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Symmetry in the Design of Large-scale Complex Control Systems: Some Initial Results using Dissipativity and Lyapunov Stability

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Abstract—**In this paper, stability conditions for large-scale systems are derived by categorizing agents into symmetry groups and applying local control laws under limited interconnections with neighbors. Particularly, stability for dissipative systems is considered. It is assumed that subsystems are dissipative and stability is studied. Conditions are derived for the max number of subsystems that may be added while preserving stability and these results may be used in the synthesis of large-scale systems with symmetric interconnections.**

I. Introduction

Symmetry is one basic feature of shapes and graphs. In contrast to classical random graph models, many real-world networks, such as the internet and US power grid, exhibit a high degree of symmetry, resulting from the process of tree-like or cyclic growing. The automorphism groups of these networks can be decomposed into direct products of symmetric groups. For purpose of analyzing or synthesizing large-scale dynamic systems, the notion of symmetry has been of interest for some time. When dealing with multi-agent systems with various information constraints and protocols, symmetry may refer to identical dynamics of subsystems or symmetric characterizations of information structure. Symmetry exists in a dynamical system if the system dynamics are invariant under transformations of coordinates. Under certain conditions such systems can be expressed as or decomposed into interconnections of lower dimensional systems, which may lead to better understanding of system properties such as stability and controllability. Furthermore, we can construct a symmetric large-scale system without reducing performance if certain properties of low dimensional systems hold.

Passivity and dissipativity in systems, together with decomposition into lower order subsystems have been used in the study of large-scale systems. [16] concentrates on Lyapunov stability using vector Lyapunov functions, as well as input-output stability results with dissipative subsystems. [17] studies linear interconnections of dissipative subsystems and specializes to interconnections of special types of dissipative systems, namely finite gain systems, passive systems, and conic systems. [6] generalizes previous results to weighted Lyapunov functions and gives spectral characterization of the interconnection matrix.

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There are interesting approaches to the analysis and design of large-scale systems involving symmetries, and some are briefly outlined below.

Early research on the topic could be found in [5] and [21]. [13], [14], [15] study distributed systems containing multiple instances of identical subsystems, and show that the controllability of the entire class of systems can be determined by reducing the model and examining a lower order member of the equivalence class. [3], [18] deal with analysis, synthesis, and implementation of distributed controllers. [3] focuses on spatially interconnected systems, especially periodic interconnected subsystems. [22] studies zero dynamics of nonlinear control systems with symmetries. It is shown that the zero dynamics of a symmetric system is also symmetric and admits a special form of cascade decomposition. See also [8], [4] for decomposition of nonlinear symmetric distributed system and [2] for a mechanical example of nonholonomic affine control system, and [1] for oscillator networks. [10] addresses the problem of determining linear-quadratic optimal control problems whose solutions preserve the symmetry of the initial linear control system. See also [7] for H-∞ optimal control for symmetric linear systems. [12], [11] explore how certain interconnection topologies influence symmetry in a multiagent system's trajectories. It is shown how circulant connectivity preserves rotation and symmetric formations. [19] defines the concept of partial symmetry for nonlinear systems, which is an intermediate notion between the concepts of symmetry and controlled invariance. [20] uses similar idea and deals with quotients of fully nonlinear control systems.

In this paper, stability conditions for large-scale systems are derived by categorizing agents into symmetry groups and applying local control laws under limited interconnections with neighbors. Particularly, stability for dissipative systems is considered. It is assumed that subsystems are dissipative and stability is studied. Conditions are derived for the maximum number of subsystems that may be added while preserving stability and these results may be used in the synthesis of large-scale systems with symmetric interconnections.

The paper is organized as follows. In Section II, we introduce preliminaries and background about symmetry in dynamical systems. In Section III, preliminary or iniitial results in dissipativity and Lyapunov stability are introduced. Section IV contains our results about dissipativity. Section V contains simulation results, followed by concluding remarks and future directions in Section IV.

II. Preliminaries and Background

A. Distributed Nonlinear Control Systems

According to [13], smooth analytic systems $\sum$ are considered of the form

$$\begin{aligned}\dot{x}_1 &= f_1(x) + g_{1,1}(x)u_{1,1} + g_{2,1}(x)u_{2,1} + \cdots + g_{n,1}(x)u_{n,1}\\ \dot{x}_2 &= f_2(x) + g_{1,2}(x)u_{1,2} + g_{2,2}(x)u_{2,2} + \cdots + g_{n,2}(x)u_{n,2}\\ &\vdots\\ \dot{x}_n &= f_n(x) + g_{1,n}(x)u_{1,n} + g_{2,n}(x)u_{2,n} + \cdots + g_{n,n}(x)u_{n,n}\end{aligned}$$

where f_i and $g_{i,j}$ are smooth vector fields on M, M is a smooth manifold, $x \in M$. The notation $u_{i,j}$ denotes the jth control input associated with agent i and $g_{i,j}$ is the associated input vector field. A drift term that is a function of states in node i is denoted by f_i.

To describe the relationships between multiple agents, we have a digraph G to model the interaction topology. If agent j can receive information from agent i, then graph nodes v_i and v_j correspond i and j, and a directed edge e_{ij} represents a unidirectional information exchange link from v_i to v_j. The interaction graph represents the communication pattern at certain discrete time.

To represent a graph in matrix form, let $G = \{V, E, A\}$ be a weighted digraph (or direct graph) of n order with the set of nodes $V = \{v_1, v_2, \ldots, v_n\}$, the set of edges $E \subseteq V \times V$, and a weighted adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency elements a_{ij}. The node indices belong to a finite index set $I = \{1, 2, \ldots, n\}$. A directed edge of G is denoted by $e_{ij} = (v_i, v_j)$, where $e_{ij} \in E$ does not imply $e_{ji} \in E$. The adjacency elements corresponding to the edges of the graph are positive, i.e., $a_{ij} > 0$ if and only if $e_{ji} \in E$. Moreover, we assume $a_{ii} \neq 0$ for all $i \in I$. The set of neighbors of the node v_i is the set of all nodes which communicate to v_i, denoted by $N_i = \{v_j \in V : (v_j, v_i) \in E\}$.

A graph G is called strongly connected if there is a directed path from v_i to v_j and v_j to v_i between any pair of distinct vertices v_i and v_j. Vertex v_i is said to be linked to vertex v_j across a time interval if there exists a directed path from v_i to v_j in the union of interaction graphs in that interval. A directed tree is a directed graph where every node except the root has exactly one parent. A spanning tree of a directed graph is a tree formed by graph edges that connect all the vertices of the graph.

B. Symmetric Distributed Nonlinear Systems

To define symmetry in a distributed dynamical system, we need to define subsystems which have identical dynamics and identical interactions with other subsystems; or, mathematically, the system dynamics are invariant under transformations of coordinates. We will first introduce vector field equivalence defined in [13].

Definition 1: (Vector Field Equivalence) Two vector fields, g_1 and g_2 are equivalent, denoted by $g_1 \sim g_2$, if there exists a diffeomorphism, $\psi : M \mapsto M$, such that

$$\psi_* \circ g_1 = g_2$$

Equivalently, we can define $E_{i,j} \sim E_{k,l}$ by only considering the jth and lth components of g_i and g_k, respectively. The definition of the push forward of a vector field is $\psi_* g = T\psi \circ g \circ \psi^{-1}$. A symmetric group of order $p!$, denoted by S_p, is the group of permutations of p objects.
[13] states that if any one member of the equivalence class of symmetric distributed control systems satisfies the Lie Algebra Rank Condition("LARC"), and all bad brackets are spanned by lower order good brackets, then all larger members of the equivalence class of symmetric distributed control systems are small time locally controllable.

C. Symmetric Distributed Linear Systems

If the system $\sum$ is linear, then it can be characterized as a dynamical network involving trajectories of multiagent dynamical systems $\sum_l$ given by

$$\dot{x}_i(t) = \sum_{j=1}^{n} \phi_{ij}(x_i(t), x_j(t)) + \sum_{k=1}^{m} b_{ik}u_{ik},$$

$$x_i(t_0) = x_{i0}, \quad t \geq 0,\ i = 1, ..., n,$$

$$\phi_{ij}(x_i, x_j) = \sum_{j \in N_i} a_{ij}(x_j - x_i)$$

or in vector form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

If definition 1 is used in the linear case, then to reflect the underlying geometric symmetry in the system structure, it can be shown that

$$AT = TA$$

$$TB = BS$$

where T and S are unitary representations of G on $\mathbb{R}^n$ and $\mathbb{R}^r$, respectively. T is not necessarily an orthoganal matrix. But when T is orthogonal(i.e. $T^{-1} = T^T$) and A is a circulant matrix, circulant connectivity preserves rotation, and in particular instances, dihedral group symmetries in a formation of locally interacting planar integrators[11].

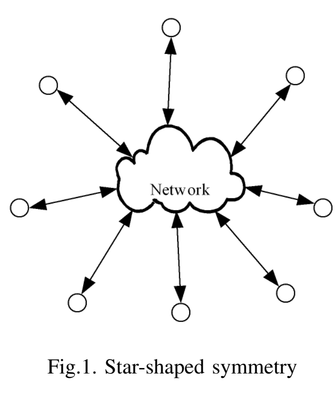

Fig.1. Star-shaped symmetry

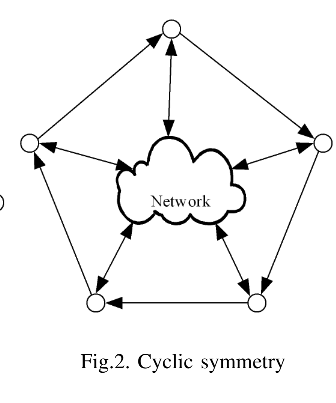

Fig.2. Cyclic symmetry

Here, for a symmetric group S_p consisted of p subsystems, we consider two types of symmetries, namely *star-shaped* symmetry and *cyclic* symmetry. Intuitively, in a star-shaped symmtric group, subsystems do not have interconnections with each other, while in cyclic symmetric group, subsystems contribute to a close related automorphism group, see Figure 1,2.

D. Dissipative Systems

When characterizing properties of systems, it is often beneficial to consider particular cases, such as disspative systems.

Let $\mathbb{U}$ be an inner product space whose elements are funcitons $u : \mathbb{R} \to \mathbb{R}$. Also let $\mathbb{U}^n$ be the space of n-tuples over $\mathbb{U}$, with inner product

$$\langle u, v \rangle = \sum_{i=1}^{n} \langle u_i, v_i \rangle$$

Then for any $u \in \mathbb{U}^n$ and any $T \in \mathbb{R}$, a truncation u_T can be defined via

$$u_T(t) = \begin{cases} u(t), \; for\, t < T \\ 0, \; otherwise \end{cases}$$

A truncated inner product is defined as $\langle u, v \rangle_T = \langle u_T, v_T \rangle$, an extended space $\mathbb{U}_e^n = \{u | u_t \in \mathbb{U}^n \; for\, all\, T \in \mathbb{R}\}$.

A system with *m* inputs and *p* outputs may now be formally defined as a relation on $\mathbb{U}_e^m \times \mathbb{U}_e^p$, that is a set of pairs ($u \in \mathbb{U}_e^m$, $y \in \mathbb{U}_e^p$), where *u* is an input and *y* the correponding output. Let $Q \in \mathbb{R}^{p \times p}$, $S \in \mathbb{R}^{p \times m}$, and $R \in \mathbb{R}^{m \times m}$ be constant matrices, with *Q* and *R* symmetric. Then we say that the above system is $(Q, S, R)-dissipative$ if

$$\langle y, Qy \rangle_T + 2 \langle y, Su \rangle_T + \langle u, Ru \rangle_T \geq 0 \quad (1)$$

for all $T \in \mathbb{R}$, and all *u* and *y* such that (u, y) is valid input-output pair.

Let $Q = -I$(where I is the unit matrix of appropriate dimension), $S = 0$ and $R = k^2 I$, for some fixed positive real number *k*. The above definition reduces to

$$\|y\|_T \leq k \|u\|_T$$

where $\|.\|_T$ is the truncated norm, defined via $\|x\|_T^2 = \langle x, x \rangle_T$. In this case we say that the system is *finite gain input-output stable*, or $L_2 - stable$ with an upper gain bound of *k*.

A linear interconnection of *N* dissipative subsystems can be described as $\sum$:

$$\dot{x}_i = f_i(x_i) + g_i(x_i) u_i$$

$$y_i = h_i(x_i)$$

$$u_i = u_{ei} - \sum_{j=1}^{N} H_{ij} y_j, \; i = 1, \cdots, n$$

where u_i is the input to subsystem i, y_i is its output, u_{ei} is an external input, and the H_{ij} are constant matrices. If we define $y = (y_1, \cdots, y_n)$, $H = [H_{ij}]$, and define u, u_e similarly, then the interconnected system can be represented by

$$u = u_e - Hy$$

Theorem 1: ([9]) If there exists a diagonal matrix $D > 0$ such that the matrix

$$\hat{Q} = -H^T DRH + DSH + H^T S^T D - DQ$$

is positive definite, i.e. $\hat{Q} > 0$, then the network $\sum$ of N interconnected (Q_i, S_i, R_i)-dissipative agents is asymptotically stable.

Remark 1: This result is a generalization of [17] and [16].

Theorem 2: (Small Gain) Let the *i*th subsystem have finite gain γ_i, for $i = 1, \cdots, N$, and suppose that each subsystem has only one input and one output. Define $\Gamma = diag\{\gamma_1, \cdots, \gamma_N\}$ and $A = \Gamma H$. Then if there exists a diagonal positive definite matrix P such that

$$P - A^T P A > 0$$

the interconnected system is stable.

Remark 2: A sufficient condition for such *P* satisfying (1) is that the matrix $\hat{A}$ with elements

$$\hat{a}_{ii} = 1 - |a_{ii}|$$

$$\hat{a}_{ij} = -|a_{ij}|, \quad i \neq j$$

has positive leading principal minors. It is called an *M*-matrix. Note that it is positive definite.

III. Initial Results in Dissipativity and Lyapunov Stability

Proposition 1: Consider a subsystem extended by *m* star-shaped symmetric subsystems with symmetric interconnection matrix $\widetilde{H}$. The whole system is stable if

$$m < \frac{\underline{\sigma}(\hat{A})}{\overline{\sigma}(\alpha \hat{A}^{-1} \alpha^T)}$$

where $\hat{A}$ is the test matrix in *Remark 2.*

Proof: When the interconnected system is extended with a single symmetric subsystem, we have the test matrix

$$\widetilde{A} = \widetilde{\Gamma}\widetilde{H} = \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} H & h_{12} \\ h_{12}^T & H \end{bmatrix}$$

$$\hat{\widetilde{A}} = \left[\hat{\widetilde{a}}_{ij}\right], \; where \; \hat{\widetilde{a}}_{ii} = 1 - |\widetilde{a}_{ii}|, \; \hat{\widetilde{a}}_{ij} = -|\widetilde{a}_{ij}|, \; i \neq j$$

The new test matrix $\hat{\widetilde{A}}$ can be written as $\hat{\widetilde{A}} = \begin{bmatrix} \hat{A} & \alpha \\ \alpha^T & \hat{A} \end{bmatrix}$, where $\alpha = \Gamma h_{12}$. since $\hat{A}$ already has positive leading principal minors, $\hat{\widetilde{A}}$ is an *M*-matrix if and only if $\hat{A} - \alpha \hat{A}^{-1} \alpha^T > 0$.

Similarly, if the system is extended with *m* symmetric subsystems, we have the form

$$\widetilde{A} = \widetilde{\Gamma}\widetilde{H} = \begin{bmatrix} \Gamma & 0 & \dots & 0 \\ 0 & \Gamma & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma \end{bmatrix} \begin{bmatrix} H & h_{12} & \cdots & h_{12} \\ h_{12}^T & H & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{12}^T & 0 & \cdots & H \end{bmatrix}$$

$$\hat{\tilde{A}} = \begin{bmatrix} \hat{A} & \alpha & \dots & \alpha \\ \alpha^T & \hat{A} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^T & 0 & \dots & \hat{A} \end{bmatrix}$$

$\hat{\tilde{A}}$ is an *M*-matrix if and only if $\hat{A} - m\alpha\hat{A}^{-1}\alpha^T > 0$, i.e. there is an upper bound on the number of such extended symmetric subsystems

$$m < \frac{\underline{\sigma}(\hat{A})}{\overline{\sigma}(\alpha\hat{A}^{-1}\alpha^T)}$$

■

Remark 3: Proposition 1 points to the fact that even with very samll gain(Theorem2), stability may be lost for large numbers of subsystems. For the more general case, let the symmetric subsystems be $(Q, S, R) - dissipative$. Then we have the following result.

Proposition 2: Consider a $(Q, S, R)-dissipative$ system extended by *m* star-shaped symmetric $(q, s, r) - dissipative$ subsystems. The whole system is asymptotically stable if

$$m < \min\left(\frac{\underline{\sigma}(\hat{Q})}{\overline{\sigma}(c^Trc + \beta(\hat{q} - b^TRb)^{-1}\beta^T)}, \frac{\hat{q}}{b^TRb}\right) \quad (2)$$

where

$$\hat{Q} = -H^TRH + SH + H^TS^T - Q > 0$$

$$\hat{q} = -h^Trh + sh + h^Ts^T - q > 0$$

$$\beta = Sb + c^Ts^T - H^TRb - c^Trh$$

Proof: By *Theorem 1*, to ensure the enlarged system is $(\widetilde{Q}, \widetilde{S}, \widetilde{R}) - dissipative$, where $\tilde{Q} = diag(Q, q, \dots, q)$,$\tilde{S} = diag(S, s, \dots, s)$, $\tilde{R} = diag(R, r, \dots, r)$, let $D = I$, we need

$$\hat{\tilde{Q}} = \widetilde{S}\widetilde{H} + \widetilde{H}^T\widetilde{S}^T - \widetilde{H}^T\widetilde{R}\widetilde{H} - \widetilde{Q} > 0 \quad (3)$$

where

$$\widetilde{H} \triangleq \begin{bmatrix} H & b & \dots & b \\ c & h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c & 0 & \dots & h \end{bmatrix}$$

This implies

$$\hat{\tilde{Q}} = \begin{bmatrix} \hat{Q} - mc^Trc & \beta & \dots & \beta \\ \beta^T & & & \\ \vdots & & \Lambda & \\ \beta^T & & & \end{bmatrix} > 0$$

where

$$\beta = Sb + c^Ts^T - H^TRb - c^Trh$$

$$\Lambda = \hat{q} \otimes I_m - b^TRb \otimes circ([1\,1 \cdots 1])$$

By linear transformations, $\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} > 0$ if and only if $D > 0$ and $A - BD^{-1}B^T > 0$. Thus, recursively, we have (2). ■

Remark 4: Proposition 2 implies that for a star-shaped symmetry group, interconnections should not exist inside the group. While for cyclic symmetry group, cyclic interconnections is represented by

$$\widetilde{H} \triangleq \begin{bmatrix} H & b & \dots & b \\ c & & & \\ \vdots & & \widetilde{h} & \\ c & & & \end{bmatrix}$$

$\widetilde{h} = circ([v_0\, v_1 \cdots v_{m-1}])$ is a circulant matrix with the first row $[v_0\, v_1 \cdots v_{m-1}]$, $\widetilde{h} = P^T\widetilde{h}P$ where

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Or $\widetilde{h}$ can be written as

$$\widetilde{h} = v_0I + v_1P + \cdots v_{m-1}P^{m-1} \quad (4)$$

.

Proposition 3: Consider a $(Q, S, R)-dissipative$ system extended by m cyclic symmetric $(q, s, r) - dissipative$ subsystems. The whole system is asymptotically stable if

$$m < \min\left(\frac{\underline{\sigma}(\hat{Q})}{\overline{\sigma}(c^Trc + \beta_m\Lambda^{-1}\beta_m^T)}, \frac{\hat{q}}{b^TRb}\right) \quad (5)$$

where

$$\Lambda = -rh^Th + sh + h^Ts - q \otimes I_m - b^TRb \otimes circ([1\,1 \cdots 1])$$

$$\hat{q} = -r\sigma(\tilde{h})\overline{\sigma(\tilde{h})} + s(\sigma(\tilde{h}) + \overline{\sigma(\tilde{h})}) - q$$

$$\beta = Sb + c^Ts^T - H^TRb - c^Tr\widetilde{h}$$

$$\beta_m = \underbrace{[\beta\,\beta \cdots \beta]}_{m}$$

$$\sigma(\widetilde{h}) = \sum_{j=0}^{m-1} v_j\lambda_i^j = \sum_{j=0}^{m-1} v_je^{\frac{2\pi ij}{m}}$$

Proof: Same as (3), we require

$$\hat{\tilde{Q}} = \begin{bmatrix} \hat{Q} - mc^Trc & \beta & \dots & \beta \\ \beta^T & & & \\ \vdots & & \Lambda & \\ \beta^T & & & \end{bmatrix} > 0$$

According to *Proposition 2* [9], requiring $\Lambda > 0$ is equivalent to assuming (q, s, r)-dissipative agents with interconnection matrix $\widetilde{h} = circ(v)$, and $\hat{Q}_{qst} - b^TRb \otimes circ([1\,1 \cdots 1]) > 0$ thus the spectral characterization of $\widetilde{h}$ should satisfy

$$\left\|\sigma(\widetilde{h}) - \frac{s}{r}\right\| < \sqrt{\frac{s^2}{r^2} - \frac{q + mb^TRb}{r}} \quad (6)$$

Then we have

$$m < \frac{-r\sigma(\tilde{h})\overline{\sigma(\tilde{h})} + s(\sigma(\tilde{h}) + \overline{\sigma(\tilde{h})}) - q}{b^T Rb}$$

It is known that if two matrices P and Q commute, so that $QP = PQ$ and if λ is a simple eigenvalue of P with eigenvector ν, then ν is also an eigenvector of Q. Thus, for every eigenvalue λ_i with eigenvector ν_i of P, $|\lambda_i| = 1$ and $\lambda_i = \sigma(P) = e^{\frac{2\pi i}{m}}$, we know that ν_i is also a eigenvector of $\widetilde{h}$. Multiply by ν_i the both sides of (4) to obtain:

$$\widetilde{h}\nu_i = (v_0 I + v_1 P + \cdots v_{m-1}P^{m-1})\nu_i$$

$$= (v_0 + v_1\lambda_i + \cdots v_{m-1}\lambda_i^{m-1})\nu_i = (\sum_{j=0}^{m-1} v_j\lambda_i^j)\nu_i$$

Hence $\sigma(\widetilde{h}) = \sum_{j=0}^{m-1} v_j\lambda_i^j$, $i = 0, 1, \ldots, m-1$.
The rest of the proof is similar as in *Proposition 2* thus omitted. ■

Remark 5: Unlike the star-shaped structure, a cyclic structure does not need the support of the original(center) system, which can be ignored if the symmetry group is the only interest. In this case, (6) is reduced to

$$\left\|\sigma(\widetilde{h}) - \frac{s}{r}\right\| < \sqrt{\frac{s^2}{r^2} - \frac{q}{r}}$$

where

$$\sigma(\widetilde{h}) = \sum_{j=0}^{m-1} v_j\lambda_i^j \leq \sum_{j=0}^{m-1}\left\|v_j\lambda_i^j\right\| \leq \sum_{j=0}^{m-1}\|v_j\|$$

$\sigma(\widetilde{h})$ is bounded and the bound will not be affected by the number of subsystems given a similar structure of interaction matrix, in which zero entries are filled when adding new symmetric subsystems.
The condition above also intuitively explains why passive systems can be arbitrarily connected. Passive systems are $(0, I, 0)-dissipative$ with $r = 0$, thus the stability condition remains unaltered.
Since the convergence rate is closely related to the eigenvalues of the interconnection matrix, we should be able to study the performance of the system. This is currently under investigation.

IV. Simulation Results

In this section we present several brief examples.

Example 1

Suppose we have $m+1$ finite gain symmetric subsystems like Figure 1, each of gain less or equal to $\frac{1}{2}$, and an interconnection matrix

$$\tilde{H} = \underbrace{\begin{bmatrix} 0.9 & -0.8 & -0.8 & \cdots & -0.8 \\ -0.8 & 0.1 & 0 & \cdots & 0 \\ -0.8 & 0 & 0.1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -0.8 & 0 & 0 & \cdots & 0.1 \end{bmatrix}}_{m+1}$$

The problem is to find how large can m be, i.e. how many symmetric subsystems can be connected without losing stability.

For such system, we know $Q = -I$, $S = 0$, $R = \frac{1}{4}I$. According to ,

$$m < \min(3.11,\ 6.25) = 3.11$$

Thus $m_{max} = 3$.

In fact, when $m = 3$, the interconnected system is finite gain input-ouput stable, since

$$\hat{\tilde{Q}} = \begin{bmatrix} 0.318 & 0.16 & 0.16 & 0.16 \\ 0.16 & 0.838 & -0.16 & -0.16 \\ 0.16 & -0.16 & 0.838 & -0.16 \\ 0.16 & -0.16 & -0.16 & 0.838 \end{bmatrix} > 0$$

But when $m = 4$,

$$\hat{\tilde{Q}} = \begin{bmatrix} 0.158 & 0.16 & 0.16 & 0.16 & 0.16 \\ 0.16 & 0.838 & -0.16 & -0.16 & -0.16 \\ 0.16 & -0.16 & 0.838 & -0.16 & -0.16 \\ 0.16 & -0.16 & -0.16 & 0.838 & -0.16 \\ 0.16 & -0.16 & -0.16 & -0.16 & 0.838 \end{bmatrix}$$

is not positive definite, and stability can no longer be guaranteed.

Example 2

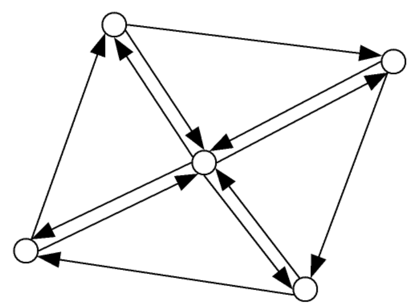

Fig.3. Cyclic interconnection structure with a center

We are considering a similar example as in [6], the rendezvous of multiple agents with damping and inertia. The dynamics for each agent are

$$M\ddot{z} + B\dot{z} = u$$

$$y = z$$

where z, $\dot{z} \in \mathbb{R}^2$ is the position and velocity, M and B are positive definite matrices. Consider the storage function $V = \dot{z}^T M^T B^{-1} M\dot{z} + \dot{z}^T Mz + \frac{1}{2}z^T Bz$. Each agent is $(Q, S, R)-dissipative$ with $Q = 0$,$S = \frac{1}{2}I$,$R = B^{-1}MB^{-1} > 0$. Assume there is an agent in the center while the other agents have cyclic interconnection structure, see Figure 3.

According to (6), the system is stable if

$$\left\|\sigma(\widetilde{h}) - \frac{s}{r}\right\| < \sqrt{\frac{s^2}{r^2} - \frac{q + mb^T Rb}{r}} < \frac{s}{r}$$

But not all $\sigma(\widetilde{h})$ satisfy the condition above, thus the stability condition is not satisfied. Actually simulation results show that such system is always unstable, no matter how many agents are extended and how small $b^T Rb$ is, see Figure 4.

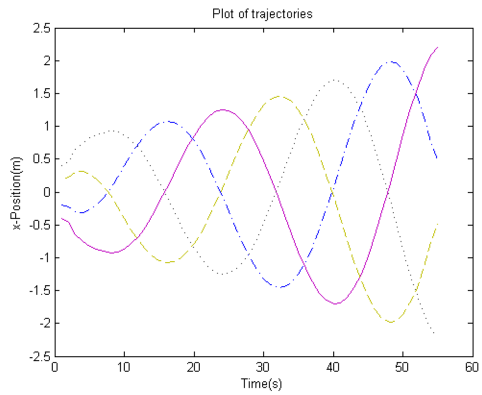

Fig.4. Trajectories of extended agents

Example 3

In Figure 3 we remove the center and consider a network of cyclic symmetric subsystems. Each subsystem is $(q, s, r) - dissipative$ with $q = -1$, $s = 0$, $r = 4$, with interaction matrix

$$\tilde{H} = \tilde{h} = \underbrace{\begin{bmatrix} 0.1 & 0.2 & 0 & \cdots & 0 \\ 0 & 0.1 & 0.2 & \cdots & 0 \\ 0 & 0 & 0.1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.2 & 0 & 0 & \cdots & 0.1 \end{bmatrix}}_{m}$$

According to (6), the system is stable if

$$\left\| \sigma(\widetilde{h}) - \frac{s}{r} \right\| = \left\| \sum_{j=0}^{m-1} v_j e^{\frac{2\pi i j}{m}} \right\| \leq 0.3 < 0.5 = \sqrt{\frac{s^2}{r^2} - \frac{q}{r}}$$

The inequality above always holds. Thus the system can be extended with infinite numbers of subsystems without losing stability.

V. Discussions and Conclusions

In this paper we introduced the notion of symmetry both in linear and nonlinear distributed systems, and we derived preliminary results about the stability conditions for dissipative systems when composing or constructing symmetric systems. It is shown that stability conditions for large-scale systems can be derived by categorizing agents into symmetry groups and applying local control laws under limited interconnections with neighbors. There exists an upper bound on the number of subsystems so to guarantee stability.

Future directions will be focused on various properties and performance of nonlinear symmetric systems.

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