Wireless Control of Passive Systems Subject to Actuator Constraints

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Abstract—Actuator constraints such as saturation can impose severe constraints on networked control systems. For instance delays in wireless control systems of unstable plants combined with actuator constraints may make it impossible to stabilize a system. In this paper the conditions are derived that show when actuator saturation, a common memoryless nonlinearity, in series with a passive system causes the loss of passivity. However, using a non-linear controller known as an inner-product recovery block, the overall passivity of the system is recovered. Furthermore, we note specific sector conditions in which strictly-input passivity and strictly-output passivity can be recovered. Finally, it is shown how the inner-product recovery block can be used to maintain an $L_2^p$-stable wireless control network.

I. INTRODUCTION

Wireless communication systems are subject to many additional disturbances which traditional wired communication systems are not exposed to. Wireless systems have to contend with random fading channels due to changes in the environment such as interference, rain, heat, and absorbing objects crossing their communication path. These time varying changes in the channel influence the data capacity of the network. If a controller (or sensor) is sending command (sensing) data to the plant (or controller) at rates which exceed the capacity of the network, then either large delays will occur in the transmission of the data and/or data will have to be dropped in order to not exceed the data capacity of the channel. Markovian jump linear systems (MILS) [1], [2] have been used to capture the dynamics of a wireless networked embedded control system (wNecs) [3], [4]. System stability may be lost when the actuator used to control a plant is subject to memoryless nonlinearities. [5, Section 3.1] discusses a motivating example showing that a discrete mean square stable control system of a continuous first order plant in which the pole is strictly in the right half plane and subject to independent Bernoulli data drop outs will be destabilized when subject to actuator saturation. Furthermore, it shows that if the single pole of the plant is not in the right half plane, then stability can be maintained in spite of actuator saturation. Such results motivated us to study the control of a large class of Lyapunov stable systems known as passive systems [6]. The $L_2^p$-stable networks depicted in [6, Fig. 2] consist of a passive plant ($G_p : e_p(t) \rightarrow f_p(t)$) and a passive controller ($G_c : f_c \rightarrow e_{oc} \Rightarrow e_{oc}[i]$). These control networks tolerate both time varying delays and data drop outs as long as the conditions of [6, Theorem 4] are met. However, memoryless input nonlinearities such as actuator saturation can eliminate the desired passivity properties of a given plant. In [7] it is shown how to use a nonlinear controller to compensate for a large class of memoryless input nonlinearities $\sigma(\cdot)$ such as actuator saturation. The nonlinear controller $\beta(\cdot)$ can then be integrated in to linear controller-plant systems such that the net system is Lyapunov stable if the linear controller and plant are both positive real. Furthermore the system will be globally asymptotically stable if the linear controller and plant are both strictly positive real. This technique has been extended to apply to systems consisting of either continuous time or discrete time controller-plant pairs in which one is passive and the other is exponentially passive [8], [9].

One of this papers fundamental results, Lemma 1 shows that the nonlinear controller, $\beta(u(x))$, as depicted in Fig. 4 recovers the value of the inner-product typically lost due to the memoryless nonlinearity $\sigma(u(x))$. Most importantly, in this paper we show how the digital control network depicted in [6, Fig. 2] is $L_2^p$-stable when plant $G_p$ and controller $G_c$ are passive (see Theorem 4). In order to do this, we show how the nonlinear controller $\beta(e_p(i))$ (which we will refer to as the inner-product recovery block) combined with the inner-product equivalent sample and hold recovers the passive mapping between $H_d : e_{op}(i) \rightarrow f_{op}(i)$ (see Theorem 3-1 and Corollary 2-1).

Section II reviews some fundamental definitions for passivity (Definition 1), the inner-product equivalent sample and hold (Definition 2), and Theorem 1 stating the preservation of passivity, strictly-input passivity, and strictly-output pas-
sivity when converting from $G_{ct}$ to $G_d$ using the inner-
product equivalent sample and hold. Section III presents
Theorem 2 showing the conditions on the passive plant $G$
(Fig. 2) when input saturation eliminates the passive input-
output mapping and conditions when the passive input-output
mapping is preserved. The interested reader is referred to
[5, Section 3.2.1] which discusses how output saturation and
other sector $[0, \infty)$ nonlinearities do not eliminate the passive
input-output mapping for certain classes of continuous LTI
passive systems. Section III-A presents the inner-product
recovery block (IPRB) which shows how it recovers the
value of the inner-product which was changed due to the
memoryless nonlinearity. Furthermore it presents Theorem 3
which shows how IPRB recovers passivity, strictly-output
passivity, and strictly-input passivity for various memoryless
input nonlinearities. Section III-B provides the necessary
new corollaries and figures which show how the IPRB is
effectively integrated with the IPESH blocks used for the
$l_2^m$-stable digital control networks which we are studying.
Which leads to Section III-C proving that Fig. 1 is a $l_2^m$
stable digital control network subject to memoryless input
nonlinearities. Finally the main results are summarized in
our concluding remarks in Section IV.

II. BRIEF REVIEW OF PASSIVITY AND THE INNER-PRODUCT
EQUIVALENT SAMPLE AND HOLD

In order to discuss our main results we state the definitions
for passivity and recall the definition of the inner-product
equivalent sample and hold (IPESH) preserves passivity [6].

Definition 1: [10], [11] Let $G: \mathcal{H}_e \rightarrow \mathcal{H}_e$ then for all
$u \in \mathcal{H}_e$ and all $T \in \mathcal{T}$:

I. $G$ is passive if there exists some constant $\beta$ such that
$\langle G(u), u \rangle_T \geq -\beta$ (1)

II. $G$ is strictly-output passive if there exists some constants
$\beta$ and $\epsilon > 0$ such that (2) holds.
$\langle G(u), u \rangle_T \geq \epsilon \|G(u)\|_T^2 - \beta$ (2)

III. $G$ is strictly-input passive if there exists some constants
$\beta$ and $\delta > 0$ such that (3) holds.
$\langle G(u), u \rangle_T \geq \delta \|u_T\|_2^2 - \beta$ (3)

Remark 1: For the discrete time case, $\mathcal{T} = \{1, 2, \ldots, \infty\},
\mathcal{H}_e \equiv l_2^m(\mathbb{R}^m)$ in which all $x[i]$ satisfy
$\sum_{i=0}^{T-1} x^T[i] x[i] < \infty, \forall T \in \mathcal{T}$. (4)

Also,
$\langle y, u \rangle_T \triangleq \sum_{i=0}^{T-1} y^T[i] u[i]$ (5)

and
$\|u_T\|_2^2 \triangleq \sum_{i=0}^{T-1} u^T[i] u[i]$. (6)

For the continuous time case, $T = [0, \infty)$ (i.e. all non-
negative real numbers), $\mathcal{H}_e \equiv L_2^m(\mathbb{R}^m)$ in which all $x(t)$ satisfy
$\int_0^T x^T(t) x(t) dt < \infty, \forall T \in \mathcal{T}$. (7)

Also,
$\langle y, u \rangle_T \triangleq \int_0^T y^T(t) u(t) dt$ (8)

and
$\|u_T\|_2^2 \triangleq \int_0^T u^T(t) u(t) dt$. (9)

Definition 2: [12], [13] Let a continuous one-port plant be denoted by the input-output mapping $G_{ct}: \mathcal{H}_e \rightarrow \mathcal{H}_e$.
Denote the continuous input as $u(t) \in \mathcal{H}_e$, the continuous output as $y(t) \in \mathcal{H}_e$, the transformed discrete input as $u[i] \in \mathcal{H}_e$, and the transformed discrete output as $y[i] \in \mathcal{H}_e$. With $T_s$ denoted as the sample and hold rate (seconds), the inner-
product equivalent sample and hold (IPESH) is implemented as follows:

I. $x(t) = \int_0^t y(\tau) d\tau$
II. $y[i] = x[(i+1)T_s] - x[iT_s]$
III. $u(t) = u[i], \forall t \in [iT_s, (i+1)T_s) $

As a result
$\langle y[i], u[i] \rangle_N = \langle y(t), u(t) \rangle_{NT_s}, \forall N \geq 1$ (10)
holds.

Theorem 1: Using the IPESH given in Definition 2, the following relationships can be stated between the continuous
one-port plant, $G_{ct}: \mathcal{H}_e \rightarrow \mathcal{H}_e$, and the discrete transformed
one-port plant, $G_d: \mathcal{H}_e \rightarrow \mathcal{H}_e$:

I. If $G_{ct}$ is passive then $G_d$ is passive.
II. If $G_{ct}$ is strictly-input passive then $G_d$ is strictly-input
passive.
III. If $G_{ct}$ is strictly-output passive then $G_d$ is strictly-
output passive.

Remark 2: Theorem 1-III is the corrected version of [6, Theorem 3-III]. The proof for Theorem 1-III which we presented at the CDC 2007 is as follows:

Proof: With $T = NT_s$, the continuous strictly-output
passive system $G_{ct}$ satisfies
$\langle y(t), u(t) \rangle_T \geq \epsilon \|y(t)\|_T^2 - \beta, \forall \tau \geq 0$ (11)

From Definition 2-II and the Schwarz’s Inequality we relate
$\|y[i]\|_N^2$ to $\|y(t)\|_T^2$ as follows:

$\|y[i]\|_N^2 = \sum_{j=1}^{m} \sum_{i=0}^{N-1} y_{j}^2[i]$ (12)

Also,
$\|y[i]\|_N^2 = \sum_{j=1}^{m} \sum_{i=0}^{N-1} \left( \int_{iT_s}^{(i+1)T_s} y_j(t) dt \right)^2$

$\leq T_s \sum_{j=1}^{m} \sum_{i=0}^{N-1} \left( \int_{iT_s}^{(i+1)T_s} y_j^2(t) dt \right)$

$\leq T_s \|y(t)\|_T^2$ (12)
Rewriting (12) as

\[ \|y(t)\|^2 \geq \frac{1}{T_s} \|y[i]\|^2 \]  

and substituting (13) into (11) results in

\[ \langle y[i], u[i] \rangle_N \geq \frac{c}{T_s} \|y[i]\|^2 - \beta, \forall N \geq 1 \]  

therefore, the transformed discrete system \( G_d \) satisfies (2).

III. Passive Systems Subject to Saturation

The input saturation block indicated in Fig. 2 is a special type of actuator input memoryless nonlinearity \( \sigma(u(x)) \) in which \( u(x) \in \mathbb{R}^m \) with components \( u_j(x), \ j \in \{1, \ldots, m\} \) which has the following form

\[ u_s(x) = \begin{cases} k_j u_j(x), & \text{if } |u_j(x)| \leq k_j u_{\max}(x) \\ k_j u_{\max} \cdot \text{sign}(u_j(x)), & \text{otherwise} \end{cases} \]  

in which each element can have separate linear gain \( k_j > 0 \), and saturation level \( k_j u_{\max} \cdot \text{sign}(u_j(x)) \). For the discussion \( G(u_s(x)) \) can be thought as either linear or nonlinear passive continuous or discrete time mapping in which \( x \in \{i, t\} \), where \( i \) is a discrete time index, \( t \) represents continuous time. We denote \( \langle u(x), y(x) \rangle_N \) as either the continuous time \((x = t, X = T)\) inner-product or the discrete time \((x = i, X = N)\) inner-product (see (8) (5) in Remark 1).

Theorem 2: For a passive system \( G(u_s(x)) \) which is subject to the input saturation nonlinearity described by (15) assume there exists an admissible \( u(x), y(x) \) waveform (sequence) in which \( \varepsilon < 0 \), and \( \delta > 0 \), \( T > 0 \) \((N > 1)\) \( \alpha = 1 \) and index \( j \) such that

\[ \alpha \epsilon = \int_{T-\delta}^{T} u_j(t)y_j(t)dt \]  

where \( u_j(t) = \alpha k_j u_{\max} \cdot \text{sign}(u_j(T - \delta)) \) or

\[ \alpha = u_j[N-1]y_j[N-1] \]  

in which \( u_j[N-1] = \alpha k_j u_{\max} \cdot \text{sign}(u_j[N-1]) \). Then the continuous (or discrete) input-output mapping \( H : \mathcal{H}_c \rightarrow \mathcal{H}_c \) is not passive. If no such reachable waveform (sequence) exists which satisfies (16) ((17)) the input-output mapping \( H \) is passive.

Proof: Assume that (16) (or (17)) condition exists, and we input a waveform (sequence) \( u(x) \) such that \( |u_j(x)| \leq k_j u_{\max}(x) \) is always satisfied up to time \( T - \delta \) (index \((N - 1) - 1\)), therefore,

\[ \langle u(t), y(t) \rangle_{T-\delta} \triangleq \beta(T - \delta) \geq -\beta(0) \]  

or

\[ \langle u(i), y(i) \rangle_{N-1} \triangleq \beta(N - 1) \geq -\beta(0) \]  

Noting that:

\[ \langle u(t), y(t) \rangle_T = \beta(T - \delta) + \int_{T-\delta}^{T} u_j(t)y_j(t)dt + \int_{T-\delta}^{T} u_{N_j}(t)^T y_{N_j}(t) \]

\[ = \beta(T - \delta) + \alpha \epsilon + \int_{T-\delta}^{T} u_{N_j}(t)^T y_{N_j}(t) \]

and

\[ \langle u[i], y[i] \rangle_N = \beta(N - 1) + u_j[N-1]y_j[N-1] + u_{N_j}[N-1]^T y_{N_j}[N-1] \]

\[ = \beta(N - 1) + \alpha \epsilon + u_{N_j}[N-1]^T y_{N_j}[N-1] \]

in which \( u_{N_j}(T)y_{N_j}(x) \) contain the remaining elements of \( u \) and \( y \) which don’t include \( j \). Since \( u_j(x) \) is saturated then the corresponding output \( y(x) \) will not change, therefore (16) and (17) will hold for all \( \alpha > 1 \). Therefore if the net system is to remain passive the following must hold for all \( \alpha > 1 \)

\[ \beta(T - \delta) + \alpha \epsilon + \int_{T-\delta}^{T} u_{N_j}(t)^T y_{N_j}(t) \geq -\beta(0) \]

\[ \alpha \leq -\frac{\beta(0) + \beta(T - \delta) + \int_{T-\delta}^{T} u_{N_j}(t)^T y_{N_j}(t)}{\epsilon} \]  

for the continuous time case or

\[ \beta(N - 1) + \alpha \epsilon + u_{N_j}[N-1]^T y_{N_j}[N-1] \geq -\beta(0) \]

\[ \alpha \leq -\frac{\beta(0) + \beta(N-1) + u_{N_j}[N-1]^T y_{N_j}[N-1]}{\epsilon} \]  

for the discrete time case. However this is clearly not possible since the expressions to the right of (24) and (25) are finite. Likewise if no such sequence exists for \( \alpha = 1 \) in which \( \epsilon < 0 \) then passivity is preserved.
Fig. 4. The nonlinear controller $\beta(u(x))$ recovers the inner-product lost due to the memoryless nonlinearity $\sigma(u(x))$.

Remark 3: In [11, Exercise VI-4.7] it was given as an exercise to show how passivity is lost for $[0,k)$ sector input nonlinearities (Appendix I) when the passive plant has the following form $G(u(s)) = \frac{1}{1+qs}$. Theorem 2 tells us that the passive plant with input saturation nonlinearity in sector $[0,u_{max}]$ will no longer be passive (Fig. 3).

Remark 4: One example of a passive system which maintains passivity when subject to actuator saturation is a positive semi-definite gain block, $G(u(x)) = K u(x)$ in which $u^T G(u(x)) u(x) = u^T K u(x) \geq 0$, $\forall u(x)$. Hence by Theorem 2 the net system $H$ is passive.

A. The Inner-Product Recovery Block.

Fig. 4 depicts how we prefer to implement the nonlinear controller $\beta(u(x))$ introduced in [7]. We choose to locate $\beta(u(x))$ at the output of the plant $G : \sigma(u(x)) \mapsto p(x)$ and explicitly analyze the map $H_p : u \mapsto y$. As indicated in Fig. 4, $\beta(u(x))$ recovers the inner-product such that $\langle y(x), u(x) \rangle_X = \langle p(x), \sigma(u(x)) \rangle_X$, therefore, we shall refer to $\beta(u(x))$ as the inner-product recovery block.

Lemma 1: Consider the inner-product recovery block $\beta(u(x))$ as defined by Definition 3 in Appendix I, using either an exact model or measurement of the memoryless nonlinearity $\sigma(u(x))$. When using the inner-product recovery block as depicted in Fig. 4, the following will always be satisfied:

$$\langle y(x), u(x) \rangle_X = \langle p(x), \sigma(u(x)) \rangle_X \tag{26}$$

Proof: The proof is straight forward using (43) to yield (29) and (42) to yield (30).

$$\langle y(x), u(x) \rangle_X = \langle \beta(u(x)) p(x), u(x) \rangle_X \tag{27}$$

$$= \langle p(x), \beta(u(x))^T u(x) \rangle_X \tag{28}$$

$$= \langle p(x), \beta(u(x)) u(x) \rangle_X \tag{29}$$

$$= \langle p(x), \sigma(u(x)) \rangle_X \tag{30}$$

Lemma 1 allows us to prove Theorem 3.

Theorem 3: Consider the inner-product recovery block $\beta(u(x))$ as defined by Definition 3 in Appendix I, using either an exact model or measurement of the memoryless nonlinearity $\sigma(u(x))$. When using the inner-product recovery block as depicted in Fig. 4, the following can be said about $H_p : u(x) \mapsto y(x)$ given $G : \sigma(u(x)) \mapsto p(x)$.

I. If $G$ is passive then $H_p$ is passive.

II. If $G$ is strictly-output passive then $H_p$ is strictly-output passive if $\sigma_{MAX}(\beta(u))^2 < \infty$, $\forall u \in \mathbb{R}^m$ (Definition 5).

III. If $G$ is strictly-input passive then $H_p$ is strictly-input passive if there exists a $\gamma > 0$ such that $\sigma^T(u) \sigma(u) \geq \gamma u^T u$, $\forall u \in \mathbb{R}^m$.

Proof: Based on our assumptions, Lemma 1 shows that (26) will hold. Furthermore:

I. if $G$ is passive then $\langle p(x), \sigma(u(x)) \rangle_X \geq -\beta$, substituting (26) yields

$$\langle y(x), u(x) \rangle_X \geq -\beta \tag{31}$$

which satisfies Definition 1 for passivity.

II. if $G$ is strictly-output passive then $\langle p(x), \sigma(u(x)) \rangle_X \geq \varepsilon \| (p(x)) x \|_2^2 - \beta$, substituting (26) yields

$$\langle y(x), u(x) \rangle_X \geq \varepsilon \| (p(x)) x \|_2^2 - \beta \tag{32}$$

next, we solve for $\| (y(x)) x \|_2^2$

$$\| (y(x)) x \|_2^2 = \| (\beta(u(x)) p(x)) x \|_2^2 \leq \sigma_{MAX}(\beta(u))^2 \| (p(x)) x \|_2^2 \tag{33}$$

and substitute (33) into (32) to yield

$$\langle y(x), u(x) \rangle_X \geq \frac{\varepsilon}{\sigma_{MAX}(\beta(u))^2} \| (p(x)) x \|_2^2 - \beta \tag{34}$$

which satisfies Definition 1 for $H_p$ to be strictly-output passive as long as $\sigma_{MAX}(\beta(u))^2 < \infty$.

III. if $G$ is strictly-input passive then $\langle p(x), \sigma(u(x)) \rangle_X \geq \delta \| (\sigma(u(x))) x \|_2^2 - \beta$, substituting (26) yields

$$\langle y(x), u(x) \rangle_X \geq \delta \| (\sigma(u(x))) x \|_2^2 - \beta \tag{35}$$

next, our assumption that $\sigma^T(u) \sigma(u) \geq \gamma u^T u$, $\forall u \in \mathbb{R}^m$ implies that

$$\| (\sigma(u(x))) x \|_2^2 \geq \gamma \| (u(x)) x \|_2^2 \tag{36}$$

and substitute (36) into (35) to yield

$$\langle y(x), u(x) \rangle_X \geq \delta \gamma \| (u(x)) x \|_2^2 - \beta \tag{37}$$

which satisfies Definition 1 for $H_p$ to be strictly-input passive.

Remark 5: Theorem 3 is written from the least restrictive case to the most restrictive case in terms of the memoryless nonlinearities $\sigma(u)$ which can be tolerated. Theorem 3-I can be applied to all types of $\sigma(u)$. Whereas Theorem 3-II can be applied to a slightly smaller class of nonlinearities within $\sigma(u)$ such as those which typically have actuator saturation. They can not include unbounded quadratic nonlinearities, $\sigma(u_i) = u_i^p$, $p \geq 2$, $p \leq -1$, however, quadratic nonlinearities can occur as long as $\sigma(u)$ is linear or saturates as $u \rightarrow \infty$ for $p \geq 1$, or a dead-zone or linearity occurs at the origin when $p \leq -1$. Also, near the origin, relay nonlinearities can not be tolerated.

$$\sigma(u_i) = \begin{cases} \text{sign}(u_i), & |u_i| > 0 \\ 0, & u_i = 0 \end{cases} \tag{38}$$

Finally, Theorem 3-III does not include quadratic nonlinearities at the origin, or saturation nonlinearities. However, all sector $[k_1, k_2]$ nonlinearities (Appendix I) in which $0 < k_1 \leq k_2$ or $k_1 \leq k_2 < 0$, will preserve strictly-input passivity.
B. Implementing the Inner-Product Recovery Block with the Inner-Product Equivalent Sample and Hold

By implementing the continuous time inner-product recovery block as depicted in Fig. 5 we can directly use Theorem 1 and Theorem 3 in order to state the following Corollary:

Corollary 1: Using the IPESH given in Definition 2 and the continuous time inner-product recovery block as depicted in Fig. 5, the following relationships can be stated between the continuous one-port plant, $G : \sigma(u(t)) \mapsto p(t)$, and the discrete transformed one-port plant, $H_{rd} : u(i) \mapsto y(i)$:

I. If $G$ is passive then $H_{rd}$ is passive for all $\sigma(u)$.

II. If $G$ is strictly-input passive then $H_{rd}$ is strictly-input passive if $\sigma_{MIN}(\beta(u))^2 > 0, \forall u \in \mathbb{R}^m$.

III. If $G$ is strictly-output passive then $H_{rd}$ is strictly-output passive if $\sigma(u)$ is a sector $[k_1, k_2]$ nonlinearity such that $\sigma_{MAX}(\beta(u))^2 = \max(k_1^2, k_2^2) < \infty$.

Remark 6: See Appendix I (Theorem 5 and Theorem 6) in order to see why $\sigma_{MAX}(\beta(u))^2 = \max(k_1^2, k_2^2)$.

Next, we note that by switching the order of the ZOH and the memoryless nonlinearity $\sigma(\cdot)$ is mathematically equivalent to the discrete time inner-product recovery block used with IPESH as depicted in Fig. 6.

Corollary 2: Using the IPESH given in Definition 2 and the discrete time inner-product recovery block as depicted in Fig. 6, the following relationships can be stated between the continuous one-port plant, $G : \sigma(u(t)) \mapsto p(t)$, and the discrete transformed one-port plant, $H_{dr} : u(i) \mapsto y(i)$:

I. If $G$ is passive then $H_{dr}$ is passive for all $\sigma(u)$.

II. If $G$ is strictly-input passive then $H_{dr}$ is strictly-input passive if $\sigma_{MIN}(\beta(u))^2 > 0, \forall u \in \mathbb{R}^m$.

III. If $G$ is LTI and strictly-output passive then $H_{dr}$ is strictly-output passive if the sector $[k_1, k_2]$ nonlinearity is such that $\sigma_{MAX}(\beta(u))^2 = \max(k_1^2, k_2^2) < \infty$.

C. $l_2^2$-stable Digital Control Networks Subject to Memoryless Nonlinearities

Theorem 4: The digital control network depicted in Fig. 1 in which $K_p > 0, K_c > 0, G_c$ are passive and the passive plant $G_p$ is subject to memoryless nonlinearities $\sigma(\cdot)$, is strictly-output passive which is sufficient for $l_2^2$-stability if

$$\langle f_{op}, e_{dac} \rangle_N \geq \langle e_{ac}, f_{op} \rangle_N$$

holds for all $N \geq 1$.

Proof: Corollary 2-1 shows that the mapping from $e_p(i)$ to $f_{op}(i)$ is passive for any memoryless actuator nonlinearity associated with the plant. Since the mapping is now passive, the strictly-output passivity and $l_2^2$-stability follows from [6, Theorem 4].

IV. CONCLUSIONS

Theorem 2 shows that actuator saturation typically eliminates the passive mapping. To address this issue this paper provides: i) Lemma 1 which states that the IPRB depicted in Fig. 4 recovers the inner-product mapping of either a continuous or discrete time system subject to a memoryless nonlinearity, ii) Theorem 3 notes that passivity is always recovered with an IPRB, however only certain classes of memoryless nonlinearities will allow a strictly-output passive or strictly-input passive mapping to be preserved, iii) Corollary 1 states that the IPESH can be used with a continuous time IPRB to preserve passivity, and conditions to also preserve either a strictly-input passive or strictly-output passive mapping. iv) Corollary 2 is similar to Corollary 1, except that it relates to implementing the IPRB in discrete time, the key to such a realization is (39) and the carefully chosen system depicted in Fig. 6. Furthermore some important connections between sector $[k_1, k_2]$ nonlinearities (see Definition 4) and the equivalent constants stated by Corollary 1 are given in the Appendix (see Theorem 5 and Theorem 6). v) Theorem 4, completes our discussion, stating how a IPRB and IPESH can be used in discrete time to implement a $l_2^2$-stable digital control network as depicted in Fig. 1.

REFERENCES


Theorem 5: Let $k_1, k_2 \in \mathbb{R}$ with $k_1 \leq k_2$. Let $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$ with $\sigma_i(u_i) = 0$, $i \in \{1, \ldots, m\}$ and $\beta(u)$ as defined by Definition 3 in Appendix I. Such that the following statements are equivalent:

i) $(k_1 I - \beta(u)) u \leq 0 \text{ and } (\beta(u) - k_2 I) u \leq 0$, $\forall u \in \mathbb{R}^m$

ii) $k_1 u^T u \leq u^T \sigma(u) \leq k_2 u^T u$, $\forall u \in \mathbb{R}^m$

Proof:

i) $\rightarrow$ ii) first we prove the left half of the inequality holds for both definitions.

$$u^T(k_1 I - \beta(u)) u \leq 0, \forall u \in \mathbb{R}^m$$

$$k_1 u^T u \leq u^T(\beta(u)u), \forall u \in \mathbb{R}^m \quad (45)$$

$$k_1 u^T u \leq u^T \sigma(u), \forall u \in \mathbb{R}^m \quad (46)$$

Note that the simplification from (45) to (46) is a direct result of application of the definition for $\beta(u)$. The proof for the right half of the inequality is as follows:

$$u^T(\beta(u) - k_2 I) u \leq 0, \forall u \in \mathbb{R}^m$$

$$u^T \sigma(u) \leq k_2 u^T u, \forall u \in \mathbb{R}^m \quad (47)$$

$$u^T \sigma(u) \leq k_2 u^T u, \forall u \in \mathbb{R}^m \quad (48)$$

ii) $\rightarrow$ i) is fairly obvious when we substitute $\sigma(u) = \beta(u)u$ and $u^T u = u^T I u$ in to (44) which yields

$$k_1 u^T I u \leq u^T(\beta(u)u) \leq k_2 u^T I u, \forall u \in \mathbb{R}^m$$

$$u^T(\beta(u) - k_2 I) u \leq 0, \forall u \in \mathbb{R}^m \quad (49)$$

$$u^T(\beta(u) - k_2 I) u \leq 0, \forall u \in \mathbb{R}^m \quad (50)$$

$$u^T(\beta(u) - k_2 I) u \leq 0, \forall u \in \mathbb{R}^m \quad (51)$$

Next we relate the maximum and minimum singular values for $\beta(u)$ to the sector $[k_1, k_2]$ bounds.

Definition 5: The maximum achievable singular value squared is $\sigma_{\text{MAX}}(\beta(u))^2 \triangleq \max(\sigma_M(\beta(u))^2)$, $\forall u \in \mathbb{R}^m$ in which $\sigma_M(\beta(u))$ denotes the maximum singular value of the resulting matrix $\beta(u)$ for a given $u$.

Definition 6: The minimum achievable singular value squared is $\sigma_{\text{MIN}}(\beta(u))^2 \triangleq \min(\sigma_m(\beta(u))^2)$, $\forall u \in \mathbb{R}^m$ in which $\sigma_m(\beta(u))$ denotes the minimum singular value of the resulting matrix $\beta(u)$ for a given $u$.

Theorem 6: For a given sector $[k_1, k_2]$ nonlinearity:

i) If either $k_1 = 0$, or $k_2 = 0$ then $\sigma_{\text{MIN}}(\beta(u))^2 = 0$.

ii) $\sigma_{\text{MAX}}(\beta(u))^2 = \max(k_1^2, k_2^2)$.

Proof: Since $\beta(u)$ is a diagonal matrix, then $\sigma_M(\beta(u)) = \max(\lambda_M(\beta(u))$, $\lambda_M(\beta(u)))$ in which $\lambda_M(\cdot)$ denotes the maximum eigenvalue and $\lambda_m(\cdot)$ denotes the minimum eigenvalue of $\beta(u)$.

i) If $k_1 \geq 0$ then $\sigma_{\text{MIN}}(\beta(u))^2 = k_1^2$, therefore $k_1 = 0$ implies that $\sigma_{\text{MIN}}(\beta(u))^2 = 0$. If $k_2 \leq 0$ then $\sigma_{\text{MIN}}(\beta(u))^2 = k_2^2$, therefore $k_2 = 0$ implies that $\sigma_{\text{MIN}}(\beta(u))^2 = 0$.

ii) From Theorem 5-i, we see that $\min(\lambda_m(\beta(u))) = k_1, \forall u \in \mathbb{R}^m$ and $\max(\lambda_M(\beta(u))) = k_2, \forall u \in \mathbb{R}^m$, therefore $\sigma_{\text{MAX}}(\beta(u))^2 = \max(k_1^2, k_2^2)$.