43rd IEEE Conference on Decision and Control
December 14-17, 2004
Atlantis, Paradise Island, Bahamas

Persistent Disturbance Attenuation Properties for Networked Control Systems

Hai Lin and Panos J. Antsaklis

Abstract— **In this paper, both the asymptotic stability and l^∞ persistent disturbance attenuation issues are investigated for a class of Networked Control Systems (NCSs) under bounded uncertain access delay and packet dropout effects. The basic idea is to formulate such NCSs as discrete-time switched systems with arbitrary switchings. Then the NCSs' stability and performance problems can be reduced to the corresponding problems of such switched systems. The asymptotic stability problem is considered first, and a necessary and sufficient condition is derived for the NCSs' asymptotic stability based on robust stability techniques. Secondly, the NCSs' l^∞ persistent disturbance attenuation properties are studied, and an algorithm is proposed to calculate the l^∞ induced gain of the NCSs. The decidability issue of the algorithm is also discussed.**

I. Introduction

By Networked Control Systems (NCSs), we mean feedback control systems where networks, typically digital band-limited serial communication channels, are used for the connections between spatially distributed system components like sensors and actuators to controllers. In traditional feedback control systems these connections are established by point-to-point cables. Compared with the point-to-point cables, the introduction of digital communication networks has many advantages, such as high system testability and resource utilization, as well as low weight, space, power and wiring requirements [12], [14]. These advantages make the networks connecting sensors/actuators to controllers more and more popular in many applications, including traffic control, satellite clusters, mobile robotics, etc. Recently, modeling, analysis and control of networked control systems with limited communication capability has emerged as a topic of significant interest to control community, see for example [4], [14], [1], [7].

Time delay typically has negative effects on the NCSs' stability and performance. There are several situations where time delay may arise. First, transmission delay is caused by the limited bit rate of the communication channels. Secondly, the channel in NCSs is usually shared by multiple sources of data, and the channel is usually multiplexed by a time-division method. Therefore, there are delays caused by a node waiting to send out a message through a busy channel, which is usually called accessing delay and serves as the main source of delays in NCSs. There are also some delays caused by processing and propagation, which are usually negligible for NCSs. Another interesting problem in NCSs is the packet dropout phenomena. Because of the uncertainties and noise in the communication channels, there may exist unavoidable errors in the transmitted packet or even loss. If this happens, the corrupted packet is dropped and the receiver (controller or actuator) uses the packet that it received most recently. In addition, packet dropout may occur when one packet, say sampled values from the sensor, reaches the destination later than its successors. In such situation, the old packet is dropped, and its successive packet is used instead. There is another important issue in NCSs, that is the quantization effect. With the finite bit-rate constraints, quantization has to be taken into consideration in NCSs. Therefore, quantization and limited bit rate issues have attracted many researchers' attention, see for example [4], [7].

In this paper, the asymptotic stability and l^∞ persistent disturbance attenuation properties for a class of NCSs under bounded uncertain access delay and packet dropout effects are investigated. The basic idea is to formulate such NCSs as discrete-time switched systems with arbitrary switching signals. Then the NCSs' stability and performance problems can be studied in the switched system framework. The strength of this approach comes from the solid theoretic results existing in the literature of switched systems. By a switched system, we mean a hybrid dynamical system consisting of a finite number of subsystems described by differential or difference equations and a logical rule that orchestrates switching between these subsystems. Properties of this type of model have been studied for the past fifty years to consider engineering systems that contain relays and/or hysteresis. Recently, there has been increasing interest in the stability analysis and switching control design of switched systems (see, e.g., [8], [6] and the references cited therein).

The paper is organized as follows. First, the assumptions on the network link layer of the NCSs are described in Section II, and the NCSs with bounded uncertain access delay and packet dropout effects are modeled as a class of discrete-time switched linear systems with arbitrary switchings in Section III. Secondly, the stability for such NCSs is studied in Section IV, and a necessary and sufficient matrix norm condition is derived for the NCSs' global asymptotic stability. Thirdly, the persistent disturbance attenuation properties for such NCSs are studied in Section V, and a non-conservative bound of the l^∞ induced gain for the NCS is calculated. The techniques are based on the recent progress on robust performance of switched systems [10]. A networked controlled perturbed integrator is used throughout the paper for illustration. Finally, concluding remarks are presented.

Notation: The letters $\mathcal{E}, \mathcal{P}, \mathcal{S} \cdots$ denote sets, $\partial\mathcal{P}$ the

The partial support of the National Science Foundation (NSF CCR01-13131) is gratefully acknowledged.

H. Lin and Panos J. Antsaklis are both with the Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA. E-mail: {hlin1,antsaklis.1}@nd.edu

0-7803-8682-5/04/$20.00 ©2004 IEEE

boundary of set $\mathcal{P}$, and $int\{\mathcal{P}\}$ its interior. A bounded polyhedral set $\mathcal{P}$ will be presented either by a set of linear inequalities $\mathcal{P} = \{x : F_i x \leq g_i,\ i = 1, \cdots, s\}$, and compactly by $\mathcal{P} = \{x : Fx \leq g\}$, or by the dual representation in terms of the convex hull of its vertex set $vert\{\mathcal{P}\} = \{x_j\}$, denoted by $Conv\{x_j\}$. For $x \in \mathbb{R}^n$, the l^1 and l^∞ norms are defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_i |x_i|$ respectively. l^∞ denotes the space of bounded vector sequences $h = \{h(k) \in \mathbb{R}^n\}$ equipped with the norm $\|h\|_{l^\infty} = \sup_i \|h_i(k)\|_\infty < \infty$, where $\|h_i(k)\|_\infty = \sup_{k\geq 0} |h_i(k)|$.

II. The Access Delay and Packet Dropout

For the network link layer, we assume that the delays caused by processing and propagation are ignored, and we only consider the access delay which serves as the main source of delays in NCSs. Dependent on the data traffic, the communication bus is either busy or idle (available). If the link is available, the communication between sender and receiver is instantaneous. Errors may occur during the communication and destroy the packet, and this is considered as a packet dropout.

For simplicity, but without loss of generality, we may combine all the time delay and packet dropout effects into the sensor to controller path and assume that the controller-actuator communicates ideally.

We assume that the plant can be modeled as a continuous-time linear time-invariant system described by

$$\begin{cases} \dot{x}(t) = A^c x(t) + B^c u(t) + E^c d(t) \\ z(t) = C^c x(t) \end{cases} \quad t \in \mathbb{R}^+ \tag{1}$$

where $\mathbb{R}^+$ stands for nonnegative real numbers, $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is control input, and $z(t) \in \mathbb{R}^p$ is the controlled output. The disturbance input $d(t)$ is contained in $\mathcal{D} \subset \mathbb{R}^r$. $A^c \in \mathbb{R}^{n\times n}$, $B^c \in \mathbb{R}^{n\times m}$ and $E^c \in \mathbb{R}^{n\times r}$ are constant matrices related to the system state, and $C^c \in \mathbb{R}^{p\times n}$ is the output matrix.

For the above NCS, it is assumed that the plant output node (sensor) is time driven. In other words, after each clock cycle (sampling time T_s), the output node attempts to send a packet containing the most recent state (output) samples. If the communication bus is idle, then the packet will be transmitted to the controller. Otherwise, if the bus is busy, then the output node will wait for some time, say $\varpi < T_s$, and try again. After several attempts or when newer sampled data become available, if the transmission still can not be completed, then the packet is discarded, which is also considered as a packet dropout. On the other hand, the controller and actuator are event driven and work in a simpler way. The controller, as a receiver, has a receiving buffer which contains the most recently received data packet from the sensors (the overflow of the buffer may be dealt with as packet dropouts). The controller reads the buffer periodically at a higher frequency than the sampling frequency, say every $\frac{T_s}{N}$ for some integer N large enough. Whenever there are new data in the buffer, the controller will calculate the new control signal and transmit it to the actuator. According to the assumption, the controller-actuator communicates without delay or packet dropouts. Upon the new control signal arrival, the actuator updates the output of the Zero-Order-Hold (ZOH) to the new value.

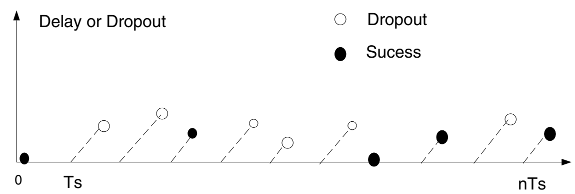

Fig. 1. The illustration of uncertain time delay and packet dropout of Networked Control Systems.

Based on the above assumptions and discussions, a typical time delay and packet dropout pattern can be shown in Figure 1. In this figure, the small bullet, $\bullet$, stands for the packet being transmitted successfully from the sensor to the controller's receiving buffer, maybe with some delay, and being read by the controller, at some time $t = kT_s + h\frac{T_s}{N}$ (k and h are integers), and the new control signal is updated in the actuator instantly. The actuator will hold this new value until the next update control signal comes. The symbol, $\circ$, denotes the packet being dropped, due to error, bus being busy, conflict or buffer overflow etc.

III. Switched System Model for NCSs

In this section, we will consider the sampled-data model of the plant. Because we do not assume the synchronization between the sampler and the digital controller, the control signal is no longer of constant value within a sampling period. Therefore the control signal within a sampling period has to be divided into subintervals corresponding to the controller's reading buffer period, $T = \frac{T_s}{N}$. Within each subinterval, the control signal is constant under the assumptions of the previous section. Hence the continuous-time plant may be discretized into the following sampled-data systems

$$x[k+1] = Ax[k] + \underbrace{[B\ B\ \cdots B]}_{N} \begin{bmatrix} u^1[k] \\ u^2[k] \\ \vdots \\ u^N[k] \end{bmatrix} + Ed[k] \tag{2}$$

where $A = e^{A^c T_s}$, $B = \int_0^{\frac{T_s}{N}} e^{A^c \eta} B^c d\eta$ and $E = \int_0^{T_s} e^{A^c \eta} E^c d\eta$. And the controlled output $z[k]$ is given by

$$z[k] = Cx[k] \tag{3}$$

where $C = C^c$. Note that for a linear time-invariant plant and constant-periodic sampling, the matrices A, B, C and E are constant.

A. Modeling Uncertain Access Delay

During each sampling period, there are several different cases that may arise.

First, if the delay $\tau = h \times T$, where $T = \frac{T_s}{N}$, and $h = 1, 2, \cdots, d_{max}$[1], then $u^1[k] = u^2[k] = \cdots = u^h[k] = u[k-1]$, $u^{h+1}[k] = u^{h+2}[k] = \cdots = u^N[k] = u[k]$, and (2) can be written as:

$$\begin{aligned} x[k+1] &= Ax[k] + [B\ B\ \cdots B] \begin{bmatrix} u[k-1] \\ \vdots \\ u[k-1] \\ u[k] \\ \vdots \\ u[k] \end{bmatrix} + Ed[k] \\ &= Ax[k] + h \cdot Bu[k-1] + (N-h) \cdot Bu[k] + Ed[k] \end{aligned} \quad (4)$$

Note that $h = 0$ implies $\tau = 0$, which corresponding to "no delay" case.

Secondly, if a packet-dropout happens, which may be due to a corrupted packet or fail in sending out with delay less than τ_{max}, then the actuator will implement the previous control signal, i.e. $u^1[k] = u^2[k] = \cdots = u^N[k] = u[k-1]$. Therefor, the state transition equation (2) for this case can be written as:

$$\begin{aligned} x[k+1] &= Ax[k] + [B\ B\ \cdots B] \begin{bmatrix} u[k-1] \\ u[k-1] \\ \vdots \\ u[k-1] \end{bmatrix} + Ed[k] \\ &= Ax[k] + N \cdot Bu[k-1] + Ed[k] \end{aligned} \quad (5)$$

In the following, we will model the uncertain multiple successive packet dropouts.

B. Modeling Packet Dropout

Here, we assume that the maximum number of the consecutive dropped packets is bounded, say by an integer D_{max}. In this subsection, we will analyze the bounded uncertain packet dropout pattern and formulate the NCSs as switched systems with arbitrary switchings.

We first consider the simplified case when the packets are dropped periodically, with period T_m. Note that T_m is integer times of the sampling period T_s, i.e. $T_m = mT_s$. In case of $m = \frac{T_m}{T_s} \geq 2$, the first $(m-1)$ packets are dropped. Then, for these first $(m-1)$ steps, the previous control signal is used. Therefore

$$\begin{aligned} x(kT_m + T_s) &= Ax(kT_m) + NBu(kT_m - T_s) + Ed(kT_m) \\ x(kT_m + 2T_s) &= A^2 x(kT_m) + N \cdot (AB + B)u(kT_m - T_s) \\ &\quad + AEd(kT_m) + Ed(kT_m + T_s) \\ &\vdots \\ x(kT_m + (m-1)T_s) &= A^{m-1}x(kT_m) + N \sum_{i=0}^{m-2} A^i Bu(kT_m - T_s) \\ &\quad + [A^{m-2}E, \cdots, E] \begin{bmatrix} d(kT_m) \\ \vdots \\ d(kT_m + (m-2)T_s) \end{bmatrix} \end{aligned}$$

Note that the integer $N = \frac{T_s}{T}$, where T is the period of the controller reading its receiving buffers. During the period $t \in [kT_m + (m-1)T_s, (k+1)T_m)$, the new packet is transmitted successfully with some delay, say $\tau = h\frac{T_s}{N}$, where $h = 0, 1, 2, \cdots, d_{max}$. Let us assume that $d(kT_m) = d(kT_m + 1) = \cdots = d(kT_m + m - 1)$, and that the controller uses just the time-invariant linear feedback control law, $u(t) = Kx(t)$. Then, we may obtain

$$\begin{aligned} x((k+1)T_m) &= [A^m + (N-h)BKA^{m-1}]x(kT_m) \\ &+ [N\textstyle\sum_{i=1}^{m-1} A^i + (N-h)NBK \sum_{i=0}^{m-2} A^i + h]BKx(kT_m - T_s) \\ &+ [(N-h)BK\textstyle\sum_{i=0}^{m-2} A^i + \sum_{i=0}^{m-1} A^i]Ed(kT_m) \end{aligned}$$

If we let $\hat{x}[k] = \begin{bmatrix} x(kT_m - T_s) \\ x(kT_m) \end{bmatrix}$, then the above equations can be written as:

$$\begin{aligned} \hat{x}[k+1] &= \begin{bmatrix} x((k+1)T_m - T_s) \\ x((k+1)T_m) \end{bmatrix} \\ &= \Phi_{(m,h)} \begin{bmatrix} x(kT_m - T_s) \\ x(kT_m) \end{bmatrix} + E_m d(kT_m) \end{aligned}$$

where $\Phi_{(m,h)}$ equals to

$$\begin{bmatrix} N\sum_{i=0}^{m-2} A^i BK & A^{m-1} \\ \begin{array}{c} N\sum_{i=1}^{m-1} A^i BK + \\ (N-h)NBK\sum_{i=0}^{m-2} A^i + h)BK \end{array} & A^m + (N-h)BKA^{m-1} \end{bmatrix}$$

and

$$E_m = \begin{bmatrix} \sum_{i=0}^{m-2} A^i E \\ (N-h)BK\sum_{i=0}^{m-2} A^i E + \sum_{i=0}^{m-1} A^i E \end{bmatrix}.$$

Here $m = \frac{T_m}{T_s} \geq 2$ in this case, and $h = 0, 1, \cdots, d_{max}$.

For the case of $m = 1$, namely no packet dropout, the following dynamic equation is derived:

$$\hat{x}[k+1] = \Phi_{(1,h)}\hat{x}[k] + E_1 d[k],$$

where
$\Phi_{(1,h)} = \begin{bmatrix} 0 & I \\ hBK & A + (N-h)BK \end{bmatrix}$, $E_1 = \begin{bmatrix} 0 \\ E \end{bmatrix}$.

For the the case of aperiodic packet dropouts, one may look the delay and packet dropout pattern (Figure 1) of the NCS as a succession of ramps of various length ($T_{m_1} + h_1$, $T_{m_2} + h_2, \cdots, T_{m_k} + h_k, \cdots$). Therefore, the NCS along with a typical aperiodic delay and packet dropout pattern can be seen as a dynamical system switching among the dynamics with different periodic delay and packet dropout pattern $\Phi_{(m,h)}$, for $m = 1, \cdots, D_{max}$ and $h = 0, 1, 2, \cdots, d_{max}$. This observation motivate us to model the NCS as a switched systems as

$$\begin{cases} \hat{x}[k+1] = \Phi_{(m,h)}\hat{x}[k] + E_m d[k] \\ z[k] = \begin{bmatrix} C & 0 \end{bmatrix} \hat{x}[k] \end{cases} \quad (6)$$

where $\Phi_{(m,h)} \in \{\Phi_{(1,0)}, \Phi_{(1,1)}, \cdots, \Phi_{(1,D_{max})}, \Phi_{(2,0)}, \cdots, \Phi_{(D_{max},0)}, \cdots, \Phi_{(D_{max},d_{max})}\}$. Here D_{max} corresponds to the maximum number of successively dropped packets, and d_{max} is the maximum access delay. For notational simplicity, let us denote $q = m + h \times D_{max}$ as the index of all the subsystems, and call the collection $\{1, 2, \cdots, D_{max} \times (d_{max} + 1)\}$ the mode set Q, $q \in Q$. Therefore, we rewrite (6) as

$$\begin{cases} \hat{x}[k+1] = \Phi_q \hat{x}[k] + E_q d[k] \\ z[k] = \begin{bmatrix} C & 0 \end{bmatrix} \hat{x}[k] \end{cases} \quad (7)$$

[1]The value of d_{max} is determined as the least integer greater than the positive scalar $\frac{\tau_{max}}{T}$, where τ_{max} stands for the maximum access delay.

Associate (7) with a class of piecewise constant functions of time $\sigma : \mathbb{Z}^+ \to Q$, which is called switching signals. Note that each switching signal σ corresponds to a (maybe aperiodic) delay and packet dropout pattern. In order to study the effects of bounded uncertain access delay and packet dropouts on the NCSs' stability and performance, one needs to consider all possible delay and packet dropout patterns, which corresponds to considering the arbitrary switching signals for (7). Therefore, the stability and performance problems for the NCS are equivalent to the corresponding problems for the switched system (7) with arbitrary switchings. To illustrate the idea, let us see an example.

Example 1: Consider the following continuous-time perturbed integrator as the plant

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} d(t) \\ z(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \end{aligned}$$

Assume that the sampling period T_s is 0.1 second. The controller reads the receiving buffer every $T = 0.01s$, i.e. $N = \frac{T_s}{T} = 10$. It is assumed that the sensor only tries to send the new sampled state value during the first $0.02s$ of each sampling period T_s, from which we may obtain that the maximum delay (if successively arrived) is $\tau_{max} = 0.02s$ and $d_{max} = \frac{0.02}{T} = 2$. Also assume that at most three successive packet-dropouts can occur, namely $D_{max} = 4$. Therefore, the above NCS can be modeled as an arbitrary switching system with $D_{max} \times (d_{max} + 1) = 12$ modes. The state matrices for each modes can be determined by plugging the following matrices

$$A = e^{A^c T_s} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \ B = \int_0^T e^{A^c t} B^c dt = \begin{bmatrix} 0.00005 \\ 0.01 \end{bmatrix}$$

$$E = \int_0^{T_s} e^{A^c t} E^c dt = \begin{bmatrix} 0.105 \\ 0.1 \end{bmatrix}, \quad K = \begin{bmatrix} -2 & -1 \end{bmatrix}$$

into the expression of $\Phi_{(m,h)}$ and E_m for all possible values of $m \in \{1, 2, 3, 4\}$ and $h = \{0, 1, 2\}$. For instance, the mode corresponding to the case of two successive packet dropouts ($m = 3$) and the third packet arriving with delay $0.02s$ ($h = 2$), i.e., the eleventh mode ($2 \times D_{max} + 3 = 11$), can be described as

$$\hat{x}[k+1] = \begin{bmatrix} -0.0220 & -0.0110 & 1.0000 & 0.2000 \\ -0.4000 & -0.2000 & 0 & 1.0000 \\ -0.1020 & -0.0510 & 0.9992 & 0.2994 \\ -0.4047 & -0.2023 & -0.1600 & 0.8880 \end{bmatrix} \hat{x}[k] + \begin{bmatrix} 0.2200 \\ 0.2000 \\ 0.2399 \\ 0.1736 \end{bmatrix} d[k]$$

$$z[k] = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \hat{x}[k]$$

Now we have modeled the NCS with uncertain access delay, packet dropout effects as a switched system (7) with arbitrary switching between its $\mathbf{N} = D_{max} \times (d_{max} + 1)$ modes. In the following sections, we will study the asymptotic stability and disturbance attenuation properties of such NCSs within the framework of switched systems. For notational simplicity, we will write $\hat{x}$ as x in the sequel.

IV. Stability Analysis

The aim of this paper is to investigate the effects of the uncertain access delay and packet dropouts on the persistent disturbance attenuation prosperities, namely the l^∞ induced norm from $d[k]$ to $z[k]$, for the NCSs (7). For such purpose, it is assumed that the disturbance $d[k]$ is contained in the l^∞ unit ball, i.e., $\mathcal{D} = \{d : \|d\|_{l^\infty} \leq 1\}$. The l^∞ induced norm from $d[k]$ to $z[k]$ is defined as

$$\mu_{inf} = \inf\{\mu : \|z[k]\|_{l^\infty} \leq \mu, \ \forall d[k], \ \|d[k]\|_{l^\infty} \leq 1\}$$

The first problem we need to answer is

Problem 1: Under what condition the l^∞ induced norm from $d[k]$ to $z[k]$ for the the NCSs with bounded uncertain access delay and packet dropouts is finite?

The answer for Problem 1 is equivalent to the condition for an arbitrarily switching system in form of (7) to have a finite l^∞ induced gain. In [10], it is shown that a necessary and sufficient condition for an arbitrarily switching system (7) to have a finite μ_{inf} is that the corresponding autonomous switched system $x[k+1] = \Phi_\sigma x[k]$ is asymptotically stable under arbitrary switching signals. Therefore, Problem 1 is transformed into a stability analysis problem for autonomous switched system under arbitrary switchings, which has been studied in the literature extensively, and is typically being dealt with by constructing a common Lyapunov function; see the survey papers [8], [6] and the references cited therein. Various attempts have been made to find a common Lyapunov function for the family of systems, ensuring the asymptotic stability of switched systems for any switching signal [5]. However, most of the work has been restricted to the case of common quadratic Lyapunov function [13], [9], which only give sufficient stability test criteria except for some special cases like pairwise community, symmetric or normal. Here, a necessary and sufficient condition is given for asymptotic stability of arbitrary switching systems.

For such purpose, let us first introduce a technical lemma [2] for the robust stability of linear time variant systems

$$x[k+1] = \Phi(k)x[k] \tag{8}$$

where $\Phi(k) \in \mathcal{A} \hat{=} Conv\{\ \Phi_1,\ \Phi_2,\ \cdots,\ \Phi_{\mathbf{N}}\}$

Lemma 1: The polytopic uncertain linear time-variant system (8) is globally asymptotically stable *if and only if* there exists a finite n such that $\|\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_n}\| < 1$ for all n-tuple $\Phi_{i_j} \in vert\{\mathcal{A}\} = \{\Phi_1, \Phi_2, \cdots, \Phi_{\mathbf{N}}\}$, for $j = 1, \cdots, n$.

Here the norm $\|\cdot\|$ stands for either 1 norm or ∞ norm of a matrix. Asymptotic stability of the switched NCS (7) can be expressed as the following proposition.

Proposition 1: A switched linear system $x[k+1] = \Phi_{\sigma(k)}x[k]$, where $\Phi_{\sigma(k)} \in \{\Phi_1,\ \Phi_2,\ \cdots,\ \Phi_{\mathbf{N}}\}$, is globally asymptotically stable under arbitrary switchings *if and only if* there exists a finite n such that

$$\|\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_n}\| < 1, \quad \forall \Phi_{i_j} \in \{\Phi_1, \Phi_2, \cdots, \Phi_{\mathbf{N}}\}, \quad (9)$$

for $j = 1, \cdots, n$.

The proof is omitted here for space limit. For the NCS example considered in the previous section, we tested the matrix norm condition

$$\|\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_{24}}\|_\infty < 1, \quad \forall \Phi_{i_j} \in \{\Phi_1, \Phi_2, \cdots, \Phi_{12}\},$$

which holds for $j = 1, \cdots, 24$. Therefore, by Proposition 1, the NCS is globally asymptotically stable with the bounded uncertain access delays and packet dropouts.

In the sequel, we limit our attention to asymptotically stable switched systems under arbitrary switchings. As a byproduct, we prove the *equivalence* between the robust asymptotic stability for polytopic uncertain linear time-variant systems and the asymptotic stability for switched linear systems with arbitrary switchings. It is quite interesting that the study of robust stability of a polytopic uncertain linear time-variant system, which has infinite number of possible dynamics (modes), is equivalent to only considering a finite number of its vertex dynamics as an arbitrary switching system. Although we only prove the equivalence in the discrete-time case, this result is also true in the continuous-time case. This fact bridges two originally distinct research fields. Therefore, existing results in the robust stability area, which has been extensively studied for over two decades, can be directly introduced to study the arbitrarily switching systems and vice versa.

V. Disturbance Attenuation Property

After the above discussion on the conditions for μ_{inf} to be finite, we shall now calculate a non-conservative bound on μ_{inf} for the NCS with bounded uncertain access delay and packet dropouts. This leads to the second problem studied in this paper.

Problem 2: Determine the minimal l^∞ induced norm (if exists) from $d[k]$ to $z[k]$ for NCSs with bounded uncertain access delay and packet dropouts.

To solve this problem, we consider the disturbance attenuation performance that the switched system (7) can preserve under arbitrary switchings. We will calculate a non-conservative bound on μ_{inf} for the arbitrarily switching system (7). The techniques are based on the positive invariant set theory and our recent results on robust performance for switched linear systems [10].

For such purpose, we first introduce the definition of a *positive disturbance invariant set* for the switched system (7) under arbitrary switching signals.

Definition 1: A set $\mathcal{P}$ in the state space is said to be *positive disturbance invariant* for the switched system (7) with arbitrary switchings if for every initial condition $x[0] \in \mathcal{P}$ we have that $x[k] \in \mathcal{P}$, $k \geq 0$, for every possible switching signal $\sigma(k)$ and every admissible disturbance $d[k] \in \mathcal{D}$.

We now formalize the definition of a limit set.

Definition 2: The limit set $\mathcal{L}$ for the switched system (7) with arbitrary switchings is the set of states x for which there exist a switching sequence $\sigma(k)$, admissible sequence $d[k]$ and a non-decreasing time sequence t_k such that

$$\lim_{k\to+\infty} \Xi(0, t_k, \sigma(\cdot), d[\cdot]) = x$$

where $\lim_{k\to+\infty} t_k = +\infty$ and $\Xi(0, t_k, \sigma(\cdot), d[\cdot])$ denotes the value at the instant t_k of the solution of (7) originating at $x_0 = 0$ and corresponding to σ and d.

The limit set $\mathcal{L}$ has the following property.

Lemma 2: Under the asymptotic stability assumption, the limit set $\mathcal{L}$ is non-empty and the state evolution of the switched system (7), for every initial condition $x[0]$, all switching sequences $\sigma(k)$ and all admissible disturbances $d[k] \in \mathcal{D}$, converges to $\mathcal{L}$. Moreover, $\mathcal{L}$ is bounded and positive disturbance invariant for the switched system (7) with arbitrary switchings.

The boundedness and convergence of the limit set come from the asymptotic stability of the switched system under arbitrary switchings. The invariance can be easily shown by contradiction. The detailed proof is omitted here due to space limitation[2].

Define now the set

$$X_0(\mu) = \{x : \|Cx\|_\infty \leq \mu\} = \{x : \begin{bmatrix} C \\ -C \end{bmatrix} x \leq \begin{bmatrix} \bar{\mu} \\ \bar{\mu} \end{bmatrix}\}$$

where $\bar{\mu}$ stands for a column vector with μ as its elements. $X_0(\mu)$ is a polytope containing the origin in its interior.

A value $\mu < +\infty$ is said to be admissible for arbitrary switching signals if $\mu > \mu_{inf}$. Clearly, given $\mu > 0$, the response of the switched system satisfies $\|z[k]\|_{l^\infty} \leq \mu$ and $\|d[k]\|_{l^\infty} \leq 1$ if and only if switched system (7) admits a positive disturbance invariant set $\mathcal{P}$ under arbitrary switching such that $0 \in \mathcal{P} \subseteq X_0(\mu)$.

In the following, we provide a procedure to compute a positive disturbance invariant set, for arbitrary switching signals, containing in $X_0(\mu)$. This is accomplished by finding the maximal positive disturbance invariant set for switched system (7) under arbitrary switchings, i.e., a set contains any other positive disturbance invariant set under arbitrary switchings in $X_0(\mu)$.

Given a compact set $\mathcal{P} \subseteq \mathbb{R}^n$, we can define its predecessor set for switched systems (7) under arbitrary switchings, $pre(\mathcal{P})$, as all the states x that can reach the set $\mathcal{P}$ in the next step in spite of disturbances or switching signals. It can be calculated as

$$\underline{pre}(\mathcal{P}) = \bigcap_{q\in Q} pre_q(\mathcal{P}), \quad (10)$$

where $pre_q(\mathcal{P})$ stands for the predecessor set of the q-th subsystem, that is the set of all states x that are mapped into $\mathcal{P}$ by the transformation $\Phi_q x + E_q d$, for all admissible $d \in \mathcal{D}$.

By recursively defining the sets $\mathcal{P}^{(k)}$, $k = 0, 1, \cdots$ as

$$\mathcal{P}^{(0)} = X_0(\mu),\ \mathcal{P}^{(k)} = \mathcal{P}^{(k-1)} \bigcap \underline{pre}(\mathcal{P}^{(k-1)}) \quad (11)$$

[2]Similar concepts and lemma were previously given in [3] for uncertain linear time-varying systems. The results developed here are direct extensions to the switched systems.

it can be shown that $\mathcal{P}^{(\infty)}$ is the maximal positive disturbance invariant set under arbitrary switching in $X_0(\mu)$. We now introduce a lemma guaranteeing that this set can be expressed by a finite set of linear inequalities (i.e. it is a polyhedral) and thus can be finitely determined.

Proposition 2: Under the asymptotic stability assumption, if $\mathcal{L} \subset int\{X_0(\mu)\}$ for some $\mu > 0$, then there exists $\mathbf{k}$ such that $\mathcal{P}^{(\infty)} = \mathcal{P}^{(\mathbf{k})}$ and this $\mathbf{k}$ can be selected as the smallest integer such that $\mathcal{P}^{(\mathbf{k+1})} = \mathcal{P}^{(\mathbf{k})}$.

In order to check whether a given performance level $\mu > 0$ is admissible for the switched system under arbitrary switchings, one may compute the maximal positive disturbance invariant set $\mathcal{P}^{(\infty)}$ in $X_0(\mu)$ and check whether or not $\mathcal{P}^{(\infty)}$ contains the origin. If yes, then $\mu > \mu_{inf}$, otherwise $\mu < \mu_{inf}$. Note that in both cases we get an answer in a finite number of steps. In the first case, this is due to the above proposition. In the second case, this comes from the fact that the sequence of closed sets $\mathcal{P}^{(k)}$ is ordered by inclusion and $\mathcal{P}^{(\infty)}$ is their intersection. Thus $0 \notin \mathcal{P}^{(\infty)}$ if and only if $0 \notin P^{(k)}$ for some k. Thus checking whether $\mu > \mu_{inf}$ can be obtained by starting from the initial set $X_0(\mu)$ and computing the sequence of sets $\mathcal{P}^{(k)}$ until some appropriate stopping criterion is met. In addition, we have another stop criterion.

Proposition 3: If the set $\mathcal{P}^{(k)} \subset int\{X_0(\mu)\}$ for some k, then the switched system (7) does not admit a positive disturbance invariant set under arbitrary switchings in $X_0(\mu)$. In other words, $\mu < \mu_{inf}$.

These results suggest the following constructive procedure for finding a robust performance bound.

Procedure 1. Checking whether $\mu > \mu_{inf}$

1) Initialization: Set $k = 1$ and set $\mathcal{P}^{(0)} = X_0(\mu)$.
2) Compute the set $\mathcal{P}^{(k)} = \mathcal{P}^{(k-1)} \bigcap \underline{pre}(\mathcal{P}^{(k-1)})$.
3) If $0 \notin \mathcal{P}^{(k+1)}$ or $\mathcal{P}^{(k)} \subset int\{X_0(\mu)\}$ then stop, the procedure has failed. Thus, the output does not robustly meet the performance level μ.
4) If the $\mathcal{P}^{(k+1)} = \mathcal{P}^{(k)}$, then stop, and set $\mathcal{P}^{(\infty)} = \mathcal{P}^{(k)}$.
5) Set $k = k + 1$ and go to step 1.

This procedure can then be used together with a bisection method on μ to approximate arbitrarily close to the optimal value μ_{inf}, which solves the Problem 2. In fact, if the procedure stops at step 3, we conclude that $\mu < \mu_{inf}$ and we can increase the value of the output bound μ. Otherwise, if the procedure stops at step 4, we have determined an admissible bound for the output, say $\mu > \mu_{inf}$, that can be decreased.

Example 2: We calculate a non-conservative bound of μ_{inf} for the switched NCSs under arbitrary switching sequences. Using the bisection method (with error tolerance $\epsilon = 0.01$), we obtain that $\mu_{inf} = 0.809$.

VI. Concluding Remarks

In this paper, we considered a class of Networked Control Systems (NCSs) affected by bounded uncertain access delay and packet dropouts, and we modeled them as discrete-time switched linear systems with arbitrary switchings. Then, the stability and persistent disturbance attenuation issues for such NCSs were studied in the framework of arbitrarily switching systems. A necessary and sufficient condition was derived for the NCSs' asymptotic stability. In addition, the equivalence between the asymptotic stability of arbitrary switching linear systems and the robust stability of a corresponding linear time-variant systems was obtained, thus bridging two originally distinct research fields. Secondly, based on recent progress in the robust performance study of switched systems, the NCSs' l^∞ persistent disturbance attenuation properties were studied as well.

VII. Acknowledgements

The first author appreciates the support from Center of Applied Mathematics Fellowship (2003-04), University of Notre Dame. The authors would like to thank Dr. Peter. H. Bauer, Mr. Lei Fang, and Dr. Guisheng Zhai for their valuable comments.

References

[1] P. Bauer, C. Lorand, M. Sichitiu, and K. Premaratne, "Total delay compensation in LAN control systems and implications for scheduling," in *Proc. of 2001 American Contr. Conf.*, pp. 4300-4305, Arlington, VA, 2001.

[2] P. Bauer, K. Premaratne, and J. Duran, "A necessary and sufficient condition for robust asymptotic stability of time-variant discrete systems," *IEEE Trans. Automat. Contr.*, vol. 38, no. 9, pp. 1427-1430, 1993.

[3] F. Blanchini, S. Miani, and M. Sznaier, "Robust performance with fixed and worst case signals for uncertain time-varying systems," *Automatica*, vol. 33, no. 12, pp. 2183-2189, 1997.

[4] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Automat. Contr.*, vol. 45, no. 7, pp. 1279-1289, 2000.

[5] W. Dayawansa and C. F. Martin, "A converse Lyapunov theorem for a class of dynamical systems which undergo switching," *IEEE Trans. Automat. Contr.*, vol. 44, no. 4, pp. 751-760, 1999.

[6] R. A. Decarlo, M.S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," in *Proccedings of the IEEE*, vol. 88, no.7, pp. 1069-1082, 2000.

[7] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Automat. Contr.*, vol. 46, no. 9, pp. 1384-1400, 2001.

[8] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Contr. Syst. Magazine*, vol.19, no. 15, pp. 59-70, 1999.

[9] D. Liberzon, J. P. Hespanha, and A. S. Morse, "Stability of switched linear systems: A Lie-algebraic condition," *Syst. Contr. Lett.*, vol. 37, no. 3, pp. 117-122, 1999.

[10] H. Lin and P. J. Antsaklis, "Disturbance attenuation properties for discrete-time uncertain linear switched systems," in *Proc. 42nd IEEE Conf. Decision Control*, pp. 5289-5294, 2003.

[11] H. Lin, G. Zhai, and P. J. Antsaklis, "Robust stability and disturbance attenuation analysis of a class of networked control systems," in *Proc. 42nd IEEE Conf. Decision Control*, pp. 1182-1187, 2003.

[12] H. Ishii and B. A. Francis, *Limited Data Rate in Control Systems with Networks*, Lecture Notes in Control and Information Sciences, vol. 275, Springer, Berlin, 2002.

[13] R. N. Shorten and K. S. Narendra, "On the stability and existence of common Lyapunov functions for stable linear switching systems," in *Proc. 37th Conf. Decision Control*, pp. 3723-3724, 1998.

[14] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Contr. Syst. Magazine*, vol. 21, no. 1, pp. 84-99, 2001.