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Wireless Control of Passive Systems Subject to Actuator Constraints

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***Abstract*— Actuator constraints such as saturation can impose severe constraints on networked control systems. For instance delays in wireless control systems of unstable plants combined with actuator constraints may make it impossible to stabilize a system. In this paper the conditions are derived that show when actuator saturation, a common memoryless nonlinearity, in series with a *passive* system causes the loss of *passivity*. However, using a non-linear controller known as a *inner-product recovery block*, the overall *passivity* of the system is recovered. Furthermore, we note specific sector conditions in which *strictly-input passivity* and *strictly-output passivity* can be recovered. Finally, it is shown how the *inner-product recovery block* can be used to maintain an l_2^m-stable wireless control network.**

I. INTRODUCTION

Wireless communication systems are subject to many additional disturbances which traditional wired communication systems are not exposed to. Wireless systems have to contend with random fading channels due to changes in the environment such as interference, rain, heat, and absorbing objects crossing their communication path. These time varying changes in the channel influence the data capacity of the network. If a controller (sensor) is sending command (sensing) data to the plant (controller) at rates which exceed the capacity of the channel, then either large delays will occur in the transmission of the data and/or data will have to be dropped in order to not exceed the data capacity of the channel. Markovian jump linear systems (*MJLS*) [1], [2] have been used to capture the dynamics of a wireless networked embedded control system (*wnecs*) [3], [4]. System stability may be lost when the actuator used to control a plant is subject to memoryless nonlinearities. [5, Section 3.1] discusses a motivating example showing that a discrete mean square stable control system of a continuous first order plant in which the pole is strictly in the right half plane and subject to independent Bernoulli data drop outs will be destabilized when subject to actuator saturation. Furthermore, it shows that if the single pole of the plant is not in the right half plane, then stability can be maintained in spite of actuator saturation. Such results motivated us to study the control of a large class of Lyapunov stable systems known as *passive* systems [6]. The l_2^m-*stable* networks depicted in [6, Fig. 2] consist of a *passive* plant ($G_p : e_p(t) \mapsto f_p(t)$) and a *passive* controller ($G_c : f_c[i] \mapsto e_{oc}[i]$). These control

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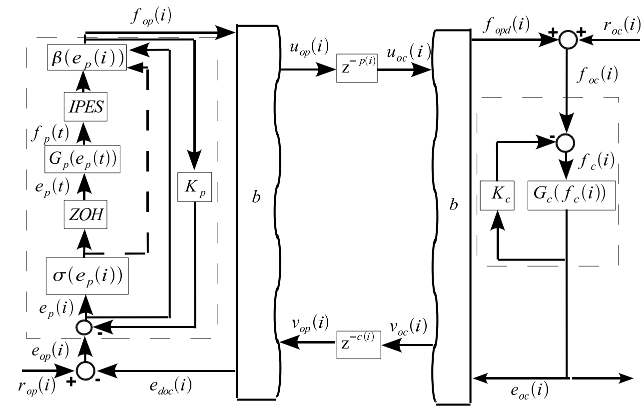

Fig. 1. l_2^m-*stable* digital control networks subject to memoryless nonlinearities in which $K_p > 0$, $K_c > 0$, G_p and G_c are *passive*.

networks tolerate both time varying delays and data drop outs as long as the conditions of [6, Theorem 4] are met. However, memoryless input nonlinearities such as actuator saturation can eliminate the desired *passivity* properties of a given plant. In [7] it is shown how to use a nonlinear controller to compensate for a large class of memoryless input nonlinearities $\sigma(\cdot)$ such as actuator saturation. The nonlinear controller $\beta(\cdot)$ can then be integrated in to linear controller-plant systems such that the net system is Lyapunov stable if the linear controller and plant are both positive real. Furthermore the system will be globally asymptotically stable if the linear controller and plant are both strictly positive real. This technique has been extended to apply to systems consisting of either continuous time or discrete time controller-plant pairs in which one is *passive* and the other is *exponentially passive* [8], [9].

One of this papers fundamental results, Lemma 1 shows that the nonlinear controller, $\beta(u(x))$, as depicted in Fig. 4 recovers the value of the inner-product typically lost due to the memoryless nonlinearity $\sigma(u(x))$. Most importantly, in this paper we show how the digital control network depicted in [6, Fig. 2] is l_2^m-stable when plant G_p and controller G_c are *passive* (see Theorem 4). In order to do this, we show how the nonlinear controller $\beta(e_p(i))$ (which we will refer to as the *inner-product recovery block*) combined with the *inner-product equivalent sample and hold* recovers the *passive* mapping between $H_d : e_{op}(i) \mapsto f_{op}(i)$ (see Theorem 3-I and Corollary 2-I).

Section II reviews some fundamental definitions for *passivity* (Definition 1), the *inner-product equivalent sample and hold* (Definition 2), and Theorem 1 stating the preservation of *passivity*, *strictly-input passivity*, and *strictly-output pas-*

sivity when converting from G_{ct} to G_d using the *inner-product equivalent sample and hold*. Section III presents Theorem 2 showing the conditions on the passive plant G (Fig. 2) when input saturation eliminates the *passive* input-output mapping and conditions when the *passive* input-output mapping is preserved. The interested reader is referred to [5, Section 3.2.1] which discusses how output saturation and other **sector**$[0, \infty)$ nonlinearities do not eliminate the *passive* input-output mapping for certain classes of continuous *LTI passive* systems. Section III-A presents the *inner-product recovery block* (*IPRB*) which shows how it recovers the value of the inner-product which was changed due to the memoryless nonlinearity. Furthermore it presents Theorem 3 which shows how *IPRB* recovers *passivity*, *strictly-output passivity*, and *strictly-input passivity* for various memoryless input nonlinearities. Section III-B provides the necessary new corollaries and figures which show how the *IPRB* is effectively integrated with the *IPESH* blocks used for the l_2^m*-stable* digital control networks which we are studying. Which leads to Section III-C proving that Fig. 1 is a l_2^m*-stable* digital control network subject to memoryless input nonlinearities. Finally the main results are summarized in our concluding remarks in Section IV.

II. Brief Review of *Passivity* and The *Inner-Product Equivalent Sample and Hold*

In order to discuss our main results we state the definitions for *passivity* and recall how the *inner-product equivalent sample and hold* (*IPESH*) preserves *passivity* [6].

Definition 1: [10], [11] Let $G : \mathcal{H}_e \mapsto \mathcal{H}_e$ then for all $u \in \mathcal{H}_e$ and all $T \in \mathcal{T}$:

I. G is *passive* if there exists some constant β such that (1) holds.
$$\langle G(u), u\rangle_T \geq -\beta \tag{1}$$
II. G is *strictly-output passive* if there exists some constants β and $\epsilon > 0$ such that (2) holds.
$$\langle G(u), u\rangle_T \geq \epsilon \|(G(u))_T\|_2^2 - \beta \tag{2}$$
III. G is *strictly-input passive* if there exists some constants β and $\delta > 0$ such that (3) holds.
$$\langle G(u), u\rangle_T \geq \delta \|u_T\|_2^2 - \beta \tag{3}$$

Remark 1: For the discrete time case, $\mathcal{T} = \{1, 2, \ldots, \infty\}$, $\mathcal{H}_e \equiv l_{2e}^m(\mathbb{R}^m)$ in which all $x[i]$ satisfy
$$\sum_{i=0}^{T-1} x^{\mathsf{T}}[i]x[i] < \infty,\ \forall T \in \mathcal{T}. \tag{4}$$
Also,
$$\langle y, u\rangle_T \triangleq \sum_{i=0}^{T-1} y^{\mathsf{T}}[i]u[i] \tag{5}$$
and
$$\|u_T\|_2^2 \triangleq \sum_{i=0}^{T-1} u^{\mathsf{T}}[i]u[i]. \tag{6}$$
For the continuous time case, $\mathcal{T} = [0, \infty]$ (i.e. all non-negative real numbers), $\mathcal{H}_e \equiv L_{2e}^m(\mathbb{R}^m)$ in which all $x(t)$ satisfy
$$\int_0^T x^{\mathsf{T}}(t)x(t)dt < \infty,\ \forall T \in \mathcal{T}. \tag{7}$$
Also,
$$\langle y, u\rangle_T \triangleq \int_0^T y^{\mathsf{T}}(t)u(t)dt \tag{8}$$
and
$$\|u_T\|_2^2 \triangleq \int_0^T u^{\mathsf{T}}(t)u(t)dt. \tag{9}$$

Definition 2: [12], [13] Let a continuous one-port plant be denoted by the input-output mapping $G_{ct} : \mathcal{H}_e \mapsto \mathcal{H}_e$. Denote the continuous input as $u(t) \in \mathcal{H}_e$, the continuous output as $y(t) \in \mathcal{H}_e$, the transformed discrete input as $u[i] \in \mathcal{H}_e$, and the transformed discrete output as $y[i] \in \mathcal{H}_e$. With T_s denoted as the sample and hold rate (seconds), the *inner-product equivalent sample and hold* (*IPESH*) is implemented as follows:

I. $x(t) = \int_0^t y(\tau)d\tau$
II. $y[i] = x[(i+1)T_s] - x[iT_s]$
III. $u(t) = u[i], \forall t \in [iT_s, i(T_s + 1))$

As a result
$$\langle y[i], u[i]\rangle_N = \langle y(t), u(t)\rangle_{NT_s}, \forall N \geq 1 \tag{10}$$
holds.

Theorem 1: Using the *IPESH* given in Definition 2, the following relationships can be stated between the continuous one-port plant, $G_{ct} : \mathcal{H}_e \mapsto \mathcal{H}_e$, and the discrete transformed one-port plant, $G_d : \mathcal{H}_e \mapsto \mathcal{H}_e$:

I. If G_{ct} is *passive* then G_d is *passive*.
II. If G_{ct} is *strictly-input passive* then G_d is *strictly-input passive*.
III. If G_{ct} is *strictly-output passive* then G_d is *strictly-output passive*.

Remark 2: Theorem 1-III is the corrected version of [6, Theorem 3-III]. The proof for Theorem 1-III which we presented at the CDC 2007 is as follows:

Proof: With $T = NT_s$, the continuous *strictly-output passive* system G_{ct} satisfies
$$\langle y(t), u(t)\rangle_T \geq \epsilon \|y(t)_T\|_2^2 - \beta, \forall \tau \geq 0 \tag{11}$$
From Definition 2-II and the *Schwarz's Inequality* we relate $\|y[i]_N\|_2^2$ to $\|y(t)_T\|_2^2$ as follows:
$$\begin{aligned}\|y[i]_N\|_2^2 &= \sum_{j=1}^{m}\left[\sum_{i=0}^{N-1} y_j^2[i]\right] \\ &= \sum_{j=1}^{m}\left[\sum_{i=0}^{N-1}\left(\int_{iT_s}^{(i+1)T_s} y_j(t)dt\right)^2\right] \\ &\leq T_s \sum_{j=1}^{m}\left[\sum_{i=0}^{N-1}\left(\int_{iT_s}^{(i+1)T_s} y_j^2(t)dt\right)\right] \\ &\leq T_s \|y(t)_T\|_2^2\end{aligned} \tag{12}$$

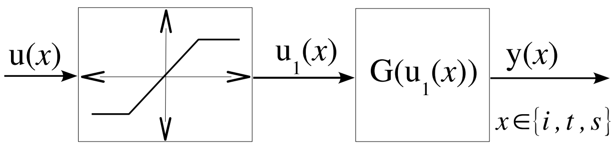

Fig. 2. Actuator saturation depicted for either continuous time (t), discrete time (i), or Laplace domain (s).

Rewriting (12) as

$$\|y(t)_T\|_2^2 \geq \frac{1}{T_s}\|y[i]_N\|_2^2 \tag{13}$$

and substituting (13) into (11) results in

$$\langle y[i], u[i]\rangle_N \geq \frac{\epsilon}{T}\|y[i]_N\|_2^2 - \beta, \forall N \geq 1 \tag{14}$$

therefore, the transformed discrete system G_d satisfies (2). ■

III. *Passive* Systems Subject to Saturation

The input saturation block indicated in Fig. 2 is a special type of actuator input memoryless nonlinearity $\sigma(u(x))$ in which $u(x) \in \mathbb{R}^m$ with components $u_j(x),\ j \in \{1,\ldots,m\}$ which has the following form

$$u_{s_j}(x) = \begin{cases} k_j u_j(x), & \text{if } |u_j(x)| \leq k_j u_{\max_j}(x) \\ k_j u_{\max_j}\mathtt{sign}(u_j(x)), & \text{otherwise} \end{cases} \tag{15}$$

in which each element can have separate linear gain $k_j > 0$, and saturation level $k_j u_{\max_j}\mathtt{sign}(u_j(x))$. For the discussion $G(u_s(x))$ can be thought as either linear or nonlinear *passive* continuous or discrete time mapping in which $x \in \{i, t\}$, where i is a discrete time index, t represents continuous time. We denote $\langle u(x), y(x)\rangle_X$ as either the continuous time ($x = t,\ X = T$) inner-product or the discrete time ($x = i,\ X = N$) inner-product (see (8) (5) in Remark 1).

Theorem 2: For a *passive* system $G(u_s(x))$ which is subject to the input saturation nonlinearity described by (15) assume there exists an admissible $u(x)$, $y(x)$ waveform (sequence) in which $\exists\epsilon < 0,$ and $\delta > 0,\ T > 0\ (N > 1)$ $\alpha = 1$ and index j such that

$$\alpha\epsilon = \int_{T-\delta}^{T} u_j(t)y_j(t)dt \tag{16}$$

where $u_j(t) = \alpha k_j u_{\max_j}\mathtt{sign}(u_j(T-\delta))$ or

$$\alpha\epsilon = u_j[N-1]y_j[N-1] \tag{17}$$

in which $u_j[N-1] = \alpha k_j u_{\max_j}\mathtt{sign}(u_j[N-1])$. Then the continuous (or discrete) input-output mapping $H : \mathcal{H}_e \mapsto \mathcal{H}_e$ is not *passive*. If no such reachable waveform (sequence) exists which satisfies (16) ((17)) the input-output mapping H is *passive*.

Proof: Assume that (16) (or (17)) condition exists, and we input a waveform (sequence) $u(x)$ such that $|u_j(x)| \leq k_j u_{\max_j}(x)$ is always satisfied up to time $T-\delta$ (index $(N-1)-1$), therefore,

$$\langle u(t), y(t)\rangle_{T-\delta} \triangleq \beta(T-\delta) \geq -\beta(0) \tag{18}$$

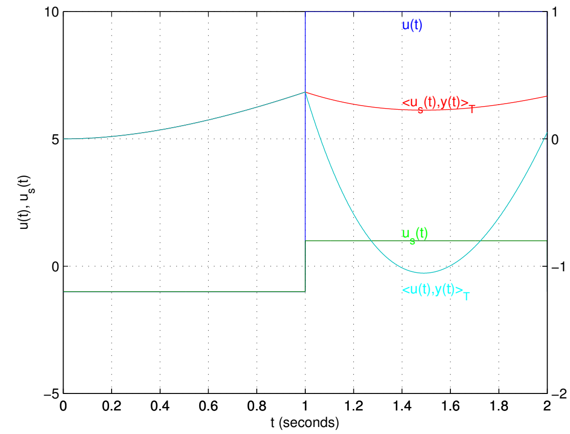

Fig. 3. Simulation of a first order plant $G(u(s)) = \frac{1}{s+1}$ with input saturation in **sector**[0,1].

or

$$\langle u(i), y(i)\rangle_{N-1} \triangleq \beta(N-1) \geq -\beta(0) \tag{19}$$

Noting that:

$$\begin{aligned}\langle u(t), y(t)\rangle_T = \ & \beta(T-\delta) + \textstyle\int_{T-\delta}^{T} u_j(t)y_j(t)dt + \\ & \textstyle\int_{T-\delta}^{T} u_{Nj}(t)^{\mathsf{T}} y_{Nj}(t) \qquad (20) \\ = \ & \beta(T-\delta) + \alpha\epsilon + \\ & \textstyle\int_{T-\delta}^{T} u_{Nj}(t)^{\mathsf{T}} y_{Nj}(t) \qquad (21)\end{aligned}$$

and

$$\begin{aligned}\langle u[i], y[i]\rangle_N = \ & \beta(N-1) + u_j[N-1]y_j[N-1] + \\ & u_{Nj}[N-1]^{\mathsf{T}} y_{Nj}[N-1] \qquad (22) \\ = \ & \beta(N-1) + \alpha\epsilon + \\ & u_{Nj}[N-1]^{\mathsf{T}} y_{Nj}[N-1] \qquad (23)\end{aligned}$$

in which $u_{Nj}(x)^{\mathsf{T}} y_{Nj}(x)$ contain the remaining elements of u and y which don't include j. Since $u_j(x)$ is saturated then the corresponding output $y(x)$ will not change, therefore (16) and (17) will hold for all $\alpha > 1$. Therefore if the net system is to remain *passive* the following must hold for all $\alpha > 1$

$$\beta(T-\delta) + \alpha\epsilon + \int_{T-\delta}^{T} u_{Nj}(t)^{\mathsf{T}} y_{Nj}(t) \geq -\beta(0)$$

$$\alpha \leq -\frac{\beta(0) + \beta(T-\delta) + \int_{T-\delta}^{T} u_{Nj}(t)^{\mathsf{T}} y_{Nj}(t)}{\epsilon} \tag{24}$$

for the continuous time case or

$$\beta(N-1) + \alpha\epsilon + u_{s_{NJ}}^{\mathsf{T}}[N-1]y_{NJ}[N-1] + \geq -\beta(0)$$

$$\alpha \leq -\frac{\beta(0) + \beta(N-1) + u_{Nj}^{\mathsf{T}}[N-1]y_{Nj}[N-1]}{\epsilon} \tag{25}$$

for the discrete time case. However this is clearly not possible since the expressions to the right of (24) and (25) are finite. Likewise if no such sequence exists for $\alpha = 1$ in which $\epsilon < 0$ then *passivity* is preserved. ■

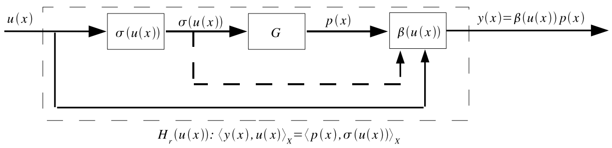

Fig. 4. The nonlinear controller $\beta(u(x))$ recovers the inner-product lost due to the memoryless nonlinearity $\sigma(u(x))$.

Remark 3: In [11, Exercise VI-4.7] it was given as an exercise to show how passivity is lost for $[0, k)$ *sector* input nonlinearities (Appendix I) when the *passive* plant has the following form $G(u(s)) = \frac{1}{1+qs}$. Theorem 2 tells us that the *passive* plant with input saturation nonlinearity in **sector** $[0, u_{\max}k]$ will no longer be *passive* (Fig. 3).

Remark 4: One example of a *passive* system which maintains *passivity* when subject to actuator saturation is a positive semi-definite gain block, $G(u(x)) = Ku(x)$ in which $u^{\mathsf{T}}(x)G(u(x))u(x) = u^{\mathsf{T}}(x)Ku(x) \geq 0,\ \forall u(x)$. Hence by Theorem 2 the net system H is *passive*.

A. The Inner-Product Recovery Block.

Fig. 4 depicts how we prefer to implement the nonlinear controller $\beta(u(x))$ introduced in [7]. We choose to locate $\beta(u(x))$ at the output of the plant $G : \sigma(u(x)) \mapsto p(x)$ and explicitly analyze the map $H_p : u \mapsto y$. As indicated in Fig. 4, $\beta(u(x))$ recovers the inner-product such that $\langle y(x), u(x)\rangle_X = \langle p(x), \sigma(u(x))\rangle_X$, therefore, we shall refer to $\beta(u(x))$ as the *inner-product recovery block*.

Lemma 1: Consider the *inner-product recovery block* $\beta(u(x))$ as defined by Definition 3 in Appendix I, using either an exact model or measurement of the memoryless nonlinearity $\sigma(u(x))$. When using the *inner-product recovery block* as depicted in Fig. 4, the following will always be satisfied:

$$\langle y(x), u(x)\rangle_X = \langle p(x), \sigma(u(x))\rangle_X \tag{26}$$

Proof: The proof is straight forward using (43) to yield (29) and (42) to yield (30).

$$\begin{aligned}\langle y(x), u(x)\rangle_X &= \langle \beta(u(x))p(x), u(x)\rangle_X && (27)\\ &= \langle p(x), \beta(u(x))^{\mathsf{T}} u(x)\rangle_X && (28)\\ &= \langle p(x), \beta(u(x))u(x)\rangle_X && (29)\\ &= \langle p(x), \sigma(u(x))\rangle_X && (30)\end{aligned}$$

■

Lemma 1 allows us to prove Theorem 3.

Theorem 3: Consider the *inner-product recovery block* $\beta(u(x))$ as defined by Definition 3 in Appendix I, using either an exact model or measurement of the memoryless nonlinearity $\sigma(u(x))$. When using the *inner-product recovery block* as depicted in Fig. 4, the following can be said about $H_r : u(x) \mapsto y(x)$ given $G : \sigma(u(x)) \mapsto p(x)$.

I. If G is *passive* then H_r is *passive*.
II. If G is *strictly-output passive* then H_r is *strictly-output passive* if $\sigma_{MAX}(\beta(u))^2 < \infty,\ \forall u \in \mathbb{R}^m$ (Definition 5).
III. If G is *strictly-input passive* then H_r is *strictly-input passive* if there exists a $\gamma > 0$ such that $\sigma^{\mathsf{T}}(u)\sigma(u) \geq \gamma u^{\mathsf{T}}u,\ \forall u \in \mathbb{R}^m$.

Proof: Based on our assumptions, Lemma 1 shows that (26) will hold. Furthermore:

I. if G is *passive* then $\langle p(x), \sigma(u(x))\rangle_X \geq -\beta$, substituting (26) yields
$$\langle y(x), u(x)\rangle_X \geq -\beta \tag{31}$$
which satisfies Definition 1 for *passivity*.

II. if G is *strictly-output passive* then $\langle p(x), \sigma(u(x))\rangle_X \geq \epsilon\|(p(x))_X\|_2^2 - \beta$, substituting (26) yields
$$\langle y(x), u(x)\rangle_X \geq \epsilon\|(p(x))_X\|_2^2 - \beta \tag{32}$$
next, we solve for $\|(y(x))_X\|_2^2$
$$\begin{aligned}\|(y(x))_X\|_2^2 &= \|(\beta(u(x))p(x))_X\|_2^2\\ &\leq \sigma_{MAX}(\beta(u))^2\|(p(x))_X\|_2^2 && (33)\end{aligned}$$
and substitute (33) into (32) to yield
$$\langle y(x), u(x)\rangle_X \geq \frac{\epsilon}{\sigma_{MAX}(\beta(u))^2}\|(y(x))_X\|_2^2 - \beta \tag{34}$$
which satisfies Definition 1 for H_r to be *strictly-output passive* as long as $\sigma_{MAX}(\beta(u))^2 < \infty$.

III. if G is *strictly-input passive* then $\langle p(x), \sigma(u(x))\rangle_X \geq \delta\|(\sigma(u(x)))_X\|_2^2 - \beta$, substituting (26) yields
$$\langle y(x), u(x)\rangle_X \geq \delta\|(\sigma(u(x)))_X\|_2^2 - \beta \tag{35}$$
next, our assumption that $\sigma^{\mathsf{T}}(u)\sigma(u) \geq \gamma u^{\mathsf{T}}u,\ \forall u \in \mathbb{R}^m$ implies that
$$\|(\sigma(u(x)))_X\|_2^2 \geq \gamma\|(u(x))_X\|_2^2 \tag{36}$$
and substitute (36) into (35) to yield
$$\langle y(x), u(x)\rangle_X \geq \delta\gamma\|(u(x))_X\|_2^2 - \beta \tag{37}$$
which satisfies Definition 1 for H_r to be *strictly-input passive*.

■

Remark 5: Theorem 3 is written from the least restrictive case to the most restrictive case in terms of the memoryless nonlinearities $\sigma(u)$ which can be tolerated. Theorem 3-I can be applied to all types of $\sigma(u)$. Whereas Theorem 3-II can be applied to a slightly smaller class of nonlinearities within $\sigma(u)$ such as those which typically have actuator saturation. They can not include *unbounded* quadratic nonlinearities, $\sigma(u_i) = u_i^p, p \geq 2,\ p \leq -1$, however, quadratic nonlinearities can occur as long as $\sigma(u)$ is linear or saturates as $u \to \infty$ for $p \geq 1$, or a dead-zone or linearity occurs at the origin when $p \leq -1$. Also, near the origin, relay nonlinearities can not be tolerated.

$$\sigma(u_i) = \begin{cases}\mathtt{sign}(u_i),\ |u_i| > 0\\ 0, u_i = 0\end{cases} \tag{38}$$

Finally, Theorem 3-III does not include quadratic nonlinearities at the origin, or saturation nonlinearities. However, all **sector**$[k_1, k_2]$ nonlinearities (Appendix I) in which $0 < k_1 \leq k_2$ or $k_1 \leq k_2 < 0$, will preserve *strictly-input passivity*.

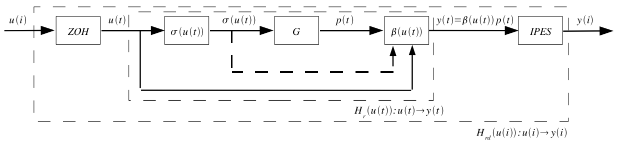

Fig. 5. Continuous time *inner-product recovery block* used with *IPESH*.

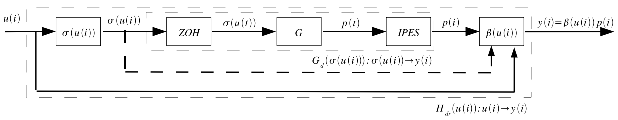

Fig. 6. Discrete time *inner-product recovery block* used with *IPESH*.

B. Implementing the Inner-Product Recovery Block with the Inner-Product Equivalent Sample and Hold

By implementing the continuous time *inner-product recovery block* as depicted in Fig. 5 we can directly use Theorem 1 and Theorem 3 in order to state the following Corollary:

Corollary 1: Using the *IPESH* given in Definition 2 and the continuous time *inner-product recovery block* as depicted in Fig. 5, the following relationships can be stated between the continuous one-port plant, $G : \sigma(u(t)) \mapsto p(t)$, and the discrete transformed one-port plant, $H_{rd} : u(i) \mapsto y(i)$:

I. If G is *passive* then H_{rd} is *passive* for all $\sigma(u)$.
II. If G is *strictly-input passive* then H_{rd} is *strictly-input passive* if $\sigma_{MIN}(\beta(u))^2 > 0, \ \forall u \in \mathbb{R}^m$.
III. If G is *strictly-output passive* then H_{rd} is *strictly-output passive* if $\sigma(u)$ is a **sector**$[k_1, k_2]$ nonlinearity such that $\sigma_{MAX}(\beta(u))^2 = \max(k_1^2, k_2^2) < \infty$.

Remark 6: See Appendix I (Theorem 5 and Theorem 6) in order to see why $\sigma_{MAX}(\beta(u))^2 = \max(k_1^2, k_2^2)$.
Next, we note that by switching the order of the *ZOH* and the memoryless nonlinearity $\sigma(\cdot)$ is mathematically equivalent

$$\sigma(\texttt{ZOH}(u(i))) = \sigma(u(t)) = \texttt{ZOH}(\sigma(u(i))) \qquad (39)$$

as depicted in Fig. 6.

Corollary 2: Using the *IPESH* given in Definition 2 and the discrete time *inner-product recovery block* as depicted in Fig. 6, the following relationships can be stated between the continuous one-port plant, $G : \sigma(u(t)) \mapsto p(t)$, and the discrete transformed one-port plant, $H_{dr} : u(i) \mapsto y(i)$:

I. If G is *passive* then H_{dr} is *passive* for all $\sigma(u)$.
II. If G is *strictly-input passive* then H_{dr} is *strictly-input passive* if $\sigma_{MIN}(\beta(u))^2 > 0, \ \forall u \in \mathbb{R}^m$.
III. If G is *LTI* and *strictly-output passive* then H_{dr} is *strictly-output passive* if the **sector**$[k_1, k_2]$ nonlinearity is such that $\sigma_{MAX}(\beta(u))^2 = \max(k_1^2, k_2^2) < \infty$.

C. l_2^m-stable Digital Control Networks Subject to Memoryless Nonlinearities.

Theorem 4: The digital control network depicted in Fig. 1 in which $K_p > 0$, $K_c > 0$, G_c are *passive* and the *passive* plant G_p is subject to memoryless nonlinearities $\sigma(\cdot)$, is *strictly-output passive* which is sufficient for l_2^m-stability if

$$\langle f_{op}, e_{doc} \rangle_N \geq \langle e_{oc}, f_{opd} \rangle_N \qquad (40)$$

holds for all $N \geq 1$.

Proof: Corollary 2-I shows that the mapping from $e_p(i)$ to $f_{op}(i)$ is *passive* for any memoryless actuator nonlinearity associated with the plant. Since the mapping is now *passive*, the *strictly-output passivity* and l_2^m-stability follows from [6, Theorem 4]. ∎

IV. Conclusions

Theorem 2 shows that actuator saturation typically eliminates the *passive* mapping. To address this issue this paper provides: i) Lemma 1 which states that the *IPRB* depicted in Fig. 4 recovers the inner-product mapping of either a continuous or discrete time system subject to a memoryless nonlinearity, ii) Theorem 3 notes that *passivity* is always recovered with an *IPRB*, however only certain classes of memoryless nonlinearities will allow a *strictly-output passive* or *strictly-input passive* mapping to be preserved, iii) Corollary 1 states that the *IPESH* can be used with a continuous time *IPRB* to preserve *passivity*, and conditions to also preserve either a *strictly-input passive* or *strictly-output passive* mapping. iv) Corollary 2 is similar to Corollary 1, except that it relates to implementing the *IPRB* in discrete time, the key to such a realization is (39) and the carefully chosen system depicted in Fig. 6. Furthermore some *important* connections between **sector**$[k_1, k_2]$ nonlinearities (see Definition 4) and the equivalent constraints stated by Corollary 1 are given in the Appendix (see Theorem 5 and Theorem 6). v) Theorem 4, completes our discussion, stating how a *IPRB* and *IPESH* can be used in discrete time to implement a l_2^m-*stable* digital control network as depicted in Fig. 1.

References

[1] Y. Ji and H. Chizeck, "Jump linear quadratic gaussian control. steady-state solution and testable conditions," *Control - Theory and Advanced Technology*, vol. 6, no. 3, pp. 289 – 319, 1990.
[2] Y. Ji, H. Chizeck, X. Feng, and K. Loparo, "Stability and control of discrete-time jump linear systems," *Control - Theory and Advanced Technology*, vol. 7, no. 2, pp. 247 – 270, 1991.
[3] P. Seiler and R. Sengupta, "Analysis of communication losses in vehicle control problems," *Proceedings of the American Control Conference*, vol. 2, pp. 1491 – 1496, 2001.
[4] ——, "An h infinity approach to networked control," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 356 – 364, 2005.
[5] N. Kottenstette, "Control of passive plants with memoryless nonlinearitites over wireless networks," Ph.D. dissertation, University of Notre Dame, August 2007. [Online]. Available: http://etd.nd.edu/ETD-db/theses/available/etd-08302007-163513/
[6] N. Kottenstette and P. J. Antsaklis, "Stable digital control networks for continuous passive plants subject to delays and data dropouts," *Proceedings of the 46th IEEE Conference on Decision and Control*, pp. 4433 – 4440, 2007.
[7] D. S. Bernstein and W. M. Haddad, "Nonlinear controllers for positive real systems with arbitrary input nonlinearities," *IEEE Transactions on Automatic Control*, vol. 39, no. 7, pp. 1513 – 1517, 1994.
[8] W. M. Haddad and V. Chellaboina, "Nonlinear controllers for nonlinear passive systems with arbitrary input nonlinearities," *Proceedings of the American Control Conference*, vol. 6, pp. 4149 – 4153, 2000.
[9] W. Haddad and V. Chellaboina, "Nonlinear control of hammerstein systems with passive nonlinear dynamics," *IEEE Transactions on Automatic Control*, vol. 46, no. 10, pp. 1630 – 1634, 2001.
[10] A. van der Schaft, *L2-Gain and Passivity in Nonlinear Control.* Secaucus, NJ, USA: Springer-Verlag New York, Inc., 1999.
[11] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties.* Orlando, FL, USA: Academic Press, Inc., 1975.

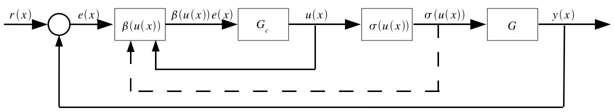

Fig. 7. Nonlinear controller arrangement studied in [7]–[9].

[12] S. Stramigioli, C. Secchi, A. van der Schaft, and C. Fantuzzi, "A novel theory for sampled data system passivity," *Proceedings IEEE/RSJ International Conference on Intelligent Robots and Systems (Cat. No.02CH37332C)*, vol. vol.2, pp. 1936 – 41, 2002.

[13] J.-H. Ryu, Y. S. Kim, and B. Hannaford, "Sampled- and continuous-time passivity and stability of virtual environments," *IEEE Transactions on Robotics*, vol. 20, no. 4, pp. 772 – 6, 2004.

[14] H. K. Khalil, *Nonlinear Systems*. Prentice-Hall, Inc., 1996.

APPENDIX I
MEMORYLESS NONLINEARITIES

[11, Definition I-1.1] provides a scalar definition for **sector** nonlinearities ($\sigma : \mathbb{R} \rightarrow \mathbb{R},\ \sigma(0) = 0$) which leads to numerous equivalent statements to describe a **sector** nonlinearity as provided in [11, Theorem I-1.2]. When the input u is a vector and ($\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m,\ \sigma(0) = 0$), such a set of equivalent statements are difficult to make. However [14, Definition 10.1] is fairly general and applies to a large class of sector nonlinearities and can be typically treated as time varying if desired. In [7]–[9] the following nonlinear control block $\beta(u(x))$ has been shown to effectively create stable control systems which are subject to an even broader class of memoryless nonlinearities, $\sigma(u(x))$, as depicted in Fig. 7. The controller has been studied when $e(x) = r(x) - y(x), x = t$ in which G and G_c are (*strictly*)-*positive real* [7]. Later it was shown when $e(x) = r(x), x = t\ (x = i)$ that G and G_c form a *passive-exponentially* (*geometrically*) *passive* pair [8], [9]. The only constraints on these memoryless nonlinearities is that $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $\forall i \in \{1, 2, \ldots, m\}$, if $u_i(x) = 0$ then $\sigma_i(u(x)) = 0$.

Definition 3: The *inner-product recovery block* (*IPRB*) $\beta(u(x))$ has the following form: $\beta(u(x)) = \mathsf{diag}(\beta_1(u(x)), \ldots, \beta_m(u(x)))$, where

$$\beta_i(u(x)) = \begin{cases} \frac{\sigma_i(u(x))}{u_i(x)},\ \text{if } u_i(x) \neq 0 \\ 1,\ \text{if } u_i(x) = 0 \end{cases} \tag{41}$$

Remark 7: In [8], [9] when $u_i(x) = 0$, $\beta_i(u(x))$ has been defined as being arbitrary, however, we chose to use $\beta_i(u(x)) = 1$, if $u_i(x) = 0$ because when $u_i(x) = 0$ then $\sigma_i(u(x)) = u_i(x)$ by assumption, hence, $\frac{\sigma_i(u(x))}{u_i(x)} = \frac{u_i(x)}{u_i(x)} = 1$ when $u_i(x) = 0$. Therefore for $x \in \{i, t\}$,

$$\beta(u(x))u(x) = \sigma(u(x)) \text{ and} \tag{42}$$

$$\beta^{\mathsf{T}}(u(x)) = \beta(u(x)). \tag{43}$$

Therefore we propose the following definition:

Definition 4: A memoryless nonlinearity $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is said to satisfy a global **sector** condition belonging to sector $[k_1, k_2]$ if

$$k_1 u^{\mathsf{T}} u \leq u^{\mathsf{T}} \sigma(u) \leq k_2 u^{\mathsf{T}} u,\ \forall u \in \mathbb{R}^m \tag{44}$$

holds any $k_1, k_2 \in \mathbb{R}$. If (44) holds with strict inequality, then $\sigma(\cdot)$ is said to belong to a sector (k_1, k_2).

Theorem 5: Let $k_1, k_2 \in \mathbb{R}$ with $k_1 \leq k_2$. Let $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^m$ with $\sigma_i(u_i = 0) = 0,\ i \in \{1, \ldots, m\}$ and $\beta(u)$ as defined by Definition 3 in Appendix I. Such that the following statements are equivalent:

i) $(k_1 I - \beta(u)) \leq 0$ and $(\beta(u) - k_2 I) \leq 0,\ \forall u \in \mathbb{R}^m$
ii) $k_1 u^{\mathsf{T}} u \leq u^{\mathsf{T}} \sigma(u) \leq k_2 u^{\mathsf{T}} u,\ \forall u \in \mathbb{R}^m$

Proof:

i) $\rightarrow$ ii) first we prove the left half of the inequality holds for both definitions.

$$u^{\mathsf{T}}(k_1 I - \beta(u))u \leq 0,\ \forall u \in \mathbb{R}^m$$

$$k_1 u^{\mathsf{T}} u \leq u^{\mathsf{T}}(\beta(u)u),\ \forall u \in \mathbb{R}^m \tag{45}$$

$$k_1 u^{\mathsf{T}} u \leq u^{\mathsf{T}} \sigma(u),\ \forall u \in \mathbb{R}^m \tag{46}$$

Note that the simplification from (45) to (46) is a direct result of application of the definition for $\beta(u)$. The proof for the right half of the inequality is as follows:

$$u^{\mathsf{T}}(\beta(u) - k_2 I)u \leq 0,\ \forall u \in \mathbb{R}^m$$

$$u^{\mathsf{T}}(\beta(u)u) \leq k_2 u^{\mathsf{T}} u,\ \forall u \in \mathbb{R}^m \tag{47}$$

$$u^{\mathsf{T}} \sigma(u) \leq k_2 u^{\mathsf{T}} u,\ \forall u \in \mathbb{R}^m. \tag{48}$$

ii) $\rightarrow$ i) is fairly obvious when we substitute $\sigma(u) = \beta(u)u$ and $u^{\mathsf{T}} u = u^{\mathsf{T}} I u$ in to (44) which yields

$$k_1 u^{\mathsf{T}} I u \leq u^{\mathsf{T}} \beta(u) u \leq k_2 u^{\mathsf{T}} I u,\ \forall u \in \mathbb{R}^m, \text{ in which}$$

$$u^{\mathsf{T}}(k_1 I - \beta(u))u \leq 0,\ \forall u \in \mathbb{R}^m \tag{49}$$

$$(k_1 I - \beta(u)) \leq 0, \forall u \in \mathbb{R}^m \text{ similarly,} \tag{50}$$

$$u^{\mathsf{T}}(\beta(u) - k_2 I)u \leq 0,\ \forall u \in \mathbb{R}^m \tag{51}$$

$$(\beta(u) - k_2 I) \leq 0, \forall u \in \mathbb{R}^m. \tag{52}$$

■

Next we relate the maximum and minimum singular values for $\beta(u)$ to the sector $[k_1, k_2]$ bounds.

Definition 5: The maximum achievable singular value squared is $\sigma_{MAX}(\beta(u))^2 \stackrel{\triangle}{=} \max(\sigma_M(\beta(u))^2),\ \forall u \in \mathbb{R}^m$ in which $\sigma_M(\beta(u))$ denotes the maximum singular value of the resulting matrix $\beta(u)$ for a given u.

Definition 6: The minimum achievable singular value squared is $\sigma_{MIN}(\beta(u))^2 \stackrel{\triangle}{=} \min(\sigma_m(\beta(u))^2),\ \forall u \in \mathbb{R}^m$ in which $\sigma_m(\beta(u))$ denotes the minimum singular value of the resulting matrix $\beta(u)$ for a given u.

Theorem 6: For a given sector $[k_1, k_2]$ nonlinearity:

i) If either $k_1 = 0$, or $k_2 = 0$ then $\sigma_{MIN}(\beta(u))^2 = 0$.
ii) $\sigma_{MAX}(\beta(u))^2 = \max(k_1^2, k_2^2)$.

Proof: Since $\beta(u)$ is a diagonal matrix, then $\sigma_M(\beta(u)) = \max(|\lambda_M(\beta(u))|, |\lambda_m(\beta(u))|)$ in which $\lambda_M(\cdot)$ denotes the maximum eigenvalue and $\lambda_m(\cdot)$ denotes the minimum eigenvalue of $\beta(u)$.

i) If $k_1 \geq 0$ then $\sigma_{MIN}(\beta(u))^2 = k_1^2$, therefore $k_1 = 0$ implies that $\sigma_{MIN}(\beta(u))^2 = 0$. If $k_2 \leq 0$ then $\sigma_{MIN}(\beta(u))^2 = k_2^2$, therefore $k_2 = 0$ implies that $\sigma_{MIN}(\beta(u))^2 = 0$.
ii) From Theorem 5-i, we see that $\min(\lambda_m(\beta(u))) = k_1, \forall u \in \mathbb{R}^m$ and $\max(\lambda_M(\beta(u))) = k_2, \forall u \in \mathbb{R}^m$, therefore $\sigma_{MAX}(\beta(u))^2 = \max(k_1^2, k_2^2)$.

■