P. J. Antsaklis and W. A. Wolovich, "Some New Bounds Related to Output Feedback Pole Placement,"
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Example

To illustrate using identification for fault detection consider the first-order digital filter

$$H(Z)=\frac{a_1 Z^{-1}}{1-b_1 Z^{-1}}=\frac{0.8Z^{-1}}{1-0.85Z^{-1}}. \tag{8}$$

For $\Delta|H[e^{j\omega T}]|=0.1$, the acceptable region for the filter coefficients is shown in Fig. 2. The identifier response to the filter is shown in Fig. 3 as b_1 changes from 0.85 to 0.4. Initially the identifier tracks the correct coefficients. When b_1 changes the error signal changes rapidly and converges to zero. The coefficient estimates converge to the new values and the b_1 coefficient of 0.4 does not allow satisfactory performance through use of the region shown in Fig. 2. A redundant filter would then be set into operation.

In the example, a noise-free simulation, the output of the identifier can be used directly to determine acceptable performance. Note that the error criterion of (7) is only valid for small coefficient changes. However, for large deviations, the performance is obviously not acceptable. In a stochastic environment, the statistics of the $\hat{c}$ vector must be utilized to determine acceptable performance.

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Some New Bounds Related to Output Feedback Pole Placement

P. J. ANTSAKLIS AND W. A. WOLOVICH

***Abstract*—A number of new results regarding linear output feedback compensation are presented. In particular, it is shown that the rank of an appropriately defined real matrix Ω represents an upper bound on the number of closed-loop poles which can be completely and arbitrarily assigned via constant gain output feedback. A new bound on the minimum number of dynamical elements required for complete and arbitrary closed-loop pole placement is also defined in terms of the observability index of a certain single-input system.**

I. Introduction

The primary purpose of this correspondence is to study the effect which linear output feedback compensation has on the closed-loop poles of linear multivariable systems. Unlike the majority of previous reports which have dealt with this question, the approach taken here will not be directly concerned with the development of any constructive procedure for arbitrarily assigning a certain number of closed-loop poles. Rather, a new matrix rank test will be outlined for determining an upper bound on the number of closed-loop poles which can be arbitrarily positioned via linear output feedback compensation.

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In Section II, we will show that the rank of a real matrix Ω represents a measure of the maximum number of poles which can be arbitrarily assigned via output feedback. We further illustrate conditions under which one cannot arbitrarily assign a number of closed-loop poles equal to the number of independent and arbitrary gain parameters when this latter number does not exceed the system order.

In Section III, we consider the employment of dynamic compensation in conjunction with linear output feedback in order to enhance closed-loop pole placement when output feedback alone is inadequate. Here we employ our earlier results to show that the observability index of an appropriately defined single-input system represents a measure of the order of dynamic compensation which is required for complete and arbitrary pole placement. A number of examples are provided throughout to illustrate and clarify the presentations, and a summation of the main results is given in the final section.

II. Linear Output Feedback Compensation

We will consider the class of all nth-order minimal (controllable and observable) state-space systems whose dynamical behavior can be represented as

$$\dot{x}(t)=Ax(t)+Bu(t), \quad y(t)=Cx(t) \tag{1}$$

with m-dimensional input $u(t)$, p-dimensional output $y(t)$, and A, B, and C real matrices of the appropriate dimensions. Rather than working directly with (1), we find it convenient to deal with a particular factored form of the strictly proper transfer matrix $T(s)$ associated with (1); i.e., it is well known [1] that under the assumptions noted,

$$T(s)=C(sI-A)^{-1}B=R(s)P^{-1}(s) \tag{2}$$

with $R(s)$ and $P(s)$ relatively right prime polynomial matrices in s, the Laplace operator, of dimensions $p\times m$ and $m\times m$, respectively. It should be noted that the (open-loop) poles of (1) are equal to the zeros of

$$\Delta(s) \triangleq |sI-A|=|P(s)|=s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0. \tag{3}$$

Furthermore, in view of (2), it is clear that if $D\triangleq d/dt$, then

$$P(D)z(t)=u(t); \quad y(t)=R(D)z(t) \tag{4}$$

represents a differential operator realization [1] of $T(s)$ with partial state $z(t)$. Since (1) and (4) are both minimal and realize the same transfer matrix, we finally note that the two representations are equivalent [1].

If *linear output feedback* (*lof*) is now defined by the control law

$$u(t)=-Hy(t)+v(t), \tag{5}$$

it follows that under lof compensation, the closed-loop poles of the system (1) or (4) are equal to the zeros of

$$\Delta_H(s)\triangleq|sI-A+BHC|=|P(s)+HR(s)| \\ =s^n+\alpha_{n-1}s^{n-1}+\cdots+\alpha_1 s+\alpha_0. \tag{6}$$

We realize, of course, that it is of considerable practical importance to determine the effect which H has on the zeros of $\Delta_H(s)$, and numerous recent papers have addressed this question with varying degrees of success [2]–[13].

To begin our discussion here, we will require some notation from Gantmacher [14]; i.e., if g_{ij} denotes the ith-row, jth-column element of a matrix $G=[g_{ij}]$, then

$$G\begin{pmatrix} i_1 i_2 \cdots i_k \\ j_1 j_2 \cdots j_k \end{pmatrix} \triangleq \begin{vmatrix} g_{i_1 j_1} & g_{i_1 j_2} & \cdots & g_{i_1 j_k} \\ g_{i_2 j_1} & g_{i_2 j_2} & \cdots & g_{i_2 j_k} \\ \vdots & & & \\ g_{i_k j_1} & g_{i_k j_2} & \cdots & g_{i_k j_k} \end{vmatrix} \tag{7}$$

represents the appropriate kth-order *minor* of G. We next note that $\Delta_H(s)=|P(s)+HR(s)|$, as given by (6), can be written as

$$\Delta_H(s)=|P(s)+HR(s)|=\left|[H \,|\, I_m]\begin{bmatrix} R(s) \\ \hline P(s)\end{bmatrix}\right| \tag{8}$$

or, in view of the Binet–Cauchy formula [14], as

$$\Delta_H(s)=\sum_{1\leq j_1<j_2\cdots<j_m\leq m+p}[H \,|\, I_m]\begin{pmatrix}1 & 2 & \cdots & m\\ j_1 & j_2 & \cdots & j_m\end{pmatrix} \cdot\begin{bmatrix} R(s) \\ \hline P(s)\end{bmatrix}\begin{pmatrix}j_1 & j_2 & \cdots & j_m\\ 1 & 2 & \cdots & m\end{pmatrix}. \tag{9}$$

In other words, $\Delta_H(s)=|P(s)+HR(s)|$ can be expressed as a sum of products of the mth-order minors of $[H \,|\, I_m]$ and the mth-order minors of $\begin{bmatrix} R(s) \\ \hline P(s)\end{bmatrix}$. Since $[H \,|\, I_m]$ has m rows and $m+p$ columns (or since $\begin{bmatrix} R(s) \\ \hline P(s)\end{bmatrix}$ has $m+p$ rows and m columns) there will be a total of

$$g \triangleq \binom{m+p}{m}=\frac{(m+p)!}{m!p!} \tag{10}$$

products of minors in (9). Furthermore, mp of these products will involve the individual elements h_{ij} of H alone, while $g-mp-1$ products will involve products of the h_{ij}. On product term, namely, $|I_m|\times|P(s)|=\Delta(s)$, will not involve any elements of H.

In view of the above, we next note that $\Delta_H(s)-\Delta(s)=\Delta_H(s)-|P(s)|$ can be expressed as the product of a $g-1$-dimensional row vector M_{HI} which consists of the elements of $[H \,|\, I_m]\begin{pmatrix}1 & 2 & \cdots & m\\ j_1 & j_2 & \cdots & j_m\end{pmatrix}$, excluding $1=|I_m|$, and a $g-1$-dimensional column vector M_{RP} which consists of an appropriate ordering (depending on the choice of M_{HI}) of the mth-order minors of $\begin{bmatrix} R(s) \\ \hline P(s)\end{bmatrix}$, excluding $|P(s)|=\Delta(s)$; i.e.,

$$\Delta_H(s)-\Delta(s)=M_{HI}M_{RP} \tag{11}$$

for an appropriate, nonunique[1] pair of vectors $\{M_{HI},M_{RP}\}$. We next note that since $T(s)=R(s)P^{-1}(s)$ is a strictly proper transfer matrix, the elements of M_{RP} will be known polynomials in s of degree strictly less than n. It therefore follows that if $S_n(s)\triangleq[1,s,\cdots,s^{n-1}]^T$, then

$$M_{RP}=\Omega S_n(s) \tag{12}$$

for some real $(g-1)\times n$-dimensional matrix Ω. If we now set $\Delta(s)=s^n+\bar{a}S_n(s)$ and $\Delta_H(s)=s^n+\bar{\alpha}S_n(s)$, where $\bar{a}=[a_0,a_1,\cdots,a_{n-1}]$ and $\bar{\alpha}=[\alpha_0,\alpha_1,\cdots,\alpha_{n-1}]$, then (11) and (12) directly imply the relation

$$\bar{\alpha}-\bar{a}=M_{HI}\Omega. \tag{13}$$

If we finally let ω denote the rank of Ω; i.e., $\omega=\rho[\Omega]$, and define q as the minimum of ω and mp; i.e.,

$$q\triangleq\min(\omega,mp), \tag{14}$$

we can state and establish the main result of this section.

Theorem 1: No more than q coefficients of $\Delta_H(s)$ can be arbitrarily assigned via H.

Proof: The proof of Theorem 1 is an immediate consequence of (13) and the fact that only mp elements of M_{HI}, namely, the individual h_{ij} terms, are independent. In particular, in view of (13) it is clear that the maximum number of elements of $\bar{\alpha}$ which can be arbitrarily assigned via H can exceed neither mp, which represents the number of independent elements of M_{HI}, nor ω, which represents the number of independent (and generally nonlinear) equations involving the h_{ij}. Theorem 1 is therefore established.

A number of remarks are now in order.

Remark 1: Since the coefficients of $\Delta_H(s)$ represent independent functions of its zeros, it follows that *no more than q closed-loop poles can be arbitrarily assigned via lof compensation.*[2] It is of interest to note that this result offers insight with respect to some "special cases" discussed in [12] and [13].

As noted earlier, the results outlined in this correspondence are not primarily concerned with the development of any constructive procedures for arbitrarily assigning either q closed-loop poles or q coefficients of $\Delta_H(s)$, since such an assignment would generally involve the simultaneous solution of nonlinear algebraic equations. Furthermore, as we illustrate in our next remarks, it is not always possible to arbitrarily assign q coefficients of $\Delta_H(s)$.

Remark 2: Since Ω has n columns, $\omega\leq n$, which implies that $q\leq n$. The condition $q=n$; i.e., $\omega=n$ and $mp\geq n$ is therefore necessary for complete and arbitrary closed-loop pole placement via lof. As noted in our prior remark, however, the condition $q=n$ is not sufficient[3] as we now illustrate.

Example 1: If

$$A=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{bmatrix},\quad B=\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix},\quad \text{and}\quad C=\begin{bmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\end{bmatrix},$$

then

$$T(s)=C(sI-A)^{-1}B=\begin{bmatrix}\frac{1}{s} & \frac{1}{s^2+1}\\ \frac{1}{s^2} & \frac{s}{s^2+1}\end{bmatrix}=\begin{bmatrix}s & 1\\ 1 & s\end{bmatrix}\begin{bmatrix}s^2 & 0\\ 0 & s^2+1\end{bmatrix}^{-1}=R(s)P^{-1}(s).$$

For example,

$$H=\begin{bmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{bmatrix},$$

and one choice for M_{HI} would be

$$M_{HI}=[h_{11},h_{12},-h_{21},-h_{22},h_{11}h_{22}-h_{12}h_{21}],$$

with corresponding

$$M_{RP}=[s^3+s,s^2+1,-s^2,-s^3,s^2-1]^T\quad\text{and}\quad\Omega=\begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 1 & 0\end{bmatrix};$$

i.e., for this example, (13) implies that

$$[\alpha_0,\alpha_1,\alpha_2-1,\alpha_3]=[h_{11},h_{12},-h_{21},-h_{22},h_{11}h_{22}-h_{12}h_{21}]\cdot\begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 1 & 0\end{bmatrix}$$

or that $h_{11}=\alpha_1$, $h_{22}=\alpha_3-h_{11}=\alpha_3-\alpha_1$, and $h_{21}=\alpha_0+\alpha_2-1-2h_{12}$ with h_{12} the solution to the quadratic equation

$$2h_{12}^2-(\alpha_0+\alpha_2)h_{12}+\alpha_0+\alpha_1\alpha_3-\alpha_1^2=ah_{12}^2+bh_{12}+c=0.$$

We therefore note that if $b^2-4ac=(\alpha_0+\alpha_2)^2-8(\alpha_0+\alpha_1\alpha_3-\alpha_1^2)$ is a negative number, which it will be for certain choices of the α_i, h_{12} will be a complex number. In other words, for certain choices of all $n=4$

[1]It should be noted that the individual elements of M_{HI} and M_{RP} are unique, although their ordering is not. Moreover, a particular ordering of the elements of one of the vectors implies a corresponding, unique ordering of the elements of the other vector.

[2]It should be noted that the "arbitrary assignment" of closed-loop poles always implies the inclusion of complex conjugates.

[3]In the case $m=1$ (or $p=1$), the condition $q=\omega=p$ (or m) is both necessary and sufficient for arbitrary assignment of p (or m) coefficients of $\Delta_H(s)$, due to the presence of only linear equations in (13).

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closed-loop poles, as well as choices "arbitrarily close" to these choices, there will be no real gain matrix H to assign these poles. Of course, for other choices of all $n=4$ closed-loop poles, all of the elements of H will be real.

Remark 3: In view of the above, it is now natural to ask 1) whether or not information can be obtained regarding arbitrary lof of pole placement without first determining Ω and 2) what can be done from the point of view of both analysis and synthesis when Ω is not of rank n? We address the analysis part of the latter question here, and the remaining questions in our subsequent discussions.

In particular, if $\omega<n$ we can readily obtain $n-\omega$ independent and linear relations which the coefficients of $\Delta_H(s)$ must satisfy independent of any choice for H. More specifically, if $\rho[\Omega]=\omega<n$, then a nonsingular $(n\times n)$ matrix K can clearly be found such that

$$\Omega K=[\Omega K_\omega \mid 0], \tag{15}$$

where K_ω denotes the first ω columns of K. In view of (13) it therefore follows that

$$(\bar{\alpha}-\bar{\alpha})K_{n-\omega}=0 \tag{16}$$

where $K_{n-\omega}$ denotes the final $n-\omega$ columns of K. We now note that the $n-\omega$ independent, linear relations given by (16) must be satisfied regardless of any lof control law, an observation which often enables one to assess the ability or inability to stabilize a system via lof compensation.

Example 2: To illustrate in light of our previous example, suppose

$$C=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}$$

instead of

$$\begin{bmatrix}0&1&1&0\\1&0&0&1\end{bmatrix}$$

with A and B unchanged. Then

$$T(s)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}s^2&0\\0&s^2+1\end{bmatrix}^{-1}=R(s)P^{-1}(s),$$

and if M_{HI} is as in Example 1,

$$\Omega=\begin{bmatrix}1&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&-1&0\\1&0&0&0\end{bmatrix}, \text{ a rank 2 matrix.}$$

We next determine that

$$K=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

is one nonsingular matrix which zeros the final $n-\omega=2$ columns of ΩK. Since $|P(s)|=\Delta(s)=s^4+s^2$, $\bar{\alpha}-\bar{a}=[\alpha_0,\alpha_1,\alpha_2-1,\alpha_3]$, and (16) would therefore imply that $\alpha_1=\alpha_3=0$ regardless of H. It is thus clear that in this case, asymptotic stabilization via lof alone is impossible.

Remark 4: If $T(s)=C(sI-A)^{-1}B=R(s)P^{-1}(s)$ is a nonsingular, diagonal transfer matrix; i.e., if $T(s)$ represents the transfer matrix of a "dynamically decoupled" system, then the number of nonzero mth-order minors of $\begin{bmatrix}R(s)\\ \hline P(s)\end{bmatrix}$ will be 2^m. In such cases, therefore, all but 2^m-1 elements of M_{RP} will be zero, which in view of (12) clearly implies that

III. Dynamic Compensation

In view of Theorem 1, we now note that if $q<n$ it would be impossible to arbitrarily assign all n coefficients of $\Delta_H(s)$ and, therefore, all n closed-loop poles of a given system via lof alone. Under the circumstances, it is well known that "dynamic compensation" can be used to enhance closed-loop pole placement. The purpose of this section will be to investigate the effect which dynamic compensation has with respect to the pole placement question. To begin, we define a *dynamic compensator of order* k, in view of (1), via the (k) additional state equations.

$$\dot{x}_{n+i}(t)=u_{m+i}(t); \quad y_{p+i}(t)=x_{n+i}(t), \tag{18}$$

for $i=1,2,\cdots,k$, noting that each (ith) additional state equation represents a new input-output pair which requires one dynamical element (integrator) for physical implementation. The original $p\times m$ open-loop transfer matrix, $T(s)=R(s)P^{-1}(s)$, is therefore augmented to become the "extended" $(p+k)\times(m+k)$ transfer matrix,

$$T_e(s)=\begin{bmatrix}T(s)&0\\0&\frac{1}{s}I_k\end{bmatrix}=R_e(s)P_e^{-1}(s), \tag{19}$$

with $R_e(s)=\begin{bmatrix}R(s)&0\\0&I_k\end{bmatrix}$ and $P_e(s)=\begin{bmatrix}P(s)&0\\0&sI_k\end{bmatrix}$.

To investigate the effect which dynamic compensation has on the "closed-loop" characteristics of the system, we first define *dynamic linear output feedback* (*dlof*) by (18) and the control law

$$u_e(t)=-H_ey_e(t)+v_e(t) \tag{20}$$

where

$$u_e(t)=\begin{bmatrix}u(t)\\u_{m+1}(t)\\ \cdot\\ \cdot\\ \cdot\\ u_{m+k}(t)\end{bmatrix}, \quad y_e(t)=\begin{bmatrix}y(t)\\y_{p+1}(t)\\ \cdot\\ \cdot\\ \cdot\\ y_{p+k}(t)\end{bmatrix},$$

and H_e is an $(m+k)\times(p+k)$ constant but arbitrary gain matrix. Under dlof it now follows that the closed-loop poles of the dynamically compensated system are given by the zeros of

$$\Delta_{H_e}(s)=|P_e(s)+H_eR_e(s)|=\left|[H_e \mid I_{m+k}]\begin{bmatrix}R_e(s)\\ \hline P_e(s)\end{bmatrix}\right|. \tag{21}$$

In view of the results presented in the previous section, and (11) in particular, we now observe that $\Delta_{H_e}(s)-|P_e(s)|=\Delta_{H_e}(s)-s^k|P(s)|$ can be represented as the product of a nonunique $g_e-1=\binom{m+p+2k}{m+k}-1$-dimensional row vector M_{H_eI}, consisting of the $(m+k)$th-order minors of $[H_e \mid I_{m+k}]$ and an appropriate column vector $M_{R_eP_e}$, consisting of the $(m+k)$th-order minors of $\begin{bmatrix}R_e(s)\\ \hline P_e(s)\end{bmatrix}$; i.e.,

$$\Delta_{H_e}(s)-|P_e(s)|=\Delta_{H_e}(s)-s^k\Delta(s)=M_{H_eI}M_{R_eP_e}. \tag{22}$$

We further note that in view of the diagonalized extension of $R(s)$ and $P(s)$ in (19), the nonzero rows of $M_{R_eP_e}$ will consist entirely of elements of the form s^jM_{RP} for $j=0,1,2,\cdots,k$, as well as $\Delta(s),s\Delta(s),\cdots,s^{k-1}\Delta(s)$; i.e., if a total of k parallel integrators is employed, then for some nonsingular matrix J,

$$JM_{R_eP_e}=\begin{bmatrix}M_{RP}\\sM_{RP}\\ \cdot\\ \cdot\\ \cdot\\ s^kM_{RP}\\ \Delta(s)\\ s\Delta(s)\\ \cdot\\ \cdot\\ \cdot\\ s^{k-1}\Delta(s)\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{bmatrix}=J\Omega_eS_{n+k}(s) \tag{23}$$

for some real (nonunique) matrix Ω_e.

If we now let ν_t denote the *observability index* [1] of the single input–multiple output system with transfer vector

$$t(s)=\frac{M_{RP}}{\Delta(s)}, \tag{24}$$

we can state and formally establish the main result of this section.

Theorem 2: Consider the minimal system (1), with $T(s)=C(sI-A)^{-1}B=R(s)P^{-1}(s)$, which directly implies a single system (24) with observability index, ν_t.

In order to arbitrarily assign all of the closed-loop poles via dynamic linear output feedback, at least λ integrators must be employed, when λ is the least integer which satisfies both 1) $\lambda \geqslant \nu_t - 1$ and 2) $(m+\lambda)(p+\lambda) \geqslant n+\lambda$.

Proof: We first note that the system with $t(s)$ given by (24) does have an observability index ν_t since $R(s)$ and $P(s)$ were assumed to be relatively right prime. Therefore, in view of [1, theorem 7.3.30] it follows that Ω_e, as given by (23), has full (column) rank $n+k$ if and only if $k \geqslant \nu_t - 1$. Finally, in view of Theorem 1, it follows that all of the coefficients of $\Delta_{H_e}(s)$ can be arbitrarily assigned only if Ω_e does have full column rank which, in view of our previous observation, directly establishes condition 1). Condition 2) is a direct consequence of the fact that the number of independent output gain parameters cannot be less than the number of desired closed-loop poles.

It should be noted that Theorem 2 extends a well-known fact regarding single-input systems to the multiinput case. In particular, it is well known [15] that all of the poles of a minimal system with observability index ν and controllability index μ can be arbitrarily assigned if one employs a dynamic compensator of dimension equal to $\min(\nu-1,\mu-1)$. We now recall that in view of Theorem 2, at least ν_t-1 integrators must be employed to arbitrarily assign all of the poles of a minimal system. Since $\nu_t \leqslant \min(\nu,\mu)^4$ in any minimal system, it follows that dlof often requires lower order dynamics for complete and arbitrary pole placement than the procedure outlined in [15], although it should be noted that the employment of ν_t-1 integrators does not always insure complete and arbitrary pole placement.

IV. Concluding Remarks

A number of new results related to linear output feedback (lof) compensation have now been presented. In particular, the rank of a $(g-1)\times n$-dimensional real matrix Ω was shown to represent an upper bound on the number of closed-loop poles which can be completely and arbitrarily assigned via constant gain output feedback. Furthermore, a new bound on the minimum number of dynamical elements necessary to completely and arbitrarily assign all of the closed-loop poles of a system via dynamic compensation was given in terms of the observability index of an appropriately defined single-input system.

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[4]This observation follows from the fact that the (numerator) elements of M_{RP} in (24) can always be combined linearly to produce the numerator elements of any transfer vector resulting from a reduction of $T(s)$ to either single-input controllable or single-output observable form.

More on the Conjecture by Šiljak

J. J. MONTEMAYOR and B. F. WOMACK

***Abstract*—Let $\tilde{A}$ be a special class of matrices with complex elements. This correspondence considers the properties of any $A \in \tilde{A}$ which will guarantee that if $G=-(A^*H+HA)$, then for any given Hermitian positive definite matrix H, there exists a unique nonsingular Hermitian matrix G. Properties of the eigenvalues of A and G are established.**

Main Development

The results presented in [1] on the specification of the conditions on $n\times n$ matrix A must satisfy for the existence of the real symmetric positive definite matrices H and G are extended to include the cases where A has complex elements, H is Hermitian positive definite, and G is either nonsingular Hermitian or Hermitian positive definite.

The specification of the conditions the matrix A must satisfy for the existence of the matrices G and H, as outlined above, is based on the following two results.

Theorem 1 [2]*:* If all eigenvalues of A have modulus less than 1 and G is a Hermitian matrix with

$$G-A^*GA=Q>0 \tag{1}$$

where $Q>0$ denotes a positive definite Hermitian matrix Q and A^* denotes the conjugate transpose of A, then G is positive definite.

Theorem 2 [2]*:* If G is a Hermitian solution of

$$G-A^*GA=Q, \qquad Q>0, \tag{2}$$

then G is nonsingular and the number of positive (negative) eigenvalues of G is equal to the number of eigenvalues of A inside (outside) the unit circle $|\lambda|<1$.

The Lyapunov matrix equations arise in a number of areas in the analysis and design of control systems, optimal control problems, and quadratic performance evaluation. For a constant linear discrete system

$$x_{k+1}=Ax_k \tag{3}$$

where A is an $n\times n$ matrix with complex elements, the equation is

$$A^*GA-G=-Q \tag{4}$$

where Q is a Hermitian positive definite matrix. It is required to find the $n\times n$ matrix G which is Hermitian. In particular, in the stability analysis of (3), G is the matrix of a quadratic Lyapunov matrix equation.

Moreover, if there are no eigenvalues λ_i, λ_j of A such that

$$\lambda_i\bar{\lambda}_j=1 \qquad (\text{all } i,j)$$

where $\bar{\lambda}$ means complex conjugate, then the solution of (4) is unique and the numbers of eigenvalues λ_i inside and outside the unit circle $|\lambda|<1$, are, respectively, equal to the numbers of positive and negative eigenvalues of G.

From (5),

$$G=-(A^*H+HA), \tag{5}$$

we get

$$G-A^*GA=Q \tag{6}$$

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