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Stability of One-Dimensional Spatially Invariant Arrays Perturbed by White Noise

Hui Fang and Panos J. Antsakls *

Abstract

For the one-dimensional spatially invariant array, a necessary and sufficient stability condition in terms of the Schur stability of a matrix over spatial frequency is obtained in this paper. Then based on the theorem on nonnegative pseudo-polynomial matrices, the frequency-dependent stability condition is converted to a finite dimensional linear matrix inequality (LMI) problem, the solution of which is easy to compute.

1. Introduction

Spatially invariant systems have been an active topic of research in recent years [1], [2], [3], [4], [5], [10], [11], [14]. Such systems are composed by similar units which directly interact with their neighbors. These systems arise in many applications, such as the control of vehicle platoons [8], airplane formation flight control [7], cross-directional control in paper processing applications [6], and recent distributed control applications at a microscopic scale based on advances in micro electro-mechanical systems (MEMS) [9].

An important aspect of many of these systems is that sensing and actuation capabilities exist at each node. Although each node may be simple, their interdependence makes the resulting system display complex behavior. This brings new challenges to control theory, since standard methods can not handle systems of such high dimension. Besides, it is not feasible to control such systems with centralized schemes since the centralized controller will need all the nodes' state information, which makes such controller impractical.

Since the spatially invariant systems can be diagonalized by a Fourier transform over the spatial domain, by the Plancherel's theorem, the control design problem with quadratic criteria can be decoupled over spatial frequency, i.e, standard finite dimensional theorems may be applied at each spatial frequency [10]. It is further shown in [10] that the optimal controller has an inherent degree of decentralization, which weighs the information coming from neighbors using a gain exponentially decaying with distance. Based on these properties, [11] presented a conservative method to impose localization in controller design for such systems, together with some sufficient condition for the H_2 problem which takes the form of Linear Matrix Inequalities (LMI) over each spatial frequency is obtained. In the one-dimensional case, by the means of the Kalman-Yakubovich-Popov(KYP) Lemma [13], these conditions can be further expressed as some LMIs independent of spatial frequency.

In another related line of work [1], analysis and synthesis results are developed for this class of systems using the l_2-induced norm as the performance criterion. With the introduction of a shift operator, a KYP-like lemma is obtained for the analysis and synthesis of controllers with rational spatial frequency dependence and H_∞ norm guarantees. Methods of structured uncertainty analysis are extended toward systems with dynamical and noncausal spatial coordinates in [3], and design techniques of robust spatially distributed controllers for paper machines are considered in [6]. The notion of loopshaping is extended to two-dimensional systems in [5]. In [15] the effect of structured uncertainty in terms of data dropouts for spatially invariant sensor-actuator networks is considered.

This paper considers the discrete time decentralized control of one-dimensional spatially invariant systems perturbed by white noise and it is motivated by [11]. Note that examples of one dimensional spatially invariant system includes platoons [8] and Cross Directional (CD) control in the chemical process industry [5]. Previous work ([1], [10] and [11]) has been more concerned with the induced gain for the overall system with the assumption that at fixed time the signal (noise, state) is square summable (l_2) in the spatial domain. We take a stochastic approach here with the assumption that the noise at each node is a white noise. A necessary and sufficient condition is obtained from the boundedness of the solution to a discrete-time Lyapunov equation across spatial frequency, which corresponds to the continuous-time stability condition in [10]. We further show how to convert those frequency-dependent stability conditions to finite dimensional LMIs using a result on non-negative pseudo-polynomial matricesfrom [12] .

This paper is organized as follows. Some preliminary concepts and the system model are introduced in section 2. In section 3, spatial frequency dependent stability condi-

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tions are derived and later converted to finite dimensional LMIs. Stability of decentralized controller are also presented. Concluding remarks and future research are described in section 4.

2. Background

Let Z denote the set of integer and N denote the non-negative integers. The space of n by m real and complex matrices are denoted $R^{n\times m}$, $C^{n\times m}$ respectively, and I_n denotes the n by n identity matrix. For a matrix M, its transpose and complex conjugate transpose are denoted by M^T and M^*. For a Hermitian matrix $(M = M^*)$, $M > (\geq) 0$ and $M < (\leq) 0$ denote (semi-)positive definiteness and (semi-)negative definiteness.

We consider a linear time-invariant dynamical systems in an array formation. We assume that the continuous states of the dynamics have been discretized in time. Let $x_{t,s}, t \in N, s \in Z$ denote the state of the s-th node at time instant t. Consider the following state-space representation for the plant (see [14], [15]):

$$x_{t+1,s} = Ax_{t,s} + A_{-1}x_{t,s-1} + A_1x_{t,s+1} + B_1w_{t,s} + B_2u_{t,s} \quad (1)$$

$$z_{t,s} = Cx_{t,s} \quad (2)$$

where the state variable $x_{t,s} \in R^n, w_{t,s} \in R^p, u_{t,s} \in R^m, z_{t,s} \in R^q$, and $A, A_{-1}, A_1, B_1, B_2, C$ are real valued matrices of apropriate dimension.

The state of the plant is $x(t) = \{\ldots, x_{s-1,t}, x_{s,t}, x_{s+1,t}, \ldots\}$, which is of infinite dimension. And $z_{t,s}$ is an output signal that can be used to characterize the overall system performance. The input disturbance $\{w_{t,s}\}$ is a two-index field of independent, identically distributed random vectors, of zero mean and unit covariance. i.e.

$$\mathbf{E}\left\{w_{t,s}w_{t,s}^T\right\} = I \quad (3)$$

With the above model for such spatially invariant systems, each node's dynamics directly depends on its neighbor's state information. We assume that the nodes are synchronized in time and that the control input $u(t,s)$ for each node is a state-feedback control based only on all the state information about itself and its direct neighbors, that is:

$$u_{t,s} = Kx_{t,s} + K_{-1}x_{t,s-1} + K_1x_{t,s+1} \quad (4)$$

With this restriction on the decentralized controllers, we are searching a subset of all possible feedback controllers which stabilize the above systems. Thus the best performance achieved by the above decentralized controllers may be only suboptimal compared to the centralized controller. However, selecting the decentralized controllers (4) greatly decreases the complexity and the computation burden for each node, which makes such controller practical and reliable.

We want to analyze the stability of the above system and minimize the induced power gain from the disturbance $w_{t,s}$ to the state $z_{t,s}$. [1], [11] have considered the setting in which for fixed t, $w_{t,\cdot}, x_{t,\cdot} \in l_2$, which are square summable for all spatial index s, the performance is evaluated in terms of the induced gain of the input signal to the output signal for the overall system $x_{t,\cdot}$. In this paper, we are more concerned with the signal power distributed at each node. The power of the output signal $z_{t,s}$ is the output energy distributed in the system averaged both in time and space, which can be written as

$$\begin{aligned} \|z(t,s)\|_p &= \mathbf{E}\left\{z_{t,s}z_{t,s}^T\right\} \\ &= Trace\left\{CP_0^tC^T\right\} \end{aligned} \quad (5)$$

where P_0^t is the covariance matrix $x_{t,s}$ for node s at time t.

$$P_{s,0}^t = \mathbf{E}\left\{x_{t,s}x_{t,s}^T\right\} \quad (6)$$

Similarly, we define $P_{s,d}^t$:

$$P_{s,d}^t = \mathbf{E}\left\{x_{t,s}x_{t,s-d}^T\right\} \quad (7)$$

$P_{s,d}^t$ is simply denoted as P_d^t because of its property of spatial invariance, and it has the following property.

$$P_{-d}^t = P_d^{t\,T} \quad (8)$$

In view of the above, we will present our main result in the next section.

3. Stability Analysis of Spatially invariant system

Definition 1 *Let $P_d^t, d \in Z$ defined as the above, the spatial power spectral density (spsd) $P^t(\omega)$ is the spatial Fourier transform of P_d^t, defined as*

$$P^t(\omega) = \sum_{d=-\infty}^{\infty} P_d^t e^{-j\omega d} \quad (9)$$

Proposition 1 *Consider the difference state equation (1) with $u_{t,s} = 0$, and $w_{t,s}$ i.i.d random vectors with zero mean and unit covariance in t,s, and $B_1B_1^T = R$. Then $P^t(\omega)$ satisfies the following difference equation:*

$$P^{t+1}(\omega) = A(\omega)P^t(\omega)A^*(\omega) + R \quad (10)$$

where

$$A(\omega) = A + A_1e^{j\omega} + A_{-1}e^{-j\omega} \quad (11)$$

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Proof:

$$\begin{aligned} P_0^{t+1} &= \mathbf{E}\left\{x_{t+1,s}x_{t+1,s}^T\right\} \\ &= \mathbf{E}\{(Ax_{t,s}+A_1x(t,s+1)+A_{-1}x(t,s-1)+d(t,s)) \\ &\quad \cdot(Ax_{t,s}+A_1x(t,s+1)+A_{-1}x(t,s-1)+d_{t,s})^T\} \\ &= AP_0^tA^T+A_1P_1^tA^T+A_{-1}P_{-1}^tA^T+0+ \\ &\quad AP_{-1}^tA_1^T+A_1P_0^tA_1^T+A_{-1}P_{-2}^tA_1^T+0+ \\ &\quad AP_1^tA_{-1}^T+A_1P_2^tA_{-1}^T+A_{-1}P_0^tA_{-1}^T+0+ \\ &\quad 0+0+0+R \end{aligned} \tag{12}$$

And for $P_k^t, k \neq 0, k \in Z$.

$$\begin{aligned} P_k^{t+1} &= \mathbf{E}\left\{x_{t+1,s}x_{t+1,s-k}^T\right\} \\ &= \mathbf{E}\{(Ax_{t,s}+A_1x(t,s+1)+A_{-1}x(t,s-1)+d_{t,s}) \\ &\quad \cdot(Ax_{t,s-k}+A_1x(t,s-k+1)+ \\ &\quad A_{-1}x(t,s-k-1)+d_{t,s-k})^T\} \\ &= AP_k^tA^T+A_1P_{k+1}^tA^T+A_{-1}P_{k-1}^tA^T+ \\ &\quad AP_{k-1}^tA_1^T+A_1P_k^tA_1^T+A_{-2}P_{k-2}^tA_1^T+ \\ &\quad AP_{k+1}^tA_{-1}^T+A_1P_{k+2}^tA_{-1}^T+A_{-1}P_k^tA_{-1}^T \end{aligned} \tag{13}$$

From (12),(13) and our definition, we can prove(10) by direct computation. Q.E.D.

With the above definition, $P^t(\omega)$ represents the spatial power spectral density of signal $x_{t,s}$ at time t. Difference matrix equation (10) describes how the spatial power spectral density evolves with time. Note that a stable state for the $P^t(\omega)$ as t goes to infinity can be reached if and only if all the eigenvalues of $A(\omega)$ lie inside the unit circle which is obtained in terms of the boundedness of the solution of the above difference Lyapunov matrix equation.

Proposition 2 *For the spatially invariant system described by (1) with $u_{t,s}=0$, and $w_{t,s}$ i.i.d random vectors with zero mean and unit covariance in t,s, and $B_1B_1^T=R$, the system is stable if and only if all the eigenvalues of $A(\omega)$ are inside the unit circle. Furthermore, the stable state spatial power spectrum density $P(\omega)$ satisfies the following algebraic Lyapunov matrix equation.*

$$P(\omega)=A(\omega)P(\omega)A^*(\omega)+R \tag{14}$$

Remark 1 *According to the discrete Lyapunov theorem [17], $A(\omega)$ is Schur stable if and only if $\forall\omega\in[-\pi,\pi],\exists$ a positive definite Hermitian matrix $X(\omega)$, such that*

$$A(\omega)X(\omega)A^*(\omega)-X(\omega)<0 \tag{15}$$

Remark 2 *If we make $X(\omega)$ to be a constant matrix over the spatial frequency ω, we can get a sufficient condition for $X(\omega)$ to be Schur stable. The solution of (14) can be obtained by a simple recursion [16], which leads to the closed formula when all the eigenvalues of $P(\omega)$ are inside the unit circle.*

$$P(\omega)=\sum_{i=0}^{\infty}A(\omega)^iR(A^*(\omega))^i \tag{16}$$

Remark 3 *Proposition 2 gives us a sufficient and necessary condition for the system to reach a steady state, at which the energy distributed at each spatial frequency is bounded. Note that Stability condition for the continuous time case in terms of the boundedness of the solution of matrix Lyapunov equations were obtained in [10]. In view of the inverse Fourier transform, the above proposition provides a way to compute the steady state power at each node by the following integral:*

$$\begin{aligned} \|x(t,s)\|_p &= Trace\{P_0\} \\ &= Trace\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}P(\omega)d\omega\right\} \end{aligned} \tag{17}$$

Before we convert the above stability condition to a ω independent one, we shall introduce a result on the characterization of non-negative pseudo-polynomial matrix on the unit circle [12].

Lemma 1 *A pseudo-polynomial matrix:*

$$\Upsilon(z)=\sum_{i=-k}^{k}P_iz^i \tag{18}$$

whose coefficient matrices satisfy $P_{-k}=P_k^*$, $P_k\in C^{m\times m}$ is nonnegative definite on the unit circle($z=e^{j\theta},\theta\in[0,2\pi]$) if and only if there exists a nonnegative definite matrix Y, such that

$$Y=Y_0+X-Z^TXZ \tag{19}$$

where X,Y_0,Z are defined as follows:

$$Y_0=\begin{bmatrix} P_0 & P_1 & \cdots & P_k \\ P_1^* & 0 & \vdots & 0 \\ \vdots & \vdots & & \vdots \\ P_n^* & 0 & \cdots & 0 \end{bmatrix} \tag{20}$$

$$Z=\begin{bmatrix} 0 & I_m & & \\ & 0 & \ddots & \\ & & \ddots & I_m \\ & & & 0 \end{bmatrix} \tag{21}$$

and X is of the following form

$$X=\left[\begin{array}{c|c} X_h & 0 \\ \hline 0 & 0 \end{array}\right] \tag{22}$$

where X_h is an $mk\times mk$ Hermitian matrix, i.e. $X_h=X_h^*$.

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Generally speaking, it is difficult to find a positive definite Hermitian matrix $X(\omega)$ for (15) to prove the Schur stability of $A(\omega)$. However, if we restrict $X(\omega)$ to be a symmetric positive matrix $X_0 > 0$ which satisfies (15), we can obtain a sufficient condition and later use the above lemma to convert this condition to its equivalent finite dimension LMI condition. The KYP lemma has been used to get finite dimensional LMIs independent of the spatial frequency ω in [11]. However, the application of KYP lemma in the discrete-time case is not that straightforward, especially when the degree of the state dependence is more than one. With Lemma 1, we obtain the following sufficient finite dimensional LMI condition for stability.

Proposition 3 *$A(\omega)$ is Schur stable if there exist a symmetric positive definite matrix $X_0 \in R^{n\times n}$, and a Hamilton matrix X, such that*

$$Y_X + X - Z^T X Z < 0 \tag{23}$$

where Y_X, X, Z are defined as follows:

$$Y_X = \begin{bmatrix} AX_0A^T + A_1X_0A_1^T + A_{-1}X_0A_{-1}^T - X_0 & (\bullet)^* & (\bullet)^* \\ A_{-1}X_0A^T + AX_0A_1^T & 0 & 0 \\ A_1X_0A_{-1}^T & 0 & 0 \end{bmatrix} \tag{24}$$

$$Z = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ 0 & 0 & 0 \end{bmatrix} \tag{25}$$

$$X = \begin{bmatrix} X_{11} & X_{12} & 0 \\ X_{12}^* & X_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{26}$$

Note that $(\bullet)^*$ here denotes the transpose of T_X's corresponding symmetric part, which is clear from context.

Proof: $A(\omega)$ is Schur stable if there exists a symmetric positive definite matrix X_0 such that

$$\Gamma(\omega) = A(\omega)X_0A(j\omega)^* - X_0 < 0 \tag{27}$$

Notice that $A(\omega) = A_0 + A_1e^{j\omega} + A_{-1}e^{-j\omega}$, and let $e^{j\omega} = z$. Then from (27), we obtain

$$\begin{aligned} \Gamma(\omega) &= (A + A_1z + A_{-1}z^{-1})X_0(A + A_1z + A_{-1}z^{-1})^* \\ &\quad -X_0 \\ &= (A + A_1z + A_{-1}z^{-1})X_0(A^T + A_1^Tz^{-1} + A_{-1}^Tz^) \\ &\quad -X_0 \\ &= AX_0A^T + A_1X_0A_1^T + A_{-1}X_0A_{-1}^T - X_0 \\ &\quad +(A_{-1}X_0A^T + AX_0A_1^T)z^{-1} \\ &\quad +(AX_0A_{-1}^T + A_1X_0A^T)z \\ &\quad +(A_1X_0A_{-1}^T)z^{-2} \\ &\quad +(A_{-1}X_0A_1^T)z^2 \end{aligned} \tag{28}$$

Now, according to Lemma 1, $\Gamma(\omega) \geq 0$ if and only if (23) holds, which guarantees that $A(\omega)$ is Schur stable. Q.E.D.

In addition to the analysis of the spatially invariant systems, we consider the synthesis problem of the decentralized controller as defined in (4), and from proposition 2, we derive the following proposition:

Proposition 4 *System (1) is stabilizable by a local controller defined by (4), if and only if there exist a positive definite Hermitian matrix $X(\omega), \omega \in [-\pi, \pi]$, and K, K_1, K_2, such that*

$$(A(\omega) + B_2K(\omega))X(\omega)(A(\omega) + B_2K(\omega))^* - X(\omega) < 0 \tag{29}$$

where

$$K(\omega) = K + K_1e^{j\omega} + K_{-1}e^{-j\omega} \tag{30}$$

From proposition 3, by the same argument, if we choose $X(\omega)$ to be independent of ω, we obtain sufficient condition for stability of the closed loop system, i.e, the closed loop system is stable if there exist a symmetric positive definite matrix X_0, such that

$$(A(\omega) + B_2K(\omega))X_0(A(\omega) + B_2K(\omega))^* - X_0 < 0 \tag{31}$$

4. Conclusion

In this paper, we considered stability conditions for one dimensional discrete-time spatially invariant systems. With the assumption that the system admits a decentralized Lyapunov matrix, this leads us to convex conditions for analysis and synthesis in the spatial frequency domain. Based on a result on nonnegative pseudo-polynomial matrices, we have shown that the stability condition can be converted to an easy to compute computable finite-dimensional LMI. Future work will continue in the controller synthesis part to guarantee global performance.

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