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Stability of Discrete-Time Plants using Model-Based Control with Intermittent Feedback

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Abstract

In this paper we study the stability of discrete-time plants in a networked control setting, employing an approach known as Model-Based Networked Control Systems (MB-NCS) with Intermittent Feedback. Model-Based Networked Control Systems use an explicit model of the plant in order to reduce the network traffic while attempting to prevent excessive performance degradation, while Intermittent Feedback consists of the loop remaining closed for some fixed interval, then open for another interval. We provide a full description of the output, as well as a necessary and sufficient condition for stability of the system. We also extend our results to the case where the full state of the plant is not known, so that we resort to a state observer. Finally, we investigate the situation where the update times are time-varying, first addressing the case where they have upper and lower bounds, then moving on to the case where their distributions are i.i.d or driven by a Markov chain.

1. Introduction

A networked control system (NCS) is a control system in which a data network is used as feedback media. NCS is an important area in control, see for example recent surveys such as [2] and [9], as well as [20], [23], and [24]. The use of networks as media to interconnect the different components of an industrial system is rapidly increasing. However, the use of NCSs poses some challenges. One of the main problems to be addressed when considering an NCS is the size of the bandwidth required by each subsystem. A particular class of NCSs is model-based networked control systems (MB-NCS), introduced by Montestruque and Antsaklis [15]. The MB-NCS architecture makes explicit use of the knowledge of the plant dynamics to enhance the performance of the system, and it is an efficient way to address the issue of reducing packet rate. Here we extend this work by taking advantage of the novel concept of intermittent feedback. In the previous work done in MB-NCS, the updates given to the model of the plant state were performed in instantaneous fashion, but with intermittent feedback the system remains in closed loop control mode for more extended intervals. This notion makes sense as it is a good representation of what occurs in both nature and industry. For example, when driving a car, when approaching a curve or hilly terrain, we pay attention to the road for a longer time, which is equivalent to staying in closed-loop mode, and we only reduce our attention -switch to open loop control- when the road is once again straight. It is worth noting that while the application of intermittent feedback to MB-NCS, the concept has been studied in different contexts, in fields such as chemical engineering [11], psychology and behavior [21],[22], and robotics [12], [19]. While intermittent control is a very intuitive notion, its combination with the MB-NCS architecture allows for obtaining important results and opening new paths in controlling NCSs effectively.

In previous work [6], we have provided results for the cases where the plant is continuous-time. While these results serve well as an initial approach, networked control systems require us to investigate what happens in the case of discrete-time plants as well. The results presented in this paper are a natural extension of the corresponding ones in continuous time but have the advantage of more closely capturing what takes place in practice, since in digital communications, packets of information are transmitted at discrete intervals. It is important to note that the parameters τ and h, which correspond to how often the loop is closed and for how long the loop is closed each time, are different from the sampling time of the digital plant, since they are tailored after the demands of use of the network, not by the internal clock of the plant. Note also that even when the loop is closed, information is being sent at discrete intervals, typically at a higher rate determined by the

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internal clock of the plant.

The rest of the paper is organized as follows. In Section 2, we describe the problem formulation in detail. In Section 3, we derive the complete description of the output of such a system. We present a necessary and sufficient condition for the stability of the system as well. In Section 4, we extend our results for the case where full information of the state of the plant is not available, so that we resort to a state observer. We once again provide the full description of the response of the system, as well as a necessary and sufficient condition for stability. In Section 5, we investigate the situations where τ and h are time-varying. Finally, in Section 6, we provide conclusions and propose future work.

2. Problem Formulation

The basic setup for discrete-time MB-NCS with intermittent feedback is essentially the same as that for continuous time; see also [6]. We make the same assumptions as in [15] for the instantaneous feedback case, where both the sensor and actuator sides are synchronized and updates occur at the same instants of time.

Consider the control of a discrete linear plant where the state sensor is connected to a linear controller/actuator via a network. In this case, the controller uses an explicit model of the plant that approximates the plant dynamics and makes possible the stabilization of the plant even under slow network conditions.

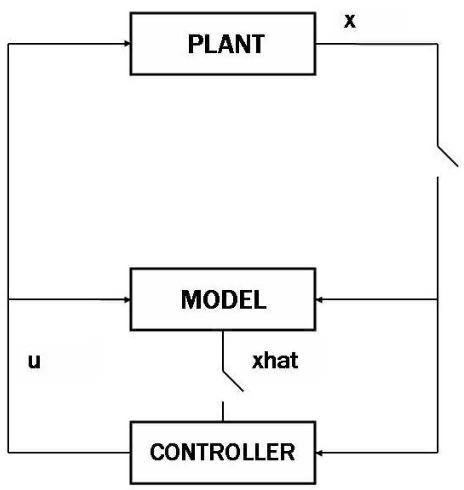

Figure 1. Basic MB-NCS architecture

In dealing with intermittent feedback, we have two key time parameters: how frequently we want to close the loop, which we shall denote by h, and how long we wish the loop to remain closed, which we shall denote by τ. Naturally, in the more general cases both h and τ can be time-varying. Unlike the continuous time formulation, h and τ are both integers here, as they represent the number of ticks of the clock in the corresponding interval.

We consider then a system such that the loop is closed periodically, every h ticks of the clock, and where each time the loop is closed, it remains so for a time of τ ticks of the clock. The loop is closed at times n_k, for $k = 1, 2, \ldots$. The system will be operating in closed loop mode for the intervals $[n_k, n_k + \tau)$ and in open loop for the intervals $[n_k + \tau, n_{k+1})$, with $n_{k+1} - n_k = h$. When the loop is closed, the control decision is based directly on the information of the state of the plant, but we will keep track of the error nonetheless.

As mentioned in the introduction, it is important to note that the parameters τ and h are different from the sampling time of the digital plant, since they are tailored after the demands of use of the network, not by the internal clock of the plant. It is also important to keep in mind that even when the loop is "closed", information is being sent at discrete intervals, the duration of which is determined by the internal clock of the plant.

The plant is given by $x(n+1) = Ax(n) + Bu(n)$, the plant model by $\hat{x}(n+1) = \hat{A}\hat{x}(n) + \hat{B}u(n)$, and the controller by $u(n) = K\hat{x}(n)$. The state error is defined as $e(n) = x(n) - \hat{x}(n)$ and represents the difference between plant state and the model state. The modeling error matrices $\tilde{A} = A - \hat{A}$ and $\tilde{B} = B - \hat{B}$ represent the plant and the model. We also define the vector $z = [x^T \; e^T]^T$.

In the next section we will derive a complete description of the response of the system as well as a necessary and sufficient condition for stability.

3. State Response of the System and Stability Condition

We will now proceed to derive the response to prove the above proposition. The approach is similar to that we used in [6] for the continuous time case. To this effect, let us separately investigate what happens when the system is operating under closed and open loop conditions.

3.1. State response of the system

During the open loop case, that is, when $n \in [n_k + \tau, n_{k+1})$, we have that

$$u(n) = K\hat{x}(n) \tag{1}$$

so

$$\begin{bmatrix} x(n+1) \\ \hat{x}(n+1) \end{bmatrix} = \begin{bmatrix} A & BK \\ 0 & \hat{A}+\hat{B}K \end{bmatrix} \begin{bmatrix} x(n) \\ \hat{x}(n) \end{bmatrix} \quad (2)$$

with initial conditions $\hat{x}(n_k+\tau) = x(n_k+\tau)$.

Rewriting in terms of x and e, that is, of the vector z :

$$z(n+1) = \begin{bmatrix} x(n+1) \\ e(n+1) \end{bmatrix} = \quad (3)$$
$$\begin{bmatrix} A+BK & -BK \\ \tilde{A}+\tilde{B}K & \hat{A}-\tilde{B}K \end{bmatrix} \begin{bmatrix} x(n) \\ e(n) \end{bmatrix}$$
$$z(n_k+\tau) = \begin{bmatrix} x(n_k+\tau) \\ e(n_k+\tau) \end{bmatrix} = \begin{bmatrix} x(n_k+\tau^-) \\ 0 \end{bmatrix},$$
$$\forall n \in [n_k+\tau, n_{k+1}) \quad (4)$$

Thus, we have

$$z(n+1) = \Lambda_{Do} z(n), \ \text{where} \ \Lambda_{Do} = \begin{bmatrix} A+BK & -BK \\ \tilde{A}+\tilde{B}K & \hat{A}-\tilde{B}K \end{bmatrix}, \quad (5)$$

$$\forall n \in [n_k+\tau, n_{k+1})$$

.

The closed loop case is a simplified version of the case above, as the difference resides in the fact that the error is always zero. Thus, for $n \in [n_k, n_k+\tau)$, we have

$$z(n+1) = \Lambda_{Dc} z(n), \ \text{where} \ \Lambda_{Dc} = \begin{bmatrix} A+BK & -BK \\ 0 & 0 \end{bmatrix}, \quad (6)$$

$$n \in [n_k, n_k+\tau)$$

. This should be clear in that the error is always zero, while the state progresses in the same way as before.

From this, it should be quite clear that given an initial condition $z(n=0) = z_0$, then after a certain time $n \in [0, \tau)$, the solution of the trajectory of the vector is given by

$$z(n) = \Lambda_{Dc}^n z_0 \, , \ n \in [0, \tau). \quad (7)$$

In particular, at time τ, $z(\tau) = \Lambda_{Dc}^{\tau} z_0$.

Once the loop is opened, the open loop behavior takes over, so that

$$z(n) = \Lambda_{Do}^{(n-\tau)} z(\tau) = \Lambda_{Do}^{(n-\tau)} \Lambda_{Dc}^{\tau} z_0 \, , \ n \in [\tau, n_1). \quad (8)$$

In particular, when the time comes to close the loop again, that is, after time h, then $z(n_1) = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} z_0$.

Notice, however, that at this instant when we close the loop again, we are also resetting the error to zero, so that we must pre-multiply by $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ before we analyze the closed loop trajectory for the next cycle. Because we wish to always start with an error that is set to zero, we should actually multiply by $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ at the beginning.

So then, after k cycles, going through this analysis yields a solution.

$$z(t_k) = \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0$$
$$= \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \, , \quad (9)$$

where $\Sigma = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau}$.

The final step is to consider the last (partial) cycle that the system goes through, that is, the time $n \in [n_k, n_{k+1})$. If the system is in closed loop, that is, $n \in [n_k, n_k+\tau)$, then the solution can be achieved merely by pre-multiplying $z(n_k)$ by $\Lambda_{Dc}^{(n-n_k)}$. In the case of the system being in open loop, that is, $n \in [n_k+\tau, n_{k+1})$, then clearly we must pre-multiply by $\Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau}$.

The results can thus be summarized in the following proposition.

Proposition 1 *The system described by (5) and (6) with initial conditions* $z(n_0) = \begin{bmatrix} x(n_0) \\ 0 \end{bmatrix} = z_0$ *has the following response:*

$$z(n) = \begin{cases} \Lambda_{Dc}^{(n-n_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \, , \\ \quad n \in [n_k, n_k+\tau) \\ \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \, , \\ \quad n \in [n_k+\tau, n_{k+1}) \end{cases} \quad (10)$$

where $\Sigma = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau}$, $\Lambda_{Do} = \begin{bmatrix} A+BK & -BK \\ \tilde{A}+\tilde{B}K & \hat{A}-\tilde{B}K \end{bmatrix}$, $\Lambda_{Dc} = \begin{bmatrix} A+BK & -BK \\ 0 & 0 \end{bmatrix}$, *and* $n_{k+1} - n_k = h$.

3.2. Stability Condition

We will present a necessary and sufficient condition for the stability of the system.

Theorem 2 *The system described by (5) and (6) is globally exponentially stable around the solution* $z = \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *if and only if the eigenvalues of* $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ *are strictly inside the unit circle, where* $\Sigma = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau}$.

Proof. Sufficiency. Taking the norm of the solution described as in Proposition #1:

$$\|z(n)\| = \left\| \Lambda_{Dc}^{(n-n_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \right\| \tag{11}$$

$$\leq \left\| \Lambda_{Dc}^{(n-n_k)} \right\| \left\| \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k \right\| \|z_0\|$$

Notice we are only doing this part for the case when $n \in [n_k, n_k + \tau)$, but the process is exactly the same for the intervals where $n \in (n_k + \tau, n_k + 1)$. Analyzing the first term on the right hand side:

$$\left\| \Lambda_{Dc}^{(n-n_k)} \right\| \leq \left(\bar{\sigma}\left(\Lambda_{Dc}\right)\right)^{n-n_k} \leq \left(\bar{\sigma}\left(\Lambda_{Dc}\right)\right)^{\tau} = K_1 \tag{12}$$

where $\bar{\sigma}\left(\Lambda_{Dc}\right)$ is the largest singular value of Λ_{Dc}. In general this term can always be bounded as the time difference $n - n_k$ is always smaller than τ. That is, even when Λ_{Dc} has eigenvalues with positive real part, $\left\| \Lambda_{Dc}^{(n-n_k)} \right\|$ can only grow a certain amount. This growth is completely independent of k.

We now study the term $\left\| \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k \right\|$. It is clear taht this term will be bounded if and only if the eigenvalues of $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ lie inside the unit circle:

$$\left\| \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k \right\| \leq K_2 e^{-\alpha_1 k} \tag{13}$$

with $K_2, \alpha_1 > 0$.

Since k is a function of time we can bounded the right term of the previous inequality in terms of t :

$$K_2 e^{-\alpha_1 k} < K_2 e^{-\alpha_1 \frac{n-1}{h}} = K_2 e^{\frac{\alpha_1}{h}} e^{-\frac{\alpha_1}{h} n} = K_3 e^{-\alpha n} \tag{14}$$

with $K_3, \alpha > 0$.

So from the above, we conclude that:

$$\begin{aligned} &\|z(n)\| \\ &= \left\| \Lambda_{Dc}^{(n-n_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \right\| \\ &\leq K_1 K_3 e^{-\alpha n} \|z_0\| . \end{aligned} \tag{15}$$

Necessity. We will now provide the necessity part of the theorem. We will do this by contradiction. Assume the system is stable and that $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ has at least one eigenvalue outside the unit circle. Let us define $\Sigma(h) = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau}$. Since the system is stable, a periodic sample of the response should converge to zero with time. We will take the samples at times n_{k+1}, that is, just before the loop is closed again. We will concentrate on a specific term: the state of the plant $x(n_{k+1})$, which is the first element of $z(n_{k+1})$. We will call $x(n_{k+1})$, $\xi(k)$.

Now assume $\Sigma(\eta)$ has the following form:

$$\Sigma(\eta) = \begin{bmatrix} W(\eta) & X(\eta) \\ Y(\eta) & Z(\eta) \end{bmatrix}.$$

Then we can express the solution $z(n)$ as:

$$\begin{aligned} &\Lambda_{Dc}^{(n-n_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma(h) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \\ &= \begin{bmatrix} W(n-n_k) & X(n-n_k) \\ Y(n-n_k) & Z(n-n_k) \end{bmatrix} \begin{bmatrix} (W(h))^k & 0 \\ 0 & 0 \end{bmatrix} z_0 \\ &= \begin{bmatrix} W(n-n_k)(W(h))^k & 0 \\ Y(n-n_k)(W(h))^k & 0 \end{bmatrix} z_0 . \end{aligned} \tag{16}$$

Now, the values of the solution at times n_{k+1}^{-}, that is, just before the loop is closed again, are

$$\begin{aligned} z(n_{k+1}) &= \begin{bmatrix} W(h)(W(h))^k & 0 \\ Y(h)(W(h))^k & 0 \end{bmatrix} z_0 \\ &= \begin{bmatrix} (W(h))^{k+1} & 0 \\ Y(h)(W(h))^k & 0 \end{bmatrix} z_0 \end{aligned} \tag{17}$$

We also know that $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ has at least one eigenvalue outside the unit circle, which means that those unstable eigenvalues must be in $W(h)$. This means that the first element of $z(n_{k+1})$, which we call $\xi(k+1)$, will in general grow with k. In other words we cannot ensure $\xi(k+1)$ will converge to zero for general initial condition x_0.

$$\begin{aligned} \|x(n_{k+1})\| &= \|\xi(k+1)\| = \left\| (W(h))^{k+1} x_0 \right\| \to \infty \\ \text{as } k &\to \infty, \end{aligned} \tag{18}$$

which clearly means the system cannot be stable. Thus, we have a contradiction. ■

4. Stability of Discrete MB-NCS with Intermittent Feedback (State Observer case)

When the full information of the state is not available, we use a state observer to estimate its value. The

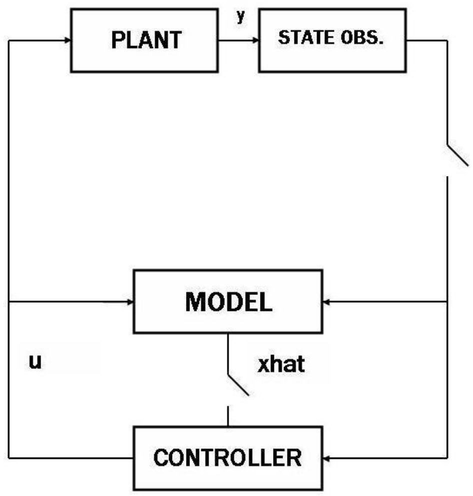

Figure 2. Model-based networked control system with state observer

corresponding architecture is showing in Figure 2 and is the same as that developed for continuous plants in [6].

The equations governing the behavior of the system can be summarized as follows:

Plant: $x(n+1) = Ax(n) + Bu(n)$,
$y(n) = Cx(n) + Du(n)$
Model: $\hat{x}(n+1) = \hat{A}\hat{x}(n) + \hat{B}u(n)$,
$y(n) = \hat{C}\hat{x}(n) + \hat{D}u(n)$
Controller: $u(n) = K\hat{x}(n)$
Observer: $\bar{x}(n+1) = (\hat{A} - L\hat{C})\bar{x}(n) + \begin{bmatrix} \hat{B} - L\hat{D} & L \end{bmatrix} \begin{bmatrix} u(n) \\ y(n) \end{bmatrix}$
Controller model state: $\hat{x}$
Observer's estimate: $\bar{x}$
Error matrices: $\tilde{A} = A - \hat{A}$, $\tilde{B} = B - \hat{B}$,
$\tilde{C} = C - \hat{C}$, $\tilde{D} = D - \hat{D}$

We will present the full description of the state response of the system as well as a necessary and sufficient condition for stability. As before, τ and h are integers.

4.1. State Response of the system (State Observer case)

The following proposition details the state response of the system for the case with state observer. The derivation of this result is similar to that of the full information case from the previous section. We will not include it here because of space limitations.

Proposition 3 *The system described above and with initial condition* $z(n_0) = \begin{bmatrix} x(n_0) \\ \bar{x}(n_0) \\ 0 \end{bmatrix} = z_0$ *has the following state response:*

$$z(n) = \begin{cases} \Lambda_{Dc}^{(n-n_k)} \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^k z_0, \\ \qquad n \in [n_k, n_k + \tau) \\ \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau} \\ \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^k z_0, \\ \qquad n \in [n_k + \tau, n_{k+1}) \end{cases} \tag{19}$$

where $\Sigma = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau}$, *and*

$$\Lambda_{Do} = \begin{bmatrix} A & BK & -BK \\ LC & \hat{A} - L\hat{C} + \hat{B}K + L\tilde{D}K & -\hat{B}K - L\tilde{D}K \\ LC & L\tilde{D}K - L\hat{C} & A - L\tilde{D}K \end{bmatrix},$$

$$\Lambda_{Dc} = \begin{bmatrix} A & BK & -BK \\ LC & \hat{A} - L\hat{C} + \hat{B}K + L\tilde{D}K & -\hat{B}K - L\tilde{D}K \\ 0 & 0 & 0 \end{bmatrix},$$

and $n_{k+1} - n_k = h$..

4.2. Stability condition (State Observer case)

We now state the following theorem characterizing the necessary and sufficient conditions for the system described in the previous section to have globally exponential stability around the solution $z = 0$.

Theorem 4 *The system described above is globally exponentially stable around the solution* $z = \begin{bmatrix} x \\ \bar{x} \\ e \end{bmatrix} = \mathbf{0}$ *if and only if the eigenvalues of* $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *are strictly inside the unit circle, where* $\Sigma = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau}$, *and* $\Lambda_{Do}, \Lambda_{Dc}$ *as before.*

The proof is similar to that of the case with full information and will be omitted for reasons of space.

5. Stability of discrete time plants with time-varying updates

Until now we have only considered the case where the parameters τ and h are constant. Let us now take a closer look at what happens when these parameters vary with time. The definitions for Lyapunov stability and mean square stability used throughout this section are the same as those in [14].

5.1. Lyapunov stability with bounded intervals

We shall first analyze the case where the parameters are time-varying, but their probability distributions are unknown. The following result describes the state response of the system. The derivation of this result is analogous to that for constant τ and h and is omitted.

Proposition 5 *The system described in (5) and (6) with initial conditions* $z = \begin{bmatrix} x(n_0) \\ 0 \end{bmatrix} = z_0$ *has the following response:*

$$z(n) = \begin{cases} \Lambda_{Dc}^{(n-n_k)} \left(\prod_{j=1}^{k} M(j) \right) z_0 \ , \ n \in [n_k, n_k + \tau) \\ \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau} \left(\prod_{j=1}^{k} M(j) \right) z_0 \ , \\ n \in [n_k + \tau, n_{k+1}) \end{cases}$$

where $M(j) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)(j)} \Lambda_{Dc}^{\tau(j)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\Lambda_{Do} = \begin{bmatrix} A+BK & -BK \\ \tilde{A}+\tilde{B}K & \hat{A}-\tilde{B}K \end{bmatrix}$, $\Lambda_{Dc} = \begin{bmatrix} A+BK & -BK \\ 0 & 0 \end{bmatrix}$, $n_{k+1} - n_k = h(k)$, *and* $\tau(j) < h(j)$.

We now present a condition for Lyapunov stability of this system.

Theorem 6 *The system described in (5) and (6) is Lyapunov asymptotically stable for* $h \in [h_{\min}, h_{\max}]$ *and* $\tau \in [\tau_{\min}, \tau_{\max}]$ *(with* $\tau_{\max} < h_{\min}$*) if there exists a symmetric positive definite matrix X such that* $Q = X - MXM^T$ *is positive definite for all* $h \in [h_{\min}, h_{\max}]$ *and* $\tau \in [\tau_{\min}, \tau_{\max}]$*, where* $M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

Proof. Note that the output norm can be bounded by

$$\begin{aligned} & \left\| \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau} \left(\prod_{j=1}^{k} M(j) \right) z_0 \right\| \\ & \leq \left\| \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau} \right\| \left\| \Lambda_{Dc}^{\tau} \right\| \left\| \prod_{j=1}^{k} M(j) \right\| \left\| z_0 \right\| \\ & \leq \bar{\sigma} \left(\Lambda_{Do}^{h_{\max}-\tau_{\min}} \right) \left\| \Lambda_{Dc}^{\tau} \right\| \left\| \prod_{j=1}^{k} M(j) \right\| \left\| z_0 \right\| \end{aligned}$$

That is, since $\Lambda_{Do}^{(n-(n_k+\tau))}$ has finite growth and will grow for at most from $\tau_{\min}$ to $h_{\max}$, then convergence of the product of matrices $M(j)$ to zero ensures the stability of the system. Such convergence to zero is guaranteed by the existence of a symmetric positive definite matrix X in the Lyapunov equation. ■

5.2. Mean square stability of discrete MB-NCS with IF with i.i.d update times

Now, let us consider the case where τ is constant, but $h(k)$ are independent identically distributed with probability distribution $F(h)$. This corresponds to the situation where we might not know how frequently we can access the network, but when we do obtain access to it, we continue to have access to it for a fixed amount of time, so as to, for example, complete a given task or transmit a certain set of packets. We present a stability condition for this case:

Theorem 7 *The system described in (5) and (6) with update times* $h(j)$ *independent identically distributed random variable with probability distribution* $F(h)$ *is globally mean square asymptotically stable around the solution* $z = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *if* $K = E\left[\left(\Lambda_{Do}^{(h-\tau)} \right)^2 \right] < \infty$ *and the maximum singular value of the expected value* $M^T M$, $\left\| E\left[M^T M \right] \right\| = \bar{\sigma}\left(E\left[M^T M \right] \right)$ *is strictly less than one, where* $M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

The proof is similar to that found in [14] for the case of instantaneous feedback and may be found in [8].

5.3. Mean square stability of discrete MB-NCS with IF with Markov chain-driven update times

We now consider the situation where the parameter h is driven by a Markov chain and provide a stability condition.

Theorem 8 *The system described in (5) and (6) with update times* $h(k) = h_{\omega_k} \neq \infty$ *driven by a finite state Markov chain* $\{\omega_k\}$ *with state space* $\{1, 2, ..., N\}$ *and transition probability matrix* Γ *with elements* $p_{i,j}$ *is globally mean square asymptotically stable around the solution* $z = \left[x^T \, e^T \right]^T = \mathbf{0}$ *if there exist positive definite matrices* $P(1), P(2), \ldots, P(N)$ *such that* $\left(\sum_{j=1}^{N} p_{i,j} \left(H(i)^T P(j) H(i) \right) - P(i) \right) < 0 \ \forall i, j \in 1, \ldots, N$ *with* $H(i) = \Lambda_{Do}^{(h_i-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

Once again, the proof follows that in [14] for the case of instantaneous feedback and may be found in [8].

6. Conclusions and future work

We have presented the full description of the response of the system as well as necessary and sufficient conditions for global exponential stability for discrete-time plants in the framework of model-based control with intermittent feedback. The results are a natural extension of the corresponding ones in continuous time but have the advantage of more closely capturing the reality of digital networks. We have also investigated the stability of the system when the parameters τ and h are time-varying.

While the focus of the present paper was on stability, we intend to investigate performance in the framework of model-based control with intermittent feedback more closely in the future. Additionally, we will seek to use intermittent feedback to improve performance, by updating the model during the times when the system is running closed loop, with the aim of enabling the user to run the system closed loop for progressively shorter intervals.

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