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A Passivity Measure Of Systems In Cascade Based On Passivity Indices

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Abstract— **Passivity indices are defined in terms of an excess or shortage of passivity, and they have been introduced in order to extend the passivity-based stability conditions to more general cases for both passive and non-passive systems. While most of the work on passivity-based stability results in the literature focuses on employing the feedback interconnection of passive or non-passive systems, our results focus on a passivity measure for cascade interconnection. In this paper, we revisit the results on secant criterion from the perspective of passivity indices and we show how to use the secant criterion to measure the excess/shortage of passivity for cascaded Input Strictly Passive/Output Strictly Passive systems; we also propose a method to measure passivity for cascaded dissipative systems, where each subsystem could be passive or non-passive. Furthermore, we study the conditions under which the cascade interconnection can be directly stabilized via output feedback.**

I. Introduction

In the early 1970's, Willems[8] introduced passivity(and dissipativity) concepts using the notions of storage function and supply rate. Passivity is the property that the rate of increase of the storage function is not higher than the supply rate. A most important passivity result states that a negative feedback loop consisting of two passive systems is passive[12],[15]; under an additional detectability condition this feedback loop is also stable[5]-[6].

What can happen when one of the systems in the negative feedback interconnection is not passive? Can an "excess of passivity" assure that the interconnection is passive? The possibility of achieving passivity of interconnections which combine systems with "excess" and "shortage" of passivity led to the introduction of the passivity index. It has been defined in terms of an excess or shortage of passivity in order to extend the passivity-based stability conditions to the feedback interconnection of dissipative systems, see[6],[9] and [10], and note that they are related to Input Strict Passivity and Output Strict Passivity introduced by Hill and Moylan[15].

In the recent paper of Arcak and Sontag[1]-[2], the "secant criterion" which has been used in the analysis of biological feedback loops [3]-[4], has been revisited and its advantages in the passivity based stability analysis for a class of output strictly passive (OSP) systems either with a cascade or with a cyclic interconnection structure has been shown. The authors show that the secant criterion developed earlier in the literature is in fact a sufficient condition for "diagonal stability" of a class of "dissipative matrices". They use the secant criterion and the diagonal stability results as a tool to construct a candidate Lyapunov-like function for the stabilization problem of a class of OSP systems with a cyclic interconnection. The reason why "diagonal stability" of the corresponding dissipative matrices is of special interest is that it enables us to choose the proper weight for each subsystem's storage function and this contributes to the construction of the composite candidate Lyapunov-like function. They have also shown the lack of input feed-forward passivity for this particular class of cascaded OSP systems.

The results in [1] and [2] motivate us to revisit the secant criterion from the perspective of passivity indices. We use passivity indices to measure the excess or shortage of passivity for each subsystem in the cascade interconnection and use the secant criterion to measure the degree of passivity for cascaded Output Strictly Passive/Input Strictly Passive systems. We further propose a way to measure passivity for cascaded Input Feed-forward Output Feedback Passive systems (which is a characterization of dissipative systems, see [6],[8] and [15]), and study how each subsystem's passivity indices affect the overall degree of passivity for the entire cascade interconnection; furthermore, we show the conditions under which the entire cascade interconnection could be stabilized by using output feedback.

The current paper not only provides a way to measure the degree of passivity for the cascade interconnection, it also suggests an alternative method to stabilize cascaded linear/nonlinear systems, in addition to the existing work in the literature by employing the negative feedback interconnection and by using "passivation" techniques, see [5]-[6], [13]-[14] and [17]-[18]. The rest of this paper is organized as follows: we briefly introduce some background material on passivity and passivity indices in Section II, which is followed by a summary of the secant criterion and some facts on quasi-dominant matrices in Section III; the main results are presented in section IV followed by an example given in section V. Finally, the conclusion is provided in Section VI.

II. Background Material

Consider the following nonlinear system:

$$H : \begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases} \tag{1}$$

where $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ and $y \in Y \subset \mathbb{R}^m$ are the state, input and output variables, respectively, and X, U and Y are the state, input and output spaces, respectively. The representation $x(t) = \phi(t,t_0,x_0,u)$ is used to denote the state at time t reached from the initial state x_0 at t_0.

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Definition 1(Supply Rate [8]). *The supply rate $\omega(t) = \omega(u(t), y(t))$ is a real valued function defined on $U \times Y$, such that for any $u(t) \in U$ and $x_0 \in X$ and $y(t) = h(\phi(t, t_0, x_0, u))$, $\omega(t)$ satisfies*

$$\int_{t_0}^{t_1} |\omega(\tau)| d\tau < \infty \tag{2}$$

Definition 2(Dissipative System [8]). *System H with supply rate $\omega(t)$ is said to be dissipative if there exists a nonnegative real function $V(x): X \to \mathbb{R}^+$, called the storage function, such that, for all $t_1 \geq t_0 \geq 0$, $x_0 \in X$ and $u \in U$,*

$$V(x_1) - V(x_0) \leq \int_{t_0}^{t_1} \omega(\tau) d\tau \tag{3}$$

where $x_1 = \phi(t_1, t_0, x_0, u)$, and $\mathbb{R}^+$ is a set of nonnegative real numbers.

Definition 3(Passive System [8]). *System H is said to be* ***passive*** *if there exists a storage function $V(x) \geq 0$ such that*

$$V(x_1) - V(x_0) \leq \int_{t_0}^{t_1} u(\tau)^T y(\tau) d\tau, \tag{4}$$

if $V(x)$ is C^1, then we have

$$\dot{V}(x) \leq u(t)^T y(t), \ \forall t \geq 0. \tag{5}$$

One can see that passive system is a special case of dissipative system with supply rate $\omega(t) = u(t)^T y(t)$.

Definition 4(Excess/Shortage of Passivity [6]). *System H is said to be:*

- *Input Feed-forward Passive (IFP) if it is dissipative with respect to supply rate $\omega(u, y) = u^T y - \nu u^T u$ for some $\nu \in \mathbb{R}$, denoted as IFP(ν).*
- *Output Feedback Passive (OFP) if it is dissipative with respect to the supply rate $\omega(u, y) = u^T y - \rho y^T y$ for some $\rho \in \mathbb{R}$, denoted as OFP(ρ).*
- *Input Feed-forward Output Feedback Passive (IF-OFP) if it is dissipative with respect to the supply rate $\omega(u, y) = u^T y - \rho y^T y - \nu u^T u$ for some $\rho \in \mathbb{R}$ and $\nu \in \mathbb{R}$, denoted as IF-OFP(ν,ρ).*

A positive ν or ρ means that the system has an excess of passivity; otherwise, the system is lack of passivity. In the case when $\nu > 0$ or $\rho > 0$, the system is said to be **input strictly passive**(ISP) or **output strictly passive**(OSP) respectively.

Definition 5(Zero-State Observability and Detectability [6]). *Consider the system H with zero input, that is $\dot{x} = f(x, 0)$, $y = h(x, 0)$, and let $Z \subset \mathbb{R}^n$ be its largest positively invariant set contained in $\{x \in \mathbb{R}^n | y = h(x, 0) = 0\}$. We say H is zero-state detectable(ZSD) if $x = 0$ is asymptotically stable conditionally to Z. if $Z = \{0\}$, we say that H is zero-state observable (ZSO).*

III. Secant Criterion and Some Facts on Diagonally Stable Matrices and Quasi-Dominant Matrices

The connection between diagonal stability and the secant criterion has been shown in [1]-[2]. We briefly summarize these results here.

Definition 4 (Diagonal Stability [7]). *A matrix $A := (a_{ij})$ is said to belong to the class of Hurwitz diagonally stable matrix if there exists a diagonal matrix $D > 0$ such that*

$$A^T D + DA < 0 \tag{6}$$

Definition 5 (Quasi-dominant matrix [7]). A square matrix is a quasi-dominant matrix, or in the class of diagonally row-sum or column-sum quasi-dominant matrices if there exists a positive diagonal matrix P=diag$\{p_1, p_2, \ldots, p_n\}$ such that $a_{ii} p_i \geq \sum_{j \neq i} |a_{ij}| p_j, \forall i$, (respectively) $a_{jj} p_j \geq \sum_{i \neq j} |a_{ji}| p_i, \forall j$. If these inequalities are strict, the matrix is referred to as strictly row-sum (respectively column-sum) quasi-dominant. If P can be chosen as the identity matrix, then the matrix is called row- or column- diagonally dominant.

Theorem 1 (Secant Criterion [2]). *A matrix of the form:*

$$A = \begin{bmatrix} -\alpha_1 & 0 & \cdots & 0 & -\beta_n \\ \beta_1 & -\alpha_2 & \ddots & & 0 \\ 0 & \beta_2 & -\alpha_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_{n-1} & -\alpha_n \end{bmatrix} \quad \alpha_i > 0, \quad \beta_i > 0, \tag{7}$$

$i = 1, \cdots, n$, is diagonally stable, that is, it satisfies (5) for some diagonal matrix $D > 0$, if and only if the secant criterion

$$\frac{\beta_1 \cdots \beta_n}{\alpha_1 \cdots \alpha_n} < sec(\pi/n)^n = \frac{1}{cos(\frac{\pi}{n})^n} \tag{8}$$

holds; here we assume $n > 2$.

Corollary 1 [19]. Every symmetric quasi-dominant matrix is positive definite.

Lemma 1 [7]. If a matrix A is diagonally stable, then A^T is also diagonally stable.

In the subsequent sections, we will show how we use the secant criterion and the properties of quasi-dominant matrix to measure the shortage/excess of passivity for cascade systems.

IV. Main Results

In this section, we show passivity measures for three classes of cascade systems, which include output strictly passive systems(OFP(ρ) with $\rho > 0$), input strictly passive systems(IFP(ν) with $\nu > 0$) and input feed-forward output feedback passive systems(IF-OFP(ν,ρ) with $\rho, \nu \in \mathbb{R}$). One should notice that while the first and the second classes assume that each interconnected subsystem is passive, the third class applies to the more general case of dissipative systems. We also show sufficient conditions under which those cascade systems can be stabilized via output feedback.

A. Passivity measure of a cascade of OSP systems

Proposition 1. *Consider the cascade interconnection shown in Fig.1, where $n \geq 2$. Let each block be OFP(ρ_i) with $\rho_i > 0$, namely there exists C^1 storage function $V_i \geq 0$ for each subsystem, such that*

$$\dot{V}_i \leq -\rho_i y_i^T y_i + u_i^T y_i, \tag{9}$$

where $u_i, y_i \in \mathbb{R}^m$. Then for some $\nu > 0$, such that

$$\nu > \frac{cos(\frac{\pi}{n+1})^{n+1}}{\rho_1\rho_2\dots\rho_n}, \tag{10}$$

the cascade system admits a storage function of the form

$$V = \sum_{i=1}^{n} d_i V_i, \quad d_i > 0, \tag{11}$$

and the cascade interconnection is IFP(-ν).

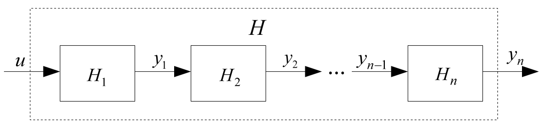

Fig. 1: Cascaded Interconnection

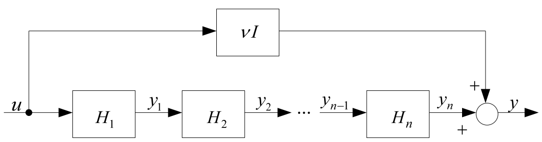

Fig. 2: Feed-forward Passivation

Proof . To show that the cascaded interconnection is IFP(-ν), we need to show that the storage function (11) satisfies:

$$\dot{V} \le \nu u^T u + u^T y_n \tag{12}$$

for some $\nu > 0$. Since $V = \sum_{i=1}^{n} d_i V_i$, and $\dot{V}_i \le -\rho_i y_i^T y_i + u_i^T y_i$, if we can show that

$$\sum_{i=1}^{n} d_i(-\rho_i y_i^T y_i + u_i^T y_i) - \nu u^T u - u^T y_n \le 0, \tag{13}$$

then (12) holds. Define

$$A = \begin{bmatrix} -1 & 0 & \cdots & 0 & -\frac{1}{\nu} \\ 1 & -\rho_1 & \ddots & & 0 \\ 0 & 1 & -\rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -\rho_n \end{bmatrix} \qquad \rho_i > 0, \quad \nu > 0 \tag{14}$$

and $D = diag\{\nu, d_1, d_2, \dots, d_n\}$. Then it can be seen that the left-hand side of (13) is equal to

$$[u^T \ y^T] DA \otimes I_m [u^T \ y^T]^T. \tag{15}$$

where $y = [y_1^T, \dots, y_n^T]^T$. According to Theorem 1, if

$$\nu > \frac{cos(\frac{\pi}{n+1})^{n+1}}{\rho_1\rho_2\dots\rho_n}, \tag{16}$$

then there exists a diagonal matrix $D > 0$ such that

$$DA + A^T D < 0, \tag{17}$$

then we have

$$(DA + A^T D) \otimes I_m < 0. \tag{18}$$

Since

$$[u^T \ y^T] DA \otimes I_m [u^T \ y^T]^T = [u^T \ y^T](DA \otimes I_m)^T [u^T \ y^T]^T \tag{19}$$

and

$$(DA \otimes I_m)^T = (DA)^T \otimes I_m^T = A^T D \otimes I_m, \tag{20}$$

we have

$$[u^T \ y^T] DA \otimes I_m [u^T \ y^T]^T = \frac{1}{2}[u^T \ y^T](A^T D + DA) \otimes I_m [u^T \ y^T]^T \tag{21}$$

so that

$$[u^T \ y^T] DA \otimes I_m [u^T \ y^T]^T < 0, \tag{22}$$

and thus

$$\dot{V} < \nu u^T u + u^T y_n. \tag{23}$$

This shows that the cascaded system is IFP(-ν). ■

Remark 1: Proposition 1 shows the shortage of passivity in cascaded OSP systems and it also shows that the shortage of passivity could be compensated by input forward νI as shown in Fig.2, where ν is determined by (10).

B. Passivity measure of a cascade of ISP systems

Proposition 2. *Consider the cascade interconnection shown in Fig.1, where $n \ge 2$. Let each block be IFP(ν_i) with $\nu_i > 0$, namely there exists a C^1 storage function $V_i \ge 0$ for each subsystem, such that*

$$\dot{V}_i \le -\nu_i u_i^T u_i + u_i^T y_i \tag{24}$$

where $u_i, y_i \in \mathbb{R}^m$. Then for some $\rho > 0$, such that

$$\rho > \frac{cos(\frac{\pi}{n+1})^{n+1}}{\nu_1\nu_2\dots\nu_n}, \tag{25}$$

the cascade system admits a storage function of the form given by (11) *and the cascade interconnection is OFP(-ρ).*

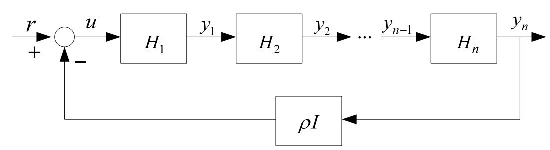

Fig. 3: Output Feedback Passivation

Proof . The proof is very similar to the proof shown in Proposition 1, thus it is omitted here. ■

Remark 2: Proposition 2 shows shortage of passivity in cascaded ISP systems and it also implies that the lack of passivity could be compensated by output feedback ρI as shown in fig.3, where ρ is determined by (25).

C. Stabilization of a cascade of OSP / ISP systems via output feedback

Proposition 3. *Consider the feedback interconnection shown in Fig.4, where $n > 2$. Let each block be OFP(ρ_i) for some*

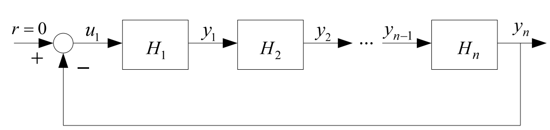

Fig. 4: Stabilized Via Output Feedback

$\rho_i > 0$ *with the storage function* $V_i \geq 0$ *such that* (9) *holds with* $u_i, y_i \in \mathbb{R}^m$ *. If*

$$\frac{1}{\rho_1\rho_2\ldots\rho_n} < \frac{1}{cos(\frac{\pi}{n})^n}\ , \tag{26}$$

and the input to the cascaded system $r = 0$*, then the closed-loop system admits a candidate Lyapunov-like function which is a weighted sum of each subsystem's storage function given by* (11) *and*

$$\dot{V} = \sum_{i=1}^{n} d_i\dot{V}_i \leq -\varepsilon \parallel y \parallel_2^2 \tag{27}$$

for some $\varepsilon > 0$ *, where* $y = [y_1^T, y_2^T, \ldots, y_n^T]^T$*. Moreover, if* $\{y = 0\}$ *is contained in the invariant set for which* $\dot{V} = 0$*, and if each subsystem is ZSD, then the equilibrium* $x_i = 0$ *of each subsystem is asymptotically stable.*

Proof . Fig.4 is a cyclic structure and since $r = 0$, we have

$$u_1 = -y_n, \quad u_2 = y_1, \ldots, \quad u_n = y_{n-1} \tag{28}$$

Moreover, since each subsystem is OFP(ρ_i) we have

$$\begin{aligned} \dot{V}_1 &\leq -y_n^T y_1 - \rho_1 y_1^T y_1 \\ \dot{V}_2 &\leq y_1^T y_2 - \rho_2 y_2^T y_2 \\ &\vdots \\ \dot{V}_n &\leq y_{n-1}^T y_n - \rho_n y_n^T y_n. \end{aligned} \tag{29}$$

If we take the time derivative of the candidate Lyapunov-like function $V = \sum_{i=1}^n d_i V_i$, we will get

$$\dot{V} \leq d_1(-y_n^T y_1 - \rho_1 y_1^T y_1) + \sum_{i=2}^{n} d_i(y_{i-1}^T y_i - \rho_i y_i^T y_i). \tag{30}$$

Define

$$A = \begin{bmatrix} -\rho_1 & 0 & \cdots & 0 & -1 \\ 1 & -\rho_2 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & -\rho_{n-1} & 0 \\ 0 & \cdots & 0 & 1 & -\rho_n \end{bmatrix} \qquad \rho_i > 0 \tag{31}$$

and $D = diag\{d_1, d_2, \ldots, d_n\}$. Notice that the right-hand side of (30) is equal to $y^T DA \otimes I_m y$. According to Theorem 1, if

$$\frac{1}{\rho_1\ldots\rho_n} < \frac{1}{cos(\frac{\pi}{n})^n} \tag{32}$$

then there exists some diagonal matrix $D > 0$ such that matrix A is diagonally stable, and we will have

$$\dot{V} \leq y^T DA \otimes I_m y = \frac{1}{2} y^T (A^T D + DA) \otimes I_m y \leq -\varepsilon \parallel y \parallel_2^2 \tag{33}$$

for some $\varepsilon > 0$. According to LaSalle's theorem [15], if $\{y = 0\}$ is contained in the invariant set for which $\dot{V} = 0$, then we have $\lim_{t\to\infty} y_i(t) = 0$, for $i = 1, \ldots, n$. Moreover, if each subsystem H_i is ZSD, then $\lim_{t\to\infty} y_i(t) = 0$ implies $\lim_{t\to\infty} x_i(t) = 0$, so each subsystem's equilibrium $x_i = 0$ is asymptotically stable. ■

Remark 3: One can show that for the case when there are only two subsystems in the cascade interconnection as shown in Fig. 5. if $\rho_1 > 0$ and $\rho_2 > 0$, which means both H_1 and H_2 are OSP, then we can simply choose the sum of their storage function as the potential Lyapunov-like function for the closed-loop system when we directly apply the unity output feedback to the cascade interconnection of H_1 and H_2.

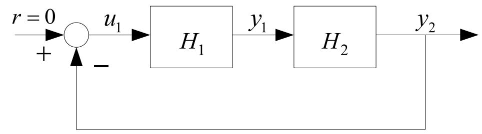

Fig. 5: Special Case When $n = 2$

We have studied the conditions under which the cascade interconnection of a class of OSP systems can be directly stabilized via output feedback. The following proposition shows similar results for a cascade of ISP systems.

Proposition 4 . *Consider the feedback interconnection shown in Fig. 4, where* $n > 2$*. Suppose that* $r = 0$ *and let each block be IFP(*ν_i*) for some* $\nu_i > 0$ *with the storage function* $V_i \geq 0$ *such that* (24) *holds with* $u_i, y_i \in \mathbb{R}^m$*. If*

$$\frac{1}{\nu_1\nu_2\ldots\nu_n} < \frac{1}{cos(\frac{\pi}{n})^n}, \tag{34}$$

then the closed-loop system admits a candidate Lyapunov-like function which is a weighted sum of each subsystem's storage function given by (11) *and*

$$\dot{V} = \sum_{i=1}^{n} d_i\dot{V}_i \leq -\varepsilon \parallel y \parallel_2^2 \tag{35}$$

for some $\varepsilon > 0$*, where* $y = [y_1^T, y_2^T, \ldots, y_n^T]^T$*. Moreover, if* $\{y = 0\}$ *is contained in the invariant set for which* $\dot{V} = 0$*, and if each subsystem is ZSD, then the equilibrium* $x_i = 0$ *of each subsystem is asymptotically stable.*

Proof . The proof is very similar to the proof shown in Proposition 3, and it is omitted here.■

Remark 4: One can show that for the case when there are only two subsystems in the cascade interconnection as shown in Fig.5, if $\nu_1 > 0$ and $\nu_2 > 0$, which means both H_1 and H_2 are ISP, then again we can simply choose the sum of their storage functions as the candidate Lyapunov-like function for the closed-loop system when we directly apply the unity output feedback to the cascade interconnection.

D. Passivity measure of a cascade of IF-OFP systems

In this section, we present our results on passivity measure for IF-OFP systems, which is an often used characterization of dissipative systems. An IF-OFP system, in general, could be passive or non-passive based on the sign of its passivity indices.

Proposition 5 . *Consider the cascade interconnection shown in Fig.1, where* $n \geq 2$*, and let each block be dissipative with respect to the supply rate given by* $\omega_i(u_i, y_i) = u_i^T y_i - \rho_i y_i^T y_i -$

$\nu_i u_i^T u_i$, *that is there exists a* C^1 *storage function* $V_i \geq 0$ *for each subsystem, such that*

$$\dot{V}_i \leq u_i^T y_i - \rho_i y_i^T y_i - \nu_i u_i^T u_i, \tag{36}$$

where $u_i, y_i \in \mathbb{R}^m$*. Here,* ν_i *and* ρ_i *are not necessarily all positive.*
Consider the symmetric matrix given by

$$A = \begin{bmatrix} -\nu_1 + \hat{\nu} & \frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\ \frac{1}{2} & -\nu_2 - \rho_1 & \frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2} & -\nu_n - \rho_{n-1} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & \frac{1}{2} & -\rho_n + \hat{\rho} \end{bmatrix}, \tag{37}$$

if $-A$ *is quasi-dominant for some* $\hat{\nu}, \hat{\rho} \in \mathbb{R}$*, then the cascade system admits a storage function of the form given by* $V = \sum_{i=1}^n V_i$ *such that the cascade interconnection is IF-OFP(*$\hat{\nu}$*,*$\hat{\rho}$*).*

Proof. To show that the cascade interconnection is IF-OFP($\hat{\nu}$, $\hat{\rho}$), we need to show that the storage function $V(x)$ satisfies:

$$\dot{V} \leq u^T y_n - \hat{\nu} u^T u - \hat{\rho} y_n^T y_n \tag{38}$$

since

$$\dot{V} \leq \sum_{i=1}^n (u_i^T y_i - \nu_i u_i^T u_i - \rho_i y_i^T y_i) \tag{39}$$

so if

$$\sum_{i=1}^n (u_i^T y_i - \nu_i u_i^T u_i - \rho_i y_i^T y_i) - u^T y_n + \hat{\nu} u^T u + \hat{\rho} y_n^T y_n \leq 0, \tag{40}$$

then (38) is true. Define

$$A_1 = \begin{bmatrix} \hat{\nu} & 0 & \cdots & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\rho_1 & \ddots & & 0 \\ 0 & \frac{1}{2} & -\rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{2} & -\rho_n \end{bmatrix} \tag{41}$$

$$A_2 = \begin{bmatrix} -\nu_1 & 0 & \cdots & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\nu_2 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2} & -\nu_n & 0 \\ 0 & \cdots & 0 & \frac{1}{2} & \hat{\rho} \end{bmatrix} \tag{42}$$

notice that the left-hand side of (40) is equal to

$$[u^T \; y^T](A_1 + A_2) \otimes I_m [u^T \; y^T]^T, \tag{43}$$

or we can rewrite it as

$$[u^T \; y^T](A_1 + A_2^T) \otimes I_m [u^T \; y^T]^T. \tag{44}$$

We can see that $A_1 + A_2^T$ is a symmetric matrix which is equal to the matrix A defined in (37). Based on Corollary 1, we can conclude that if $-A$ is quasi-dominant, then A is negative definite, and $A \otimes I_m$ is also negative definite. So we can get

$$[u^T \; y^T](A_1 + A_2^T) \otimes I_m [u^T \; y^T]^T = [u^T \; y^T] A \otimes I_m [u^T \; y^T]^T < 0 \tag{45}$$

and thus (38) holds. This shows that the entire cascade interconnection is IF-OFP($\hat{\nu}$, $\hat{\rho}$). ■

E. Stabilization of a cascade of IF-OFP systems via output feedback

In this section, we present the conditions under which a cascade of IF-OFP systems can be stabilized directly via output feedback.

Proposition 6 . *Consider the feedback interconnection shown in Fig.4, where* $n > 2$*. Let each block be IF-OFP(*ν_i*,*ρ_i*) with its storage function* $V_i \geq 0$ *such that* (36) *is satisfied, where* $u_i, y_i \in \mathbb{R}^m$*.* $\nu_i, \rho_i \in \mathbb{R}$ *are not necessarily all positive.*
Consider the symmetric matrix given by

$$A = \begin{bmatrix} -\rho_1 - \nu_2 & \frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\ \frac{1}{2} & -\rho_2 - \nu_3 & \frac{1}{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2} & -\rho_{n-1} - \nu_n & \frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & \frac{1}{2} & -\rho_n - \nu_1 \end{bmatrix}, \tag{46}$$

if $-A$ *is quasi-dominant, then the closed-loop system with* $r = 0$ *admits a candidate Lyapunov-like function given by* $V = \sum_{i=1}^n V_i$ *such that*

$$\dot{V} = \sum_{i=1}^n \dot{V}_i \leq -\varepsilon \parallel y \parallel_2^2 \tag{47}$$

for some $\varepsilon > 0$*, where* $y = [y_1^T, y_2^T, \ldots, y_n^T]^T$*. Moreover, if* $\{y = 0\}$ *is contained in the invariant set for which* $\dot{V} = 0$*, and if each subsystem is ZSD, then the equilibrium* $x_i = 0$ *of each subsystem is asymptotically stable.*

Proof . The proof is very similar to the proof shown in Proposition 5, thus is omitted here. ■

Remark 5: One can show that for the case when there are only two subsystems in the cascade interconnection ($n = 2$)as shown in Fig.5., if $\nu_1 + \rho_2 > 0$ and $\nu_2 + \rho_1 > 0$, then again we can simply choose the sum of storage functions for H_1 and H_2 as the candidate Lyapunov-like function for the closed-loop system when we directly apply the unity output feedback to their cascade interconnection.

V. Example

Example. Consider a cascade of three dissipative systems H_1, H_2 and H_3 given by

$$H_1 : \begin{cases} \dot{x}_1(t) = \dfrac{1}{20} x_1(t) + u_1(t) \\ y_1(t) = x_1(t) + 5u_1(t) \end{cases}, \quad H_2 : \begin{cases} \dot{x}_2(t) = -\dfrac{1}{6} x_2(t) + u_2(t) \\ y_2(t) = x_2(t) + 6u_2(t) \end{cases} \tag{48}$$

$$H_3 : \begin{cases} \dot{x}_{31}(t) = x_{32}(t) \\ \dot{x}_{32}(t) = -0.5 x_{31}^3(t) + 0.5 x_{32}(t) + 2u_3(t) \\ y_3(t) = x_{32}(t) + u_3(t) \end{cases} \tag{49}$$

Choosing $V_1(x) = x_1^2(t)$ as the storage function for H_1, we can obtain

$$\dot{V}_1 = u_1(t)y_1(t) - 7.5u_1^2(t) + 0.1y_1^2(t) \ , \tag{50}$$

so the passivity indices for H_1 are $\rho_1 = -0.1$ and $\nu_1 = 7.5$. We can see that H_1 is ZSD but unstable. H_2 admits a storage function given by $V_2(x) = \frac{1}{6}x_2^2$, and

$$\dot{V}_2 = y_2(t)u_2(t) - 4u_2^2(t) - \frac{1}{18}y_2^2(t) \ , \tag{51}$$

so the passivity indices for H_2 are $\rho_2 = \frac{1}{18}$ and $\nu_2 = 4$. We can see that H_2 is stable and ZSD. H_3 admits a storage function given by $V_3(x) = \frac{1}{8}x_{31}^4(t) + \frac{1}{2}x_{32}^2$, and

$$\dot{V}_3 = u_3(t)y_3(t) - 1.5u_3^2(t) + 0.5y_3^2(t) \ , \tag{52}$$

so the passivity indices for H_3 are $\rho_3 = -0.5$ and $\nu_3 = 1.5$. H_3 is neither ZSD or stable. Then the A matrix defined in Proposition 6 is given by

$$A = \begin{bmatrix} -\rho_1 - \nu_2 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\rho_2 - \nu_3 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\rho_3 - \nu_1 \end{bmatrix}. \tag{53}$$

Since

$$\nu_2 + \rho_1 = 3.9 > 1, \ \nu_1 + \rho_3 = 7 > 1, \ \nu_3 + \rho_2 = \frac{14}{9} > 1 \tag{54}$$

$-A$ is row/column diagonally dominant. Moreover, since $\{y_1 = y_2 = y_3 = 0\}$ is contained in the invariant set for which $\dot{V} = \sum_{i=1}^3 \dot{V}_i = 0$, according to Proposition 6, the cascade of H_1, H_2 and H_3 could be stabilized directly via unity output feedback. The simulation results for the closed-loop system with $r = 0$ (see Fig.4) are shown in Fig.6-Fig.7.

VI. Conclusions

In this paper, we introduce a way to measure the degree of passivity for cascaded ISP/OSP systems. We further proposed a method for passivity measure of cascaded IF-OFP systems(dissipative systems), and study the conditions under which the cascade interconnection can be stabilized via output feedback.

VII. Acknowledgments

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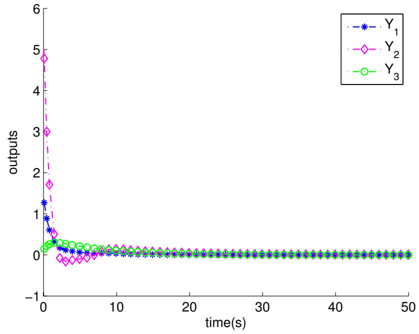

Fig. 6: Outputs of H_1,H_2 and H_3

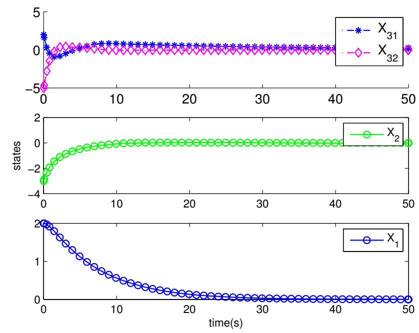

Fig. 7: States of H_1,H_2 and H_3

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