EXPLICIT HYBRID OPTIMAL CONTROLLER FOR DISTURBANCE ATTENUATION IN LINEAR HYBRID SYSTEMS

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Abstract: In this paper, the disturbance attenuation properties for classes of linear hybrid systems perturbed by exterior disturbances are investigated, and a hybrid $l^1$ robust optimal control problem is studied. First, a procedure is developed to determine the minimal $l^1$ norm of linear hybrid systems. However, for general hybrid systems, the termination of the procedure is not guaranteed. Then, the decidability issues are briefly discussed. Finally, we study the robust $l^1$ optimal controller synthesis problem. It is shown that the optimal performance level can be achieved by a piecewise linear state feedback control law for the linear hybrid systems, and a systematic approach to design such feedback control is proposed.

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Keywords: Piecewise linear controllers, State feedback, Hybrid modes, Disturbance rejection, Robust performance.

1. INTRODUCTION

The last decade has seen increasing research activities in hybrid/switched systems. However, the literature on robust control of hybrid/switched systems is still relatively sparse. In this paper, we will focus on the induced gain analysis and robust optimal control for classes of linear hybrid/switched systems which are perturbed by exterior disturbances.

There are some related works in the literature on analyzing the induced gain in switched systems. In (Zhai et al., 2001), the $L_2$ gain of continuous-time switched linear systems was studied using an average dwell time approach and piecewise quadratic Lyapunov functions, and the results were extended to discrete-time case in (Zhai et al., 2002). In (Hespanha, 2003), the root-mean-square (RMS) gain of a continuous-time switched linear system with slow switching was computed in terms of the solutions to a collection of Riccati equations. Both of these robust performance problems are in the signal's energy sense, and assume that the disturbances are constrained to have finite energy, i.e., bounded $L_2$ norm. In practice, there are disturbances that do not satisfy this condition and act more or less continuously over time. Such disturbances are called persistent, and can not be treated in the above framework. In this paper we consider $l^\infty$ induced gains to deal with the robust performance problems in the signal's magnitude sense, i.e., time domain specifications.

The persistent disturbance attenuation properties for hybrid/switched systems have been considered in our previous work (Lin and Antsaklis, 2003a; Lin and Antsaklis, 2003b; Lin and Antsaklis, 2004). In (Lin and Antsaklis, 2003c), a class of uncertain switched linear systems affected by both parameter variations and exterior disturbances was considered, and the uniformly ultimate boundedness control problem was studied for both discrete-time and continuous-time case. Under the assumption that each subsystem admits a finite persistent disturbance at-

In linear hybrid systems, it was shown in [Lin and Antsaklis, 2003c] that, by proper switching, the closed-loop switched systems can reach a better disturbance attenuation level than any single subsystem's. An optimal disturbance attenuation property for uncertain switched systems and its decidability issue were discussed in [Lin and Antsaklis, 2003a]. The results for optimal disturbance attenuation property analysis were extended to classes of general uncertain hybrid systems in [Lin and Antsaklis, 2004]. All of these previous works are analysis results on the disturbance attenuation property. This paper is an extension of [Lin and Antsaklis, 2004] and addresses the robust optimal hybrid controller synthesis problem.

This paper is organized as follows. In Section 2, we first define the linear hybrid systems with persistent external disturbances. Then, the $l^\infty$ induced gain analysis problem for the linear hybrid system and its robust optimal controller synthesis problem are formulated. After introducing some necessary preliminary results in Section 3, the $l^\infty$ induced gain analysis problem is investigated in Section 4, and a bisection based procedure is proposed to determine a non-conservative bound on the optimal disturbance attenuation level. The decidability issues of the proposed procedure are briefly discussed. The robust optimal controller synthesis problem is studied in Section 5. A systematic approach to design an explicit hybrid state feedback control law is introduced. It is interesting that the optimal performance level can be achieved by a piecewise linear state feedback control law, which is deduced from a conic partition of a (non-convex) performance region, for linear hybrid systems. The techniques are based on polyhedral algebra and linear programming. Finally, concluding remarks are made.

Notation: The letters $E, P, S \ldots$ denote sets, $\partial P$ the boundary of set $P$, and $\text{int}(P)$ its interior. A polytope (bounded polyhedral set) $P$ will be presented either by a set of linear inequalities $P = \{x : F_ix \leq g_i, i = 1, \ldots, s\}$, and compactly by $P = \{x : Fx \leq g\}$, or by the dual representation in terms of the convex hull of its vertex set $\{x_i\}$, denoted by $\text{Conv}(x_i)$. For $x \in \mathbb{R}^n$, the $l^1$ and $l^\infty$ norms are defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_i |x_i|$ respectively, $l^\infty$ denotes the space of bounded vector sequences $h = \{h(k) \in \mathbb{R}^n\}$ equipped with the norm $\|h\|_\infty = \sup_{k \geq 0} \|h_k\|_\infty < \infty$, where $\|h_k\|_\infty = \sup_{k \geq 0} |h_k|$.

2. PROBLEM FORMULATION

We consider discrete-time piecewise linear hybrid systems of the form

$$x(t+1) = A_x x(t) + B_x u(t) + Ed(t), \text{ if } x \in P_q \quad (1)$$

where $x(t) \in \mathbb{R}^n \ (t \in \mathbb{Z}^+)$ is the state variable, $u(t) \in \mathcal{U}_i \subset \mathbb{R}^m$ is the control input, and the disturbance input $d(t)$ is contained in $\mathcal{D} \subset \mathbb{R}^r$, the $l^\infty$ unit ball, i.e., $\mathcal{D} = \{d : \|d\|_\infty \leq 1\}$. It is assumed that $\mathcal{U}_i$ is polytopes assigned to each mode $q$. Let the finite set $Q$ stand for the collection of discrete modes $q$. The partition of the state space $\mathcal{X}$ is given as a finite set of polyhedra $\{P_q : q \in Q\}$, where $P_q \subset \mathcal{X}$ and $\bigcup_{q \in Q} P_q = \mathcal{X}$.

A possible evolution of the piecewise linear hybrid systems from a given initial condition $x_0 \in \mathcal{X}$ can be described as follows. First, there exists at least one discrete mode $q_0 \in Q$ such that $x_0 \in P_{q_0}$; let the mode $q_0$ be called the feasible mode for state $x_0$. Then the next continuous variable state is given by the transition $x_1 = A_{q_0} x_0 + B_{q_0} u + Ed$ for some possible disturbance $d \in \mathcal{D}$ and specific $u \in \mathcal{U}_{q_0}$. Then the above procedure is repeated for state $x_1$ to determine the next possible state $x_2$, and so on.

Associated with the piecewise linear hybrid system (1), a controlled output $z(t)$ is considered.

$$z(t) = Cx(t) \quad (2)$$

where $C \in \mathbb{R}^{p \times n}$ and $z(t) \in \mathbb{R}^p$.

For this linear hybrid system (1)-(2), we are interested in determining a non-conservative bound for the $l^\infty$ induced norm from $d(t)$ to $z(t)$, which is defined as

$$\mu_{inf} = \inf\{\mu : \exists q \in Q, u \in \mathcal{U}_q : \|z(t)\|_\infty \leq \mu \|d(t)\|_\infty \leq 1\}$$

The first problem considered in this paper can be formulated as follows.

**Problem 1.** (Robust Performance Analysis). Given the piecewise linear hybrid system (1)-(2), determine the minimal $l^\infty$ induced gain from $d(t)$ to $z(t)$ that can be achieved by some admissible control law.

The second problem is to construct such a control law, including a rule for active mode selecting and an admissible continuous variable control signal, that guarantees the $l^\infty$ induced gain from $d(t)$ to $z(t)$, $\mu_{inf}$, being obtained (assume proper initial conditions). This problem can be formulated as follows.

**Problem 2.** (Robust Optimal Control). Given linear hybrid systems (1)-(2), construct an admissible control law, such that the minimal $l^\infty$ induced gain from $d(t)$ to $z(t)$, $\mu_{inf}$, is achieved.

The robust optimal controller synthesis problem is studied in Section 5. It is interesting to notice that the optimal performance level can be achieved by a piecewise linear state feedback control law.

3. PRELIMINARY RESULTS

The basic idea employed in this paper is to translate the required level of performance into consider.
4. ROBUST PERFORMANCE ANALYSIS

In this section, we will focus on the first problem and determine the minimal l∞ induced gain from
\( d(t) \) to \( x(t) \) that can be achieved by some admissible control laws for the closed-loop linear hybrid
systems. For such purpose, we first introduce the performance level \( \mu \) set as

\[
\Omega_\mu = \{ x : \| C x \|_\infty \leq \mu \} = \{ x : \begin{bmatrix} C \\ -C \end{bmatrix} x \leq \begin{bmatrix} \bar{\mu} \\ \bar{\mu} \end{bmatrix} \},
\]

where \( \bar{\mu} \) stands for a column vector with \( \mu \) as its elements. Note that \( \Omega_\mu \) is a polytope containing
the origin in its interior.

A value \( \mu < +\infty \) is said to be admissible if \( \mu > \mu_{\text{inf}} \). Clearly, a sufficient condition for \( \mu \) to be
admissible is that the hybrid performance level set \( \Omega_\mu \) is controlled robust invariant. Therefore, the
\( l^\infty \) induced gain analysis problem is transformed into checking the controlled robust invariance of
the disturbance attenuation performance level set.

In order to get necessary and sufficient condition for the admissibility of \( \mu \), we introduce the following
definition.

Definition 5. The set \( C_\infty(\Omega_\mu) \) is the maximal controlled robust invariant set contained in \( \Omega_\mu \) for
the linear hybrid system (1)-(2) if \( C_\infty(\Omega_\mu) \) is controlled robust invariant and contains all the
controlled robust invariant sets contained in \( \Omega_\mu \).

The uniqueness of the maximal controlled robust invariant set \( C_\infty(\Omega_\mu) \), if non-empty, follows immediately
from the fact that the union of two controlled robust invariant sets is still controlled robust invariant, and
that \( C_\infty(\Omega_\mu) \) is a subset of \( \Omega_\mu \). In order to calculate the maximal controlled robust invariant set in \( \Omega_\mu \), we introduce the one-step
controllable set of \( \Omega_\mu \) as

\[
C_1(\Omega_\mu) = \text{pre}(\Omega_\mu) \cap \Omega_\mu,
\]

and recursively define the i-step controllable set \( C_i(\Omega_\mu) \) as

\[
C_i(\Omega_\mu) = C_{i-1}(C_{i-1}(\Omega_\mu)) \cap C_{i-1}(\Omega_\mu)
\]

for \( i \geq 1 \). The sequence of finite-step controllable sets \( C_i(\Omega_\mu) \) has the following property.

Proposition 6. The sequence of finite step controllable sets \( C_i(\Omega_\mu) \) is decreasing in the sense of

\[
C_i(\Omega_\mu) \subseteq C_{i-1}(\Omega_\mu),
\]

for \( i \geq 1 \) and \( C_0(\Omega_\mu) = \Omega_\mu \). The maximal controlled invariant set in \( \Omega_\mu \) for the piecewise
linear hybrid system (1) is given by

\[
C_\infty(\Omega_\mu) = \cap_{i=0}^{\infty} C_i(\Omega_\mu).
\]

Based on the maximal controlled robust invariant set \( C_\infty(\Omega_\mu) \), we state now the basic result of
this section which will be used to give a solution
5. HYBRID CONTROLLER DESIGN

Our objective in this section is to design a hybrid control law, \( (q(t), u(t)) \), such that the closed-loop hybrid systems achieve the possible minimal \( \sup_{d(t)} \) induced gain from \( d(t) \) to \( z(t) \), \( \mu_{inf} \). It has been shown in the previous section that the disturbance attenuation problem is solved if and only if the set \( \Omega_\mu \) has nonempty controlled invariant subset, \( C_{\infty}(\Omega_\mu) \). In addition, we know that the robust optimal control problem can be solved if and only if the closed loop trajectories remain in the maximal invariant subset of the performance level set \( C_{\infty}(\Omega_{\mu_{inf}}) \). In this section, we will present a systematic procedure for the hybrid controller design, which robustly drives the system, with proper initial conditions, to guarantee that the states remain within \( C_{\infty}(\Omega_{\mu_{inf}}) \) despite the disturbances. For notational simplicity, we denote the maximal controlled invariant subset \( C_{\infty}(\Omega_{\mu_{inf}}) \) as \( C \) in the sequel. Note that \( C \) is a (maybe non-convex) piecewise linear set.

A similar invariant control problem has been considered in (Lin and Antsaklis, 2003b), in which a receding horizon control based approach was proposed to obtain the appropriate discrete modes and control signals by solving a collection of linear programming problems at each step. However, the computation burden is usually heavy for practical applications. Therefore, an explicit state feedback controller is desirable, especially for large dimensional systems or applications with fast dynamics.

In this section, we will design the hybrid control law in an explicit state feedback form, i.e., \( (q(x(t)), u(x(t))) \). For such purpose, we partition the region \( C \) into a finite number of convex sub-regions. First, we coarsely divide \( C \) into a finite union of convex piecewise linear sets \( C_i \), i.e., \( C = \bigcup_{i=1}^{m} C_i \), which satisfy the property

\[
\text{pre}_q(\bigcup_{i=1}^{m} C_i) = \bigcup_{i=1}^{m} \text{pre}_q(C_i),
\]

for all feasible mode \( q \). It is easy to show the existence of such partition. It is assumed that the polytopic region \( C_i \) can be represented as

\[
C_i = \{ x : F_i^t x \leq g_i \},
\]

with proper dimensional matrix \( F_i \) and vector \( g_i \), for all \( i = 1, 2, \cdots, m \).

Secondly, we refine the above polyhedral partition by subdividing each polyhedra \( C_i \) into

\[
C_q^{i,j} = \text{pre}_q(C_i) \bigcap C_j, \quad i, j = 1, \cdots, m, \quad q \in Q.
\]

Note that \( C_q^{i,j} \) is a convex polyhedral set and it has the following property.

**Proposition 8.** A controlled invariant set \( C \) can be written as the following polytopic partition

...
\[ C = \bigcup_{q \in Q} C_{q,i}^j , \]  
(7)

where \( C_{q,i}^j \) is defined in (6).

**Proof:** First, \( \bigcup_{q \in Q} C_{q,i}^j = \bigcup_{q \in Q} (\text{pre}_q(C^i) \cap C^j) = \bigcup_{q \in Q} (\text{pre}_q(C^i)) \cap \bigcup_{q \in Q} C^j = \text{pre}_q(C^i) \cap C^j ) = \text{pre}_q(C) \cap C^j . \) Secondly, because \( \text{pre}_q(C) \cap C \subseteq C \), for all \( q \in Q \), so \( \bigcup_{q \in Q} \text{pre}_q(C) \cap C = C \). On the other hand, \( C \) is controlled invariant, so \( C \subseteq \text{pre}_q(C) = \bigcup_{q \in Q} \text{pre}_q(C) \). □

These polytopic subregion \( C_{q,i}^j \) has the following property. For all the states \( x \) contained in the polytopic subregion \( C_{q,i}^j \), there exist admissible control signals \( u \) such that \( x \) within \( C^i \) (not \( C_{q,i}^j \) itself) along mode \( q \) for all admissible disturbances. This property comes from the definition that \( C_{q,i}^j = \text{pre}_q(C^i) \cap C^j \), so \( x \in C_{q,i}^j \) implies \( x \in \text{pre}_q(C^i) \). The possible next step state \( x' \), which is guaranteed to be contained in \( C^i \), also falls into another polytopic subregion \( C_{q',i'}^j \), for some \( q' \in Q \) and \( i' \in \{1, 2, \ldots, m \} \). For the state \( x' \), there also exist control signals \( u \in U_q \) to drive \( x' \) into \( C^i \) along mode \( q' \). The procedure repeated for the next step state, and so on. Therefore, the state trajectories under such control signals are contained in the region \( C = \bigcup_{q \in Q} C^i \) despite disturbances. This observation suggests that one may pick the mode \( q \) as the active mode for a polytopic subregion \( C_{q,i}^j \), and the existence of the admissible continuous-variable control signal \( u \) in \( U_q \) is guaranteed, which makes the region \( C (= \bigcup_{q \in Q} C_{q,i}^j) \) robust controlled invariant. In the sequel, we will propose a systematic method to construct such continuous-variable control signals. In particular, a linear state feedback control law is designed for each polytopic subregion \( C_{q,i}^j \).

For such purpose, an optimization problem for each vertex of the polytopic subregion \( C_{q,i}^j \) is formulated to calculate an admissible control signal for the vertex \( x_q^k \in \text{vert}(C_{q,i}^j) \). Notice that the vertices can be easily determined by solving some linear programming problems once the polytopic region \( C_{q,i}^j \) is specified. The control signal for the vertex \( x_q^k \) can be selected as the solution to the following minimax optimization problem:

\[
\min_{u \in U_q} \max_{d \in D_q} \left\| P^i [A_q x_q^k + B_q u + Ed] \right\|_\infty \\
\text{s.t.} \begin{cases} 
F^i B_q u \leq g^i - F^i A_q x_q^k - \delta_q^i \\
u \in U_q
\end{cases}
\]  
(8)

where \( \delta_q^i = \max_{d \in D_q} (F^i Ed) \) componentwise, which incorporates the worst effects of the disturbance \( d \). The optimal action of the controller is one that tries to minimize the maximum cost, and tries to counteract the worst disturbance and to keep the next step state inside the region \( C^j \) (not \( C_{q,i}^j \) itself). Of course, one may choose another cost functional, but the key point here is that the above constraints are linear inequalities in \( u \) and nonempty, i.e., feasible. This is simply because that \( x_q^k \in C_{q,i}^j \subseteq \text{pre}_q(C^i) \).

Because of the guaranteed optimality of the above optimization problem for each vertex of the polytope \( C_{q,i}^j \), the admissible control signals for each vertex \( x_q^k \) exist, which may be denoted as \( u_q^k \). In the next step, we will construct the continuous variable control signals for the state contained in region \( C_{q,i}^j \) from the control signals at the vertices. The arguments are obtained through convexity.

Note that any \( x \in C_{q,i}^j \) can be (not uniquely) written as the convex combination of the vertices of \( C_{q,i}^j \) that is \( x = \sum_k a_q^k(x) x_q^k \), where the convex combination coefficients \( a_q^k(x) \geq 0 \) and \( \sum k a_q^k(x) = 1 \). We set the control signal \( u(x) \) for state \( x \) simply as the convex combination of the control signals at the vertex \( u_q^k \). In particular,

\[
u(x) = \sum_k a_q^k(x) u_q^k \]  
(9)

and \( u(x) \in U_q \) comes from the convexity of \( U_q \). And

\[
F^i [A_q x_q^k + B_q u(x)] = F^i [A_q \sum_k a_q^k(x) x_q^k + B_q \sum_k a_q^k(x) u_q^k] = \sum_k a_q^k(x) F^i [A_q x_q^k + B_q u_q^k] \leq \sum_k a_q^k(x) [g^i - \delta_q^i] = g^i - \delta_q^i
\]

holds. In other words, for any \( x \in C_{q,i}^j \), the control signal \( u(x) \) given in (9) will drive the next state in \( C^i \) (not \( C_{q,i}^j \) itself) despite disturbances. Therefore, the control law of the form (9) solves the robust controlled invariance problem.

In summary, to make the performance level set \( C \) controlled invariant, the control law is given as follows. For \( x(t) \in C_{q,i}^j \), the discrete mode is selected as \( q(t) = q \). This is always possible since \( x(t) \in C_{q,i}^j = \text{pre}_q(C^i) \cap C^j \subset P_q \).

Secondly, the continuous variable control signal \( u(t) \), is of the form (9). In this expression, \( a_q^k(x) \) is the convex combination coefficients of \( x(t) \) by the vertices of \( C_{q,i}^j \), and \( u_q^k \) is the control signal for the corresponding vertices of \( C_{q,i}^j \). It has been shown that the vertex control signal \( u_q^k \) can be derived by solving a linear programming problem, which can be solved off-line.

A control law of the above form (9) can be implemented as a **piecewise linear state feedback controller**. For example, let \( X_{q,i} \) be a matrix whose columns are formed by the vertex vector of \( C_{q,i}^j \). The columns of matrix \( U_{q,i} \) are the calculated continuous variable control vector, \( u_q^k \), corresponding to each vertex of \( C_{q,i}^j \). A piecewise linear state
feedback controller is then obtained by applying the control
\[ u(x) = \sum_{k} \alpha^k(x) u^k = U^k_{ij} (X^T_{ij} X_{ij})^{-1} X^T_{ij} x \]
where \((\cdot)^T\) stands for transpose, and \((\cdot)^{-1}\) inverse of matrix. The convex combination coefficients \(\alpha^k(x)\) can be calculated as \((X^T_{ij} X_{ij})^{-1} X^T_{ij} x\) if \((X^T_{ij} X_{ij})\) is invertible. Otherwise another procedure is needed to generate the convex combination vector coefficients \(\alpha^k(x)\). Note that all the calculations to derive the matrix \(X^T_{ij} X_{ij}\) and \(U^k_{ij}\) can be done off-line by linear programming techniques. The implementation of the control law only needs to calculate the convex combination coefficients vector \(\alpha^k(x)\), which can be done by solving some linear equations. Therefore, this computational advantage makes the above method a good candidate to deal with high dimensional hybrid systems.

Some remarks are in order. First, for some states \(x(t) \in C\), there may exist more than one feasible modes and admissible control signals. Then some criteria could be designed for the selection of \((q(x(t)),u(x(t)))\), e.g. the magnitude or energy of \(u(x(t))\). This flexibility may also lead to optimal control with respect to other kinds of cost functions. Secondly, the procedure developed here answers the robust optimal control problem in a decidable way even for the general hybrid systems (1)-(2) under the assumption that \(C\) is controlled invariant. In addition, although we only synthesize a hybrid control law to guarantee optimal \(\inf\) induced gain \(\mu_{\inf}\) here, the procedure can be directly used to achieve any admissible disturbance attenuation level, \(\mu > \mu_{\inf}\).

6. CONCLUDING REMARKS

In this paper, we put the robust performance analysis problems of linear hybrid switched systems into the framework of invariant set theory. The robust performance problem was transformed into robust controlled invariant problems for a specific region decided by the disturbance attenuation level. A bisection based procedure was proposed to determine the optimal disturbance attenuation level \(\mu_{\inf}\). The decidability issue of the robust performance analysis problem was briefly discussed. Finally, a systematic procedure for explicit hybrid \(L^1\) optimal controller design was given, which was based on polyhedral algebra and linear programming techniques. The robust optimal controller is in the form of a piecewise linear state feedback control law. It is interesting to note that in (Bemporad et al., 2002) piecewise linear state feedback control law, which is obtained through multi-parametric programming, solves a variety of optimal control problems of piecewise linear systems.

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