. There exists a unique rational matrix $\underline{Q}(s)$ of the form $\underline{Q}(s)=N(s)D(s)^{-1}$, where the polynomial matrix $N(s)$ has column degrees $\partial_{ci}[N(s)]=d_i$, for which

$$\underline{Q}(s_j)c_j = b_j \qquad j = 1,\dots,\ell \tag{4}$$

Notice that in Theorem 1 the polynomial matrix $Q(s)$ with $\partial_{ci}[Q(s)]=d_i$ is uniquely determined from $Q(s)=QS(s)$ with Q satisfying

$$Q\,S_\ell = [b_1,\dots,b_\ell] = B_\ell{}^{+} \tag{5}$$

It is clear that there are many matrices $Q(s)$ with column degrees greater than d_i which satisfy (3). This additional freedom can be exploited so that $Q(s)$ satisfies requirements in addition to (3). Similar comments apply to the rational matrix $\underline{Q}(s)$ of Theorem 2. Such is the case in [10,11] where a (rational) matrix $\underline{Q}(s)$ must satisfy $||\underline{Q}(jw)|| < 1$ in addition to interpolation constraints of the form

$$\underline{Q}(z_k) = Q_k \qquad k = 1,\dots,\bar{k} \tag{6}$$

Note that these conditions are a special case of conditions (4) (or(3)). To see this, consider (4) where $s_i=z_1$, $c_i=e_i$, with $i=1,\dots,m$, $s_{m+i}=z_2$ with $c_{m+i}=e_i$ and so on, where the entries of e_i are zero except the ith entry which is 1; then $Q_1 = [b_1,\dots,b_m]$ and so on. Conditions of the form (6) are derived if (polynomial) rational function interpolation is applied to each entry of $(Q(s))$ $\underline{Q}(s)$ and the results are combined in matrix form.

+It is clear that alternative polynomial bases (other than $1,s,\dots$), which might offer computational advantages, can be used.

The exact relationship between a non-singular polynomial matrix $Q(s)$ and its characteristic value and vector pairs (s_j,a_j) has been established in [4] where the case of repeated characteristic values has been treated in detail. If the characteristic values and vectors of $Q(s)$ satisfy

$$Q(s_j)a_j=0 \qquad j = 1,\dots,n \tag{7}$$

where $n=\deg|Q(s)|$ (e.g. when s_j are distinct) then, a $(p\times m)$ polynomial matrix $A(s)$ can be written as $A(s) = \hat{A}(s).Q(s)$ with $\hat{A}(s)$ some polynomial matrix, if and only if

$$A(s_j)a_j=0 \qquad j = 1,\dots,n \tag{8}$$

The basic interpolation results outlined above have been used to study and develop methods to resolve a number of system and control questions. These interpolation methods have been successfully implemented via an interactive computer program [9] and some of the results obtained are presented in the following section.

Some Applications of Interpolation

(A) Determine solutions (X,Y) of the Diophantine equation

$$[X(s),Y(s)]\begin{bmatrix}D(s)\\N(s)\end{bmatrix} = M(s)L(s) = Q(s). \tag{9}$$

Let $M(s)=M_0+\dots+M_rs^r$, write $L_r(s)=[L(s)^T,\dots,s^rL(s)^T]^T$ and consider interpolation points (s_j,a_j,b_j) $j=1,\dots,\ell$ where

$$\ell = \sum_1^m d_i + m(r+1), \tag{10}$$

$L(s)$ is $(t\times m)$, $b_j=Q(s_j)a_j$ and (s_j,a_j) satisfy the conditions of Theorem 1 (for $\partial_{ci}[M(s)L(s)] = \partial_{ci}[L(s)]+r = d_i+r$). Then if $M=[M_0,\dots,M_r]$ and $B_\ell=[b_1,\dots,b_\ell]$ the equation to be solved is

$$ML_{r,\ell} = B_\ell \tag{11}$$

where $L_{r,\ell}=[L_r(s_1)a_1,\dots,L_r(s_\ell)a_\ell]$ $((r+1)t\times\ell)$.

Theorem 3: Assume r satisfies

$$\partial_{ci}[Q(s)] \le d_i + r. \tag{12}$$

Then, any and all solutions $M(s)$ of (9) of degree r are derived by solving (11).

Clearly, solution $M(s)$ of degree r might not exist; Theorem 3 guarantees that searching for degree r solutions of (9) is equivalent to solving (11). Assume (N,D) right prime. For solutions of degree r to exist, r must satisfy certain inequality, in addition to (12) (e.g. $r \ge \nu-1$ where ν the observability index of ND^{-1}). Solutions with $X^{-1}Y$ proper are also obtained by specifying part of the structure of X via the interactive computer program (Note that the method does not require any special structure for the given matrices).

If $D(s) = \begin{bmatrix}s^2 & 0\\1 & -s+1\end{bmatrix}$, $N(s) = \begin{bmatrix}s+1 & 0\\1 & 1\end{bmatrix}$

and $Q(s) = \begin{bmatrix}s^3+2s^2-3s-5, & -5s-5\\-2s^2-5s-4, & -s^2-3s-2\end{bmatrix}$

then [9] $r \ge 1$, $\ell = 5+2r$, and for $r = 1$

$X(s) = \begin{bmatrix}s+5 & 5\\2 & s+4\end{bmatrix}$, $Y(s) = \begin{bmatrix}-3s & -10\\-4s-2 & -6\end{bmatrix}$

where $X^{-1}Y$ proper. Note that (s_j,a_j) are almost arbitrarily chosen, the only restriction being to satisfy condition of Theorem 1. Similarly a unimodular matrix U such that $U\begin{bmatrix}D\\N\end{bmatrix} = \begin{bmatrix}I\\0\end{bmatrix}$ is derived.

(B) If $|Q(s)|$ is given and not $Q(s)$, as in the eigenvalue assignment problem, a proper $X^{-1}Y$ is derived by solving equations of the form

$$[XD + YN](s_j)a_j = 0. \tag{13}$$

where s_j are the desired eigenvalues and a_j (almost) arbitrary vectors; the choices for a_js correspond to different $Q(s)$ satisfying $Q(s_j)a_j=0$. Eigenvalue assignment is also accomplished using directly the transfer matrix $P=ND^{-1}$ and the equation

$$[X + YP](s_j)a_j = 0. \tag{14}$$

(C) Solutions (X,Y) of

$$[X,Y]\begin{bmatrix}D\\N\end{bmatrix} = WR \tag{15}$$

where R given and W arbitrary are also obtained. If D, N as in (A) and $R = \mathrm{diag}[s+1]$ then for (s_j,a_j) satisfying $R(s_j)a_j = 0$ [9]

$X(s) = \begin{bmatrix}s+3/2 & 1/2\\s+1/2 & s+1/2\end{bmatrix}$ and $Y(s) = \begin{bmatrix}s+1 & s\\s & 1\end{bmatrix}$.

(D) Stable rational matrices $\underline{X}$ such that $(I-\underline{X}N)D^{-1}$ is stable are obtained via interpolation. Note that this condition must be satisfied when characterizing the stabilizing feedback controllers using the parameter $\underline{X}$ [12]. Write $\underline{X}(s) = X(s)/d(s)$ where $d(s)$ Hurwitz and solve

$$[I - (1/d(s_i))X(s_i)N(s_i)]a_i = 0 \tag{16}$$

where $D(s_i)a_i=0$ with s_i all the unstable poles of D^{-1}.

(E) Eigenvalue assignment via state and constant output feedback is also accomplished.

The interactive computer program [9] is written in Fortran 77 and it is run on an IBM-3033 computer. LINPACK is used for solving the linear equations and testing the singular values and condition number of the matrices. A complete description of the polynomial matrix interpolation theory with applications is under preparation and will be available shortly.

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