Condition (3.6) follows from (3.8c) and (3.10). Condition (3.5) then follows from (2.4) and (3.2). ▽▽▽

The following corollary demonstrates that for $\gamma = 1$, the internally balanced structure minimizes J in (3.4) while for $\gamma = S/N$ the optimal low noise structures [1], [2] minimize J. Both these structures are optimal for $\gamma = 1$ and $S = N$.

Corollary:

i) $\gamma = 1$ implies $K_T = W_T$ in (3.5). For $U = V = I$ in (3.2a), the optimal $T = \Sigma^{-1/2}$ which implies

$$K_T = W_T = \Sigma. \tag{3.11}$$

ii) For $\gamma = S/N$ and $U = I$, the optimal $T = (S/N)^{1/2}\Sigma^{-1/2}V'$ which implies

$$K_T = \left(\frac{N}{S}\right) V\Sigma V'; \; W_T = \left(\frac{S}{N}\right) V\Sigma V' \tag{3.12}$$

in which case

$$\mathrm{tr}\ [W_T] = \frac{S^2}{N}. \tag{3.13}$$

Furthermore, there exists a transformation T such that in addition

$$(K_T)_{jj} = 1 \qquad \text{for all } j. \tag{3.14}$$

iii) If $S = N$, then both the internally balanced structure (3.11) and the optimal low-noise structure of Mullis and Roberts [1] defined by (3.13) and (3.14) minimize tr $[K_T + W_T]$.

Proof: Part i) follows directly from (3.5) and (3.6), while part iii) is obvious once ii) is established. Now from (3.6) $\gamma = S/N$ and $U = I$ implies $\Pi^{-2} = (N/S)\Sigma$. Hence, (3.12) follows from (2.4) and (3.1). Also since tr $(\Pi^{-2}) = N$ there exists [2] a unitary matrix V such that

$$(V\Pi^{-2}V')_{jj} = 1 \qquad \text{for all } j. \qquad ▽▽▽$$

Under the action of the bilinear transformation, the discrete system (2.1) is transformed to

$$\dot{x} = \tilde{A}x + \tilde{B}u$$
$$y = \tilde{C}x \tag{3.15}$$

where

$$\tilde{A} = (I+A)^{-1}(A-I); \; \tilde{B} = \sqrt{2}(I+A)^{-1}B$$

$$\tilde{C} = \sqrt{2}C(I+A)^{-1}; \; \tilde{D} = D - C(I+A)^{-1}B.$$

Since this transformation preserves both the controllability and observ-

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Proper Stable Transfer Matrix Factorizations and Internal System Descriptions

P. J. ANTSAKLIS

Abstract—**The exact relations between coprime proper stable factorizations of $P(s)$ and coprime polynomial matrix factorizations are derived, and they directly lead to relations with internal descriptions of the plant in differential operator or state-space form. It is shown that obtaining any right or left proper stable coprime factorization is equivalent to state-feedback stabilization or to designing a full-order full-state observer, respectively. Solving the Diophantine equation is shown to be equivalent to designing a full or reduced-order observer of a linear functional of the state and to designing a stable inverse system; and this suggests new computational methods to solve the Diophantine.**

I. Introduction

Proper stable factorizations of a transfer matrix $P = N'D'^{-1}$ were introduced in [1] and [2], extending the polynomial factorizations $P = ND^{-1}$ [3], [4] to factorizations over more general rings and in recent years, they have gained popularity in the control literature.

The polynomial matrix factorization $P = ND^{-1}$ corresponds to the controllable internal description of the plant $Dz = u$, $y = Nz$ (in differential operator form) which is related, via equivalence, to state-space internal descriptions. The ability to work with N, D which are so closely related to the transfer matrix P, while in fact working with internal descriptions, is one of the most important advantages of the polynomial matrix approach to control and the main reason for its acceptance as an analysis and synthesis tool. When working with proper and stable factorizations N', D', this advantage appears to be lost since the relation between N', D', and N, D, or other internal descriptions has not been adequately explained in the literature. Recent work on computing the solutions to the Diophantine equation in the state-space (see [5], [6]) has not shed adequate light into the problem since the main thrust there is to show, via transfer matrix identities, that certain state-space expressions are solutions of the Diophantine equation; no attempt is being made to examine the generality of these solutions or to clarify the underlying

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relations. The purpose of this paper is to establish the exact relations between proper stable factorizations and internal descriptions, thus bridging the obvious gap in the literature.

The relations between proper stable factorizations and polynomial matrix factorizations of P are first established (Theorem 1). It is then shown that deriving any right coprime proper stable factorization is equivalent to solving a state-feedback stabilization problem (Theorem 2). Similarly, it is shown that deriving any left factorization is equivalent to designing a full-order full-state observer. The results are then used to study the Diophantine equation and it is shown that obtaining a solution is equivalent to designing a full or reduced-order observer of a linear functional of the state (Theorem 3); it is also equivalent to determining proper stable inverses of a stable system. This shows that the vast literature on observers and inverse systems can be tapped to derive efficient computational methods to solve the Diophantine equation and derive low order solutions.

The fact that such relations, as described here, do exist has been long suspected by researchers in the area and implied in a number of published works. For example, similar results to Theorem 2, but for strictly proper plants, can be seen by combining results in [7, Theorems 4.2 and 5.4]; also in [3, Chapter 7], observers are determined by solving a polynomial Diophantine equation. It is the first time, however, that the important problem of determining the exact relations between proper stable factorizations and internal descriptions is fully and directly addressed and explained; additional insight is also gained here by using constructive proofs.

II. Relation to Polynomial Factorizations

Consider linear, lumped, time-invariant multivariable systems and let $P(s)$ be the $(p \times m)$ transfer matrix of an m-input, p-output plant. Assume P to be proper, i.e., $\lim_{s\to\infty} P(s) < \infty$.

Write

$$P = ND^{-1} \tag{1}$$

where N, D are right coprime (r.c.) polynomial matrices, that is $(N, D) \in M(\mathbb{R}[s])$ and r.c. in $R[s]$. Also let

$$P = N'D'^{-1} \tag{2}$$

where N', D' are proper and stable rational matrices, denoted here as $(N', D') \in M(S)$, that is matrices with elements in S, the set of all proper and stable rational functions.

Let (N', D') be right coprime (r.c.) in S; that is, there exists $(x_1', x_2') \in M(S)$ such that the Diophantine equation (or Bezout identity)

$$x_1'D' + x_2'N' = I \tag{3}$$

is satisfied [1]. Coprimeness of (N', D') in (2) implies that D'^{-1} is proper, that is, D' is biproper, and that D'^{-1} cannot have unstable poles other than the unstable poles of the plant P. This can be seen by writing (3) as $x_1' + x_2'P = D'^{-1}$; notice that the left-hand side is proper and that the only possible unstable poles are poles of P.

Theorem 1: The pair $(N', D') \in M(S)$ defines a right coprime factorization (2) of P in S if and only if

$$\begin{bmatrix} D' \\ N' \end{bmatrix} = \begin{bmatrix} D \\ N \end{bmatrix} \Pi \tag{4}$$

where Π and $\Pi^{-1} \in M(\mathbb{R}(s))$ with Π and Π^{-1} stable and $D\Pi$ biproper.

Proof: Consider (2) with $(N', D') \in M(S)$ and let $[D'^T, N'^T]^T = [D_1^T, N_1^T]^T \hat{D}_1^{-1}$, an r.c. polynomial factorization. Notice that $P = N'D'^{-1} = N_1D_1^{-1} = ND^{-1}$. Since (N, D) r.c. and (N_1, D_1) are not necessarily coprime polynomial matrices, $D_1 = D\hat{N}_1$ and $N_1 = N\hat{N}_1$ where $\hat{N}_1$ is a greatest common right divisor (grd) of (N_1, D_1) [3]. Therefore

$$\Pi = \hat{N}_1\hat{D}_1^{-1} \tag{5}$$

with $(\hat{N}_1, \hat{D}_1)$ r.c.. Since $|\hat{N}_1| \neq 0$, Π and $\Pi^{-1} \in M(\mathbb{R}(s))$; also in view of $\hat{D}_1^{-1}$ stable and D' proper, Π is stable and $D\Pi$ is proper. If now (N', D') are r.c. in S, (3) is satisfied and D'^{-1} is also proper; that is, $D\Pi$ is biproper. Furthermore, (3) can be written as $x_1'D + x_2'N = \Pi^{-1}$; that is, Π^{-1} is stable. Conversely, assume that Π satisfies the conditions of the theorem and define N', D' from (4). Then $N'D'^{-1} = (N\Pi)(D\Pi)^{-1} = P$ and $D' = D\Pi$, $N' = N\Pi = (D\Pi)P$; that is, $(N', D') \in M(S)$. To show that (N', D') are r.c. in S, it suffices to show that $[D'(s)^T, N'(s)^T]^T$ has full column rank wherever $Re\, s \geqslant 0$ and at $s = \infty$ [6], [8]; and this is true in view of (4) and the fact that Π^{-1} is stable and $D\Pi$ is biproper. Q.E.D.

Corollary: The pair $(N', D') \in M(S)$ defines a right factorization (2) of P if and only if (4) is satisfied where Π and $\Pi^{-1} \in M(\mathbb{R}(s))$ with Π stable and $D\Pi$ proper.

Remarks: 1) The corollary characterizes all proper stable factorizations $P = N'D'^{-1}$ which are not necessarily r.c. If $(N', D') \in M(S)$ are also r.c., then two additional conditions must be added to obtain the conditions of the theorem; namely Π^{-1} stable and $(D\Pi)^{-1}$ proper.

2) In view of $D' = D\Pi = D(\hat{N}_1\hat{D}_1^{-1})$ biproper $\delta[D'^T, N'^T]^T = \delta\hat{D}_1^{-1}(= \delta\Pi) = \delta P + \deg|\hat{N}_1|$ where $\delta(\cdot)$ denotes the McMillan degree or order of the transfer matrix; to minimize the McMillan degree of the coprime proper stable factorizations, $\hat{N}_1$ must be chosen to be unimodular ($\deg|\hat{N}_1| = 0$). In this case $\Pi = \hat{D}_1^{-1}$ with D_1, N_1 r.c., which is the case in a) of the example below.

Example: Let $P = (s-1)/(s-2)(s+1)$ where $N = s-1$, $D = (s-2)(s+1)$.

a) If $\Pi = \hat{N}_1\hat{D}_1^{-1} = 1/(s+1)^2$ then $N' = N\Pi$, $D' = D\Pi$, and

$$x_1'D' + x_2'N' = \frac{s-5}{s+1}\frac{s-2}{s+1} + 9\frac{s-1}{(s+1)^2} = 1.$$

b) If $\Pi = \hat{N}_1\hat{D}_1^{-1} = (s+2)/(s+1)^2(s+3)$, $x_1'D' + x_2'N' =$

$$\frac{(s-5)(s+3)}{(s+1)(s+2)}\frac{(s-2)(s+2)}{(s+1)(s+3)} + \frac{9(s+3)}{(s+2)}\frac{(s-1)(s+2)}{(s+1)^2(s+3)} = 1.$$

Notice that Π, Π^{-1} stable and $D\Pi$ biproper in both a) and b).

All of the above results involve right factorizations of P. It is clear that similar (dual) results exist for left factorizations. To illustrate, let

$$P = \tilde{D}^{-1}\tilde{N} \tag{6}$$

be a left coprime (l.c.) polynomial matrix factorization and

$$P = \tilde{D}'^{-1}\tilde{N}' \tag{7}$$

where the pair $(\tilde{N}', \tilde{D}') \in M(S)$ defines an l.c. factorization of P in S; that is, there exist $(\tilde{x}_1', \tilde{x}_2') \in M(S)$ such that

$$\tilde{D}'\tilde{x}_1' + \tilde{N}'\tilde{x}_2' = I. \tag{8}$$

Theorem 1 (Dual): The pair $(\tilde{N}', \tilde{D}') \in M(S)$ defines a left coprime factorization (7) of P in S if and only if

$$[\tilde{N}', \tilde{D}'] = \tilde{\Pi}[\tilde{N}, \tilde{D}] \tag{9}$$

where $\tilde{\Pi}$ and $\tilde{\Pi}^{-1} \in M(\mathbb{R}(s))$ with $\tilde{\Pi}$ and $\tilde{\Pi}^{-1}$ stable and $\tilde{\Pi}\tilde{D}$ biproper.

The relations shown in this section are, in effect, the relations between coprime proper, stable factorizations, and internal descriptions since $Dz = u$, $y = Nz$ and $\tilde{D}\tilde{z} = \tilde{N}u$, $y = \tilde{z}$ are internal descriptions of the plant in differential operator form. This fact will be used in the following to establish the relation between the right, left proper and stable factorizations and the control, filtering problems, respectively.

III. Right Factorizations and Control

Let $P = N'D'^{-1}$ with $(N', D') \in M(S)$ r.c. in S. Consider $D_1 \in M(\mathbb{R}[s])$ (see proof of Theorem 1) and assume without loss of generality that D_1 is column reduced (column proper). Note that since N', D' are r.c., $D' = D_1\hat{D}_1^{-1}$ is biproper; that is,

$$\lim_{s\to\infty} D' = \lim_{s\to\infty} D_1\hat{D}_1^{-1} = L, \qquad |L| \neq 0. \tag{10}$$

Write $L\hat{D}_1D_1^{-1} = I - FD_1^{-1}$ where FD_1^{-1} is strictly proper; notice that F

$\in M(\mathbb{R}[s])$ is unique and col $\deg_i(F) <$ col $\deg_i(D_1)$ for $i = 1, \cdots, m$. Let

$$D_F := D_1 - F = L\hat{D}_1 \tag{11}$$

and write

$$\begin{bmatrix} D' \\ N' \end{bmatrix} = \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} D_F^{-1}L. \tag{12}$$

Note that $D_1z = u$, $y = N_1z$ is a controllable realization of P; it is also detectable since $\hat{N}_1$, a grd of D_1, N_1, satisfies $\hat{N}_1^{-1}$ stable [see Theorem 1 and (5)]. Linear state feedback (lsf) control is defined for such system description by $u = Fz + Lr$ where col $\deg_i(F) <$ col $\deg_i(D_1)$ $i = 1, \cdots, m$ [3]; consider L real satisfying $|L| \neq 0$. If lsf is applied to $\{D_1, I, N_1, 0\}$, the compensated system is described by $D_Fz = Lr$, $y = N_1z$ and the transfer matrices between y and r, u and r are given by

$$y = T_Fr = N_1D_F^{-1}Lr, \quad u = M_Fr = [FD_F^{-1} + I]Lr = D_1D_F^{-1}Lr.$$

Consequently, for any proper stable r.c. factorization $P = N'D'^{-1}$

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} M_F \\ T_F \end{bmatrix} r = \begin{bmatrix} D' \\ N' \end{bmatrix} r = \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} D_F^{-1}Lr; \tag{13}$$

that is D', N' are equal to the transfer matrices M_F, T_F obtained when a stabilizing lsf control is applied to a controllable and detectable realization $\{D_1, I, N_1, 0\}$ of P. Conversely, if a stabilizing lsf is applied to any controllable and detectable realization $\{D_1, I, N_1, 0\}$ of P, then (M_F, T_F) will be a proper, stable r.c. factorization (D', N') of P since: $M_F = D_1D_F^{-1}L = D\hat{N}_1D_F^{-1}L = D\Pi$, $T_F = N_1D_F^{-1}L = N\hat{N}_1D_F^{-1}L = N\Pi$ where $\Pi = \hat{N}_1D_F^{-1}L$, Π^{-1} are stable and $D\Pi$ biproper (see Theorem 1). The following theorem has therefore been proved.

Theorem 2: All proper stable r.c. factorizations $P = N'D'^{-1}$ of $y = Pu$ are obtained by applying stabilizing lsf to controllable and detectable realizations of P.

Remarks:

1) The r.c. factorizations $D' = D\Pi$, $N' = N\Pi$ derived using Theorem 2 have $\Pi = \hat{N}_1D_F^{-1}L$; the poles of Π (poles of D_F^{-1}) are determined by the lsf control law, while its zeros (poles of $\hat{N}_1^{-1}$), are the uncontrollable but stable modes of the chosen realization $\{D_1, I, N_1, 0\}$ of P.

2) The factorization D', N' of P can be written in terms of its state-space realization. In particular, if $\{A_c, B_c, C_c, E_c\}$ is an equivalent description to $\{D_1, I, N_1, 0\}$ in controllable companion form, then using the structure theorem [3], it can be easily shown that D', N' are the transfer matrices of $\{A_c + B_cF_c, B_cL, F_c, L\}$, $\{A_c + B_cF_c, B_cL, C_c + E_cF_c, E_cL\}$, respectively, where F_c satisfies $u = Fz + Lr = F_cx_c + Lr$. If $\{A, B, D, E\}$ is a description equivalent to $\{A_c, B_c, C_c, E_c\}$ and $u = Fx + Lr$ is the lsf in this case, then

$$D' = [F[sI - (A + BF)]^{-1}B + I]L$$

and

$$N' = [(C + EF)[sI - (A + BF)]^{-1}B + E]L. \tag{14}$$

Equation (14) was also given in [5] (with $L = I$) where $\{A, B, C, E\}$ is any stabilizable and detectable realization of P. Here (A, B) is controllable; observe that if there are any uncontrollable modes, they will never appear in either D' or N' since they will cancel between $[sI - (A + BF)]$ and B. It is therefore clear that as long as F assigns, to the controllable modes, values in the LHP, any uncontrollable and detectable realization can be used. These results strengthen the results in [5] and [6]; note that in [5] particular state-space expressions for D', N', x_1', x_2' are given and it is shown, using matrix identities, that they satisfy the Diophantine (3) (dual results are also given in [6]).

The contribution of this section is the relation between r.c. factorizations $P = N'D'^{-1}$ and lsf maps and mainly the fact that all such factorizations can be derived in this way. It is perhaps of interest to notice at this point that the N' of factorizations (N', D') can be chosen to have certain desirable properties. In particular, N' can be taken to be H_P, the Hermite canonical form of P under dynamic compensation. This can be easily seen in view of Theorem 2 and [10].

IV. Left Factorizations and Filtering

Consider proper stable l.c. factorizations $P = \tilde{D}'^{-1}\tilde{N}'$. It is clear that results dual to the ones derived for $P = N'D'^{-1}$ can be obtained. In particular, consider P^T and apply stabilizing feedback $(\tilde{F}^T, \tilde{L}^T)$ to any controllable and detectable realizations of P^T, to obtain results involving stabilizable and observable realizations of P and observers. In the following, these results will be shown directly, using direct constructive proofs instead of duality. In this way additional insight will be gained.

Theorem 2 (Dual): All proper stable l.c. factorizations $P = \tilde{D}'^{-1}\tilde{N}'$ are obtained by using full-order observers to estimate the full state in stabilizable and observable realizations of P.

Proof: Consider $P = \tilde{D}'^{-1}\tilde{N}'$ where $(\tilde{N}', \tilde{D}') \in M(S)$ l.c. in S. In a manner analogous to Section III, write

$$[\tilde{N}', \tilde{D}'] = \tilde{\Pi}[\tilde{N}, \tilde{D}] = \tilde{L}\tilde{D}_F^{-1}[\tilde{N}_1, \tilde{D}_1] \tag{15}$$

where $\tilde{L} = \lim_{s\to\infty} \tilde{D}'$, $|\tilde{L}| \neq 0$, and $\tilde{D}_1$ in row reduced form; $\tilde{D}_F := \tilde{D}_1 - \tilde{F}$ with row $\deg_j(\tilde{F}) <$ row $\deg_j(\tilde{D}_1)$. Note that $\tilde{D}_1\tilde{z} = \tilde{N}_1u$, $y = \tilde{z}$ is a stabilizable and observable realization of P. Using the structure theorem [3], state-space descriptions equivalent to $\{\tilde{D}_F, \tilde{N}_1, \tilde{L}, 0\}$ and $\{\tilde{D}_F, \tilde{D}_1, \tilde{L}, 0\}$ in observable canonical form can be derived; switching to $\{A, B, C, E\}$, an equivalent to $\{\tilde{D}_1, \tilde{N}_1, I, 0\}$ realization of P, the transfer matrices in (15) can be written as

$$\tilde{N}' = \tilde{L}[C[sI - (A + \tilde{F}C)]^{-1}(B + \tilde{F}E) + E]$$

$$\tilde{D}' = \tilde{L}[C[sI - (A + \tilde{F}C)]^{-1}\tilde{F} + I] \tag{16}$$

which are similar to expressions in [5] and [6] (when $\tilde{L} = I$). Note that (16) satisfies $\tilde{D}'y - \tilde{N}'u = 0$; it is now shown that they can be obtained by applying a full-order full-state observer to $\{A, B, C, E\}$.

Let $\dot{\hat{x}} = A\hat{x} + Bu + \tilde{F}[\hat{y} - y]$, where $\hat{y} := C\hat{x} + Eu$, be such an observer ($\tilde{F}$ is such that all the eigenvalues of $A + \tilde{F}C$ are in the LHP). Consider the error $e = \tilde{L}(\hat{y} - y)$ where $|\tilde{L}| \neq 0$. Working in the s-domain, it can be shown that

$$e = \tilde{L}(\hat{y} - y) = \tilde{T}u - \tilde{M}y + \tilde{L}C[sI - (A + \tilde{F}C)]^{-1}\hat{x}(0) \tag{17}$$

where $\tilde{T} := \tilde{N}'$, $\tilde{M} := \tilde{D}'$ of (16). Note that for initial conditions $x(0)$ and $\hat{x}(0)$, (17) implies $\tilde{T}u - \tilde{M}Pu = 0$ or $\tilde{T} = \tilde{M}P$, since it is true for all u, as expected; this can also be seen if $x(0) = \hat{x}(0) = 0$ in (17), since in this case $\hat{y}(t) = y(t)$. Consequently, $\tilde{D}'$ and $\tilde{N}'$ can be obtained using a full-order full-state observer and considering the signal $e = \tilde{L}(\hat{y} - y)$; $\tilde{D}'$ and $\tilde{N}'$ are then the transfer matrices $\tilde{M}$ and $\tilde{T}$ between y, e and u, e, respectively. Finally note that N', D' are l.c.; this can be shown by observing that $[\tilde{N}'(s), \tilde{D}'(s)]$ has full row rank whenever $Re s \geq 0$ and at $s = \infty$. Q.E.D.

Remarks:

1) It is now quite straightforward to establish the relation between l.c. factorizations $P = \tilde{D}'^{-1}\tilde{N}'$ and filters of the plant output. Note that $y = Pu + C(sI - A)^{-1}x(0)$ from which $\tilde{y} := \tilde{D}'y = \tilde{N}'u + \tilde{L}C[sI - (A + \tilde{F}C)]^{-1}x(0)$. That is, $\tilde{D}'$ can be seen as filtering the output y to obtain a new signal $\tilde{y}$. $\tilde{N}'$ is such that when it filters u one obtains the same signal $\tilde{y}$ (assuming the initial conditions to be zero). In view of (15), internal descriptions of the systems $\tilde{D}'$ and $\tilde{N}'$ are: $\tilde{D}_Fz_1 = \tilde{D}_1y$, $\tilde{y} = \tilde{L}z_1$ and $\tilde{D}_F\tilde{z}_1 = \tilde{N}_1u$, $\tilde{y} = \tilde{L}\tilde{z}_1$, respectively. In state-space terms an appropriate equivalent description for $\tilde{D}'$ is: $\dot{x}_1 = (A + \tilde{F}C)x_1 + \tilde{F}y$, $\tilde{y} = \tilde{L}Cx_1 + \tilde{L}y$ which, in cascade with the plant $\dot{x} = Ax + Bu$, $y = Cx + Eu$, gives an overall transfer matrix equal to $\tilde{N}'$ in (16).

2) Notice that if $\{A, B, C, E\}$ is unobservable, no unobservable modes will appear in $\tilde{N}'$, $\tilde{D}'$ of (16) as they will cancel between C and $[sI - (A + \tilde{F}C)]$ [similar observations were made for D', N' in (14)]. One could therefore start with any stabilizable and detectable realization $\{A, B, C, E\}$ and use (14) and (16) to derive both right and left factorizations of P.

In the following the solutions of the Diophantine equation are studied.

V. Solutions of the Diophantine

Consider the Diophantine equation (Bezout identity):

$$x_1'D' + x_2'N' = I \tag{3}$$

where $(N', D') \in M(S)$ are given; $P = N'D'^{-1}$ is a proper stable r.c. factorization. All solutions $[x_1', x_2'] \in M(S)$ of (3) can be determined from observers (full or reduced-order) of the linear functional of the state Fz. To show this, assume that (3) is satisfied. Consider the controllable and detectable realization of $P D_1 z = u, y = N_1 z$ and the lsf controller $u = Fz + Lr$ [see (10)–(12)]; write (3) as $x_1'D_1 + x_2'N_1 = L^{-1}D_F = L^{-1}(D_1 - F)$ or as $[I - Lx_1']D_1 + [-Lx_2']N_1 = F$. Postmultiply by the (partial) state z to obtain

$$[I - Lx_1', \ -Lx_2'] \begin{bmatrix} u \\ y \end{bmatrix} = Fz \tag{18}$$

which shows that $[I - Lx_1', -Lx_2']$ is an observer of the linear functional of the state $F(D)z(t)(=F_c x_c(t) = Fx(t))$ where $D := d/dt$. Conversely, given D', N' notice that L ($|L| \neq 0$) and $Fz, F \in M(\mathbb{R}[s])$, are uniquely determined (10)–(12). Assume that an observer of Fz, $[R_1, R_2] \in M(S)$ has been found; that is, $R_1 u + R_2 y = Fz$. Let $x_1' = L^{-1}[I - R_1]$, $x_2' = -L^{-1}R_2$ which implies that (18) is satisfied. In view of $u = D_1 z$, and $y = N_1 z$ (satisfied for all z) (3) is true. The following result has therefore been shown.

Let $P = N'D'^{-1}$ a proper stable r.c. factorization; use (10)–(12) to determine $(F(D), L)$. Then the following is true.

Theorem 3: $(x_1', x_2') \in M(S)$ are solutions of the Diophantine equation (3) if and only if $[I - Lx_1', -Lx_2']$ is an observer of the linear functional of the state Fz.

Remarks:

1) If a full-order observer is used, the order of $[x_1', x_2']$ will be the order of the plant P (see 2) below). Low order solutions $[x_1', x_2']$ can be obtained by using reduced-order observers for the linear functional of the state Fz; such methods do exist in the literature [3], [11], [12] and they will not be discussed here.

2) A direct state-space method to construct a full-order observer of Fz is as follows. The process involves two steps (see also Remark 2) of Theorem 2). First a description $\{A_c, B_c, C_c, E_c\}$ equivalent to $\{D_1, I, N_1, 0\}$ is used with $u = Fz + Lr = F_c x_c + Lr$ and H_c the observer gain and then the transfer matrices are written in terms of an equivalent description $\{A, B, C, E\}$ with $u = Fx + Lr$ and H the observer gain; the intermediate step is omitted here for brevity. Consider the full-order full-state observer $\dot{\hat{x}} = (A + HC)\hat{x} + (B + HE)u - Hy$; write it as $x = \hat{x} = [sI - (A + HC)]^{-1}[(B + HE)u - Hy]$ (assume zero initial conditions). Premultiply by F to obtain

$$F[sI - (A + HC)]^{-1}[B + HE, \ -H] \begin{bmatrix} u \\ y \end{bmatrix} = Fx \tag{19}$$

which describes an observer for Fx. In view of Theorem 3

$$x_1' = L^{-1}[I - F[sI - (A + HC)]^{-1}(B + HE)]$$

$$x_2' = L^{-1}[F[sI - (A + HC)]^{-1}H]. \tag{20}$$

These are the expressions (with $L = I$) used in [5]; note that the gains F and H can be found by using methods from LQG control theory. Notice that lower order $[x_1', x_2']$ could have been derived if a reduced-order observer for Fx had been used.

3) It is known that the controller $u = -Cy$ where $C = x_1'^{-1}x_2'$ with $[x_1', x_2']$ a solution of the Diophantine (3) stabilizes the plant $y = Pu$ [1]. Notice that if P is strictly proper, then x_1' is always biproper and therefore C is proper. If P is not strictly proper, care should be exercised to choose x_1' biproper. If (20) is used, x_1' is biproper, C is proper; note that in this case the closed loop eigenvalues will be at the zeros of $|sI - (A + BF)|$ and of $|sI - (A + HC)|$. An explicit description for C can be derived in this case as follows. Substitute $u = F\hat{x}$ in the observer $\dot{\hat{x}} = (A + HC)\hat{x} + (B + HE)u - Hy$ [see (19)]; then $\dot{\hat{x}} = (A + HC + BF + HEF)\hat{x} - Hy = \hat{A}\hat{x} - Hy$ from which $\hat{x} = -(sI - \hat{A})^{-1}Hy$ and $u = -F(sI - \hat{A})^{-1}Hy$. Therefore

$$C = F[sI - (A + HC + BF + HEF)]^{-1}H. \tag{21}$$

Inverses: It is clear that the solutions $[x_1', x_2']$ of (3) are the proper and stable left inverses of $[D'^T, N'^T]^T$ which, in view of (13), is a stable system with input r and output $[u^T, y^T]^T$. Notice that $\lim_{s\to\infty} [D'^T, N'^T]^T$ is of full column rank which implies that proper left inverses do exist; furthermore, this system has no multivariable zeros which implies that stable inverses exist [12]–[14]. In view of these observations it is rather obvious that solutions to (3) can be derived via any method which constructs proper and stable left inverses: to derive minimum order such inverses one could use the method suggested in [12]. If working in the state-space is desirable, observe that in view of (13), (14)

$$\dot{x} = (A + BF)x + BLr$$

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} F \\ C + EF \end{bmatrix} x + \begin{bmatrix} I \\ E \end{bmatrix} Lr \tag{22}$$

is an internal description of $[D'^T, N'^T]^T$ and all proper inverses can be characterized using [13]; for proper and stable inverses the method described in [14] can also be used.

Dual Results: Consider the Diophantine

$$\tilde{D}'\tilde{x}_1' + \tilde{N}'\tilde{x}_2' = I. \tag{8}$$

Similar results to the ones developed for x_1', x_2' are true here. In particular $[\tilde{x}_1'^T, \tilde{x}_2'^T]^T$ are now proper and stable right inverses of $[\tilde{D}', \tilde{N}']$; and methods dual to the ones described above can be used to obtain solutions. Futhermore, in view of the discussion of left factorizations and filtering, $(\tilde{x}_1', \tilde{x}_2')$ can be seen as a filter driven by the error $e = \tilde{L}(\hat{y} - y)$ to give y and u; in particular, in view of (17)

$$\begin{bmatrix} -y \\ u \end{bmatrix} = \begin{bmatrix} \tilde{x}_1' \\ \tilde{x}_2' \end{bmatrix} \tilde{L}(\hat{y} - y). \tag{23}$$

It is clear that one could also work with P^T and use Theorem 3. Expressions analogous to (20) can be derived in this way; namely:

$$\tilde{x}_1' = [I - (C + E\tilde{H})[sI - (A + B\tilde{H})]^{-1}\tilde{F}]\tilde{L}^{-1}$$

$$\tilde{x}_2' = [\tilde{H}[sI - (A + B\tilde{H})]^{-1}\tilde{F}]\tilde{L}^{-1}. \tag{24}$$

Here $\{A, B, C, E\}$ is a stabilizable and observable (or detectable) realization of P; $\tilde{F}$, $\tilde{L}$ are determined from $\tilde{D}'$, $\tilde{N}'$ as in (15) and (16), while $\tilde{H}$ is so that $A + B\tilde{H}$ has eigenvalues in the LHP.

Λ-Approach: In [9] the transformation $\lambda = 1/(s + \alpha)$ where $-\alpha$ real in the LHP is used; this transformation maps the stable region in the s-plane into a "stable" region in the λ-plane and $s = \infty$ to $\lambda = 0$. This approach corresponds to working with proper stable factorizations N', D' with all the poles of N', D' at $-\alpha$ (here $\Pi = (1/(s + \alpha)^n)I$) and it requires only polynomial matrix manipulations. Note that all the poles of the solutions of the corresponding proper and stable Diophantine will also be at $-\alpha$ in this case, and stabilizing controllers obtained via this method will tend to assign multiple closed loop eigenvalues at $-\alpha$. As an illustration, consider $P = (s - 1)/(s - 2)(s + 1)$ (see example above) and let $\lambda = 1/(s + 1)$; then $\hat{P}(\lambda) = (1 - 2\lambda)\lambda/(1 - 3\lambda)$. The polynomial Diophantine, in λ, is solved to obtain

$$\hat{x}_1\hat{D} + \hat{x}_2\hat{N} = (-6\lambda + 1)(1 - 3\lambda) + 9(1 - 2\lambda)\lambda = 1.$$

The corresponding Diophantine (3) is obtained if we let $\lambda = 1/(s + 1)$ in $\hat{x}_1$, $\hat{D}$, $\hat{x}_2$, and $\hat{N}$; then x_1', D', x_2', and N' of case a) of the example are derived. If the controller $C(s)(= \hat{C}(\lambda) = \hat{x}_1^{-1}\hat{x}_2$ with $\lambda = 1/(s + 1)) = 9(s + 1)/(s - 5)$ is used, all three closed loop eigenvalues will be at -1.

VII. Concluding Remarks

The relations between proper stable factorizations and polynomial factorizations of a proper transfer matrix P were established; and they directly led to relations with internal descriptions in differential operator or state-space forms. This result connects the proper and stable factorizations to the standard system theory approach. Right factorizations are identified with control problems and left with filtering. The solutions to the Diophantine are identified with full or reduced-order observers of a linear functional of the state; this immediately suggests a variety of computational methods to solve the Diophantine drawn from the observer literature. Similarly, the relation to the inverse problem is shown. The results developed in this work lead to better utilization of proper stable factorizations in control; they also clearly establish the fact that this approach is an extension of the accepted system and control theory rather than an independent methodology.

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A Further Simplification in the Proof of the Structural Controllability Theorem

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Abstract—**The structural controllability theorem characterizes generic controllability of parameter-dependent systems. A further simplification of its proof is presented.**

I. INTRODUCTION

Consider a linear time-invariant system

$$\dot{x} = Ax + Bu$$

where A and B are structured matrices of dimensions $n \times n$ and $n \times m$, respectively. The structural controllability theorem asserts equivalence of the following statements:

i) (A, B) is structurally controllable;

ii) the structured matrix (A, B) is irreducible and has generic rank n;

iii) the graph $G(A, B)$ induced by (A, B) is spanned by a cacti.

Since the result has first been presented [1]–[3], there has been much effort in simplifying the rather complicated original proofs [4]–[8]. The implications i) $\Rightarrow$ ii) and ii) $\Leftrightarrow$ iii) are relatively easy to prove, and we refer to [7] for this part. In all the above references, the proof of the missing implication [either ii) $\Rightarrow$ i) or iii) $\Rightarrow$ i)] is rather involved. This note presents a simple proof of the implication iii) $\Rightarrow$ i).

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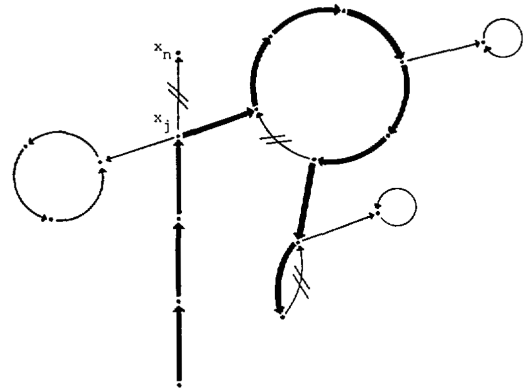

Fig. 1. How to obtain a cacti from C'. The arcs to be deleted are indicated. The final stem is bold.

II. THE PROOF

The proof is by induction on n. Every structured system with $n = 1$ whose graph contains a cacti is certainly structurally controllable. Now assume that this is also true for every $(n-1)$-dimensional system. Also, let (A, B) be a n-dimensional system such that $G(A, B)$ contains a cacti C. Note that C contains a vertex (the top of any stem) which has no outgoing arc. Without loss of generality one can assume that x_n (the vertex corresponding to the "last" state component) has this property. Delete x_n in C and its adjacent arc, say (x_j, x_n). The resulting graph C' with $n-1$ state vertices contains a cacti. Indeed, it is a cacti except that the "new top" x_j might have outgoing arcs. In this case, there exists a bud which is attached to x_j. This bud can be included into the stem by deleting an appropriate arc. If the "new stem" still has a top with outgoing arcs, one can further increase the stem by including a bud. Since there are only finitely many buds, this process has to stop, and one obtains a cacti in C', c.f. Fig. 1.

The previous argument together with the induction hypothesis shows that the structured system (A', B') corresponding to C' is structurally controllable. Now assign values to the parameters of (A, B) such that for the resulting system $(\bar{A}, \bar{B})$

- every parameter value corresponding to an arc not in C is zero,
- the system $(\bar{A}', \bar{B}')$ corresponding to C' is controllable,
- the rank of $(\bar{A}, \bar{B})$ is n [this is possible because of the equivalence ii) $\Leftrightarrow$ iii)].

Then

$$\bar{A} = \begin{bmatrix} \bar{A}' & 0 \\ \bar{a} & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}' \\ \bar{b} \end{bmatrix}$$

and the rank of $[sI - \bar{A}, \bar{B}]$ is n for all complex numbers s.

III. CONCLUSION

A significant simplification of the proof of the structural controllability theorem has been presented. The author believes that the proof is now in a form which makes it understandable also for nonspecialists. One might even consider including it in an advanced textbook on linear control systems.