ISIS Technical Report: Self-Triggered Real-Time Output Feedback Control Strategy For Stabilization of Passive and Output Feedback Passive Systems

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Abstract

In this paper, we propose a robust self-triggered real-time scheduling strategy for stabilization of passive or output feedback passive systems under unstructured uncertainties and in the presence of non-trivial actuation update delays. We assume that the unstructured uncertainty is a $\mathcal{L}_2$ stable dynamic system in a feedback or feedforward interconnection with the model of the plant, we took the structure uncertainty into account to design the output feedback controller and we derived the self-triggered real-time scheduling strategy for both cases. Stability and expected inter-sampling time of the system under the proposed scheduling strategy is analyzed in detail. Simulations are also provided to validate these results.

I. Introduction

In traditional implementations of the control tasks, one first designs the controllers under the assumption of no-delayed actuation updates and then determines the maximum admissible interval between two consecutive actuation updates. However, the control strategy obtained based on this approach is conservative in the sense that resource usage(i.e., sampling rate, CPU time) is more frequent than necessary to assure a specified performance level, since stability is guaranteed under sufficiently fast periodic execution of control action. To overcome the drawback of periodic paradigm, several researchers suggested the idea of event-based control, although the heterogeneous terminology refers to the triggering mechanism as event-based-sampling[10], event-driven sampling[11], Lebesgue sampling[4], deadband control[12], level-crossing sampling[13], state-triggered sampling[6] with slightly different meaning, all refer to the situation where the control signals are kept constant until the violation of a condition on certain signals of the plant triggers the re-computation of the control signals. One should be aware that event-triggered technique reduces resource usage while providing a high degree of robustness, since embedded hardware is used to monitor the state of the plant and examine the triggering conditions continuously.

Self-triggered real-time scheduling strategy is studied in [5], [7],[8], [9]. It takes the advantage of the event-triggered technique without resorting to extra hardware. The key idea of self-triggered control is to compute the next instants of time at which the control action is to be recomputed based on the current or the last state' measurements of the plant. A first attempt to explore self-triggered paradigm for linear systems was developed in [5], by discretizing the plant, and in [8] for linear full-state information $\mathcal{H}_\infty$ controllers. A study on self-triggered scheduling for nonlinear dynamic systems is shown in [7] and [9], where a simple self-triggering condition based on the norm of the current states is proposed by exploiting the properties of the trajectories of homogeneous control systems. However, most of these results on self-triggered control assume that the feedback law provides input-to-state stability(ISS) in the sense of [15] with respect to measurement errors which is quite restrictive in general, although some results on designing such control laws are available [16]-[19].

The key complications in applying self-triggered control to the nonlinear context are that the propagation of the measurement error is not as straightforward to characterize and that a suitable form of robustness with respect to this error is needed from the controller. In this paper, we propose a robust self-triggered real-time scheduling strategy for stabilization of passive/output feedback passive systems. We treat the structure uncertainty as a $\mathcal{L}_2$ stable dynamic system in a feedback/feedforward interconnection with the model of the plant and we derived the self-triggered real-time scheduling strategies for both cases. We have also taken non-trivial actuation update delays into consideration. The inter-sampling time under the proposed scheduling strategy is shown to be strictly positive based on the analysis shown in this paper.

The rest of this paper is organized as follows. We first introduce some background on passive/output feedback passive systems in section II; the problem statement is made in section III; the self-triggered scheduling strategy when the uncertainty is in a feedback interconnection with the model of the plant is proposed in section IV.A and in section IV.B we proposed a self-triggered scheduling strategy when the uncertainty is in a feed-forward interconnection with the model of the plant; examples are provided in section V; finally, conclusion is made in section VI.

II. Notation and Background Material

We first introduce some basic concepts on passive systems and output feedback passive systems.

Consider the following control system, which could be linear or nonlinear:

$$H : \begin{cases} \dot{x} = f(x,u) \\ \\ y = h(x,u) \end{cases} \tag{1}$$

where $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ and $y \in Y \subset \mathbb{R}^m$ are the state, input and output variables, respectively, and X, U and Y are the state, input and output spaces, respectively. The representation $x(t) = \phi(t, t_0, x_0, u)$ is used to denote the state at time t reached from the initial state x_0 at t_0.

Definition 1(Supply Rate)[1]: The supply rate $\omega(t) = \omega(u(t), y(t))$ is a real valued function defined on $U \times Y$, such that for any $u(t) \in U$ and $x_0 \in X$ and $y(t) = h(\phi(t, t_0, x_0, u))$, $\omega(t)$ satisfies

$$\int_{t_0}^{t_1} |\omega(\tau)| d\tau < \infty. \tag{2}$$

Definition 2(Dissipative System)[1]: System H with supply rate $\omega(t)$ is said to be **dissipative** if there exists a nonnegative real function $V(x) : X \to \mathbb{R}^+$, called the storage function, such that, for all $t_1 \geq t_0 \geq 0$, $x_0 \in X$ and $u \in U$,

$$V(x_1) - V(x_0) \leq \int_{t_0}^{t_1} \omega(\tau) d\tau, \tag{3}$$

where $x_1 = \phi(t_1, t_0, x_0, u)$ and $\mathbb{R}^+$ is a set of nonnegative real numbers.

Definition 3(Passive System)[1]: System H is said to be **passive** if there exists a storage function $V(x) \geq 0$ such that

$$V(x_1) - V(x_0) \leq \int_{t_0}^{t_1} u(\tau)^T y(\tau) d\tau, \tag{4}$$

if $V(x)$ is $\mathcal{C}^1$, then we have

$$\dot{V}(x) \leq u(t)^T y(t), \ \forall t \geq 0. \tag{5}$$

One can see that passive system is a special case of dissipative system with supply rate $\omega(t) = u(t)^T y(t)$.

Definition 4(Output Feedback Passive System)[2]: System H is said to be **Output Feedback Passive**(OFP) if it is dissipative with respect to the supply rate

$$\omega(u, y) = u^T y - \rho y^T y, \tag{6}$$

for some $\rho \in \mathbb{R}$.

Remark 1: Note that if $\rho > 0$, then H is strictly output passive[2], and H is said to have excessive output feedback passivity of ρ, we denote it as OFP(ρ); if $\rho < 0$, H is said to lack output feedback passivity, and we denote it as OFP($-|\rho|$). One can verify that a OFP($-|\rho|$) system can be rendered passive by a negative feedback $|\rho| I$. And clearly, if a system is OFP(ρ), then it is also OFP($\rho - \varepsilon$), $\forall \varepsilon > 0$. ■

Definition 5[2]: Consider the system H with zero input, that is $\dot{x} = f(x, 0)$, $y = h(x, 0)$, and let $Z \subset \mathbb{R}^n$ be its largest positively invariant set contained in $\{x \in \mathbb{R}^n | y = h(x, 0) = 0\}$. We say H is **Zero-State Detectable**(ZSD) if $x = 0$ is asymptotically stable conditionally to Z. If $Z = \{0\}$, we say that H is **Zero-State Observable** (ZSO).

III. Problem Statement

We first consider a passive control system H as given in (1), we know that if H is zero-state detectable(ZSD), then under the feedback control law

$$u(t) = -Ky(t) \tag{7}$$

where $K > 0$ could be a scalar or an $m \times m$ positive definite matrix, the origin of H is asymptotically stable. For the rest of the paper, we will assume that $K > 0$ is a scalar for simplicity.

In real time, the implementation of the output feedback control law (7) on an embedded processor is typically done by sampling the output at time instants $t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ \ldots$, updating the actuator at time instants $t_0 + \Delta_0,\ t_1 + \Delta_1,\ t_2 + \Delta_2,\ t_3 + \Delta_3,\ t_4 + \Delta_4,\ \ldots$, where $\Delta_i, i = 0, 1, 2, \ldots$ represents the time required to read the output from the sensor, compute the control action and update the actuators. Between actuator updates, the control law $u(t)$ is held constant according to

$$u(t) = -Ky(t_i), t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}). \tag{8}$$

The implementation of the control action for sensor-actuator network is shown in Fig.1.

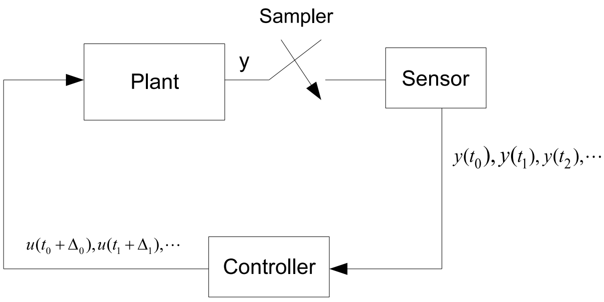

Fig. 1: Implementation of the Feedback Control Action (we assume that the actuator and the controller are collocated with the controller)

If we define the output novelty error induced by the network at the actuator to be

$$e(t) = y(t) - y(t_i), t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}),\ \forall t \geq 0,\ \forall i, \tag{9}$$

since system H is passive, based on (5), (8) and (9) we can obtain

$$\begin{aligned} \dot{V}(x) &\leq u(t)^T y(t) = -K(y(t) - e(t))^T y(t) \\ &= -Ky(t)^T y(t) + Ke(t)^T y(t) \\ &\leq K\|e(t)\|_2 \|y(t)\|_2 - K\|y(t)\|_2^2,\ t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}),\ \forall i. \end{aligned} \tag{10}$$

So if $\|e(t)\|_2 \leq \|y(t)\|_2,\ \forall t \geq 0$, we will have $\dot{V}(x) \leq 0,\ \forall t \geq 0$, and stability of the origin follows from LaSalle's invariance principle [14] and the assumption that system H is ZSD.

If we denote "sampling" as *event* in the loop, the above discussion gives us an idea on when a new sampled output information $y(t_i)$ should be obtained for re-computation of the control action: i.e., for the case when there is no network induced delay in the loop($\Delta_i = 0, \forall i$), the *event time* is implicitly defined by the following event triggering condition

$$\|e(t)\|_2 = \|y(t)\|_2. \tag{11}$$

So when the triggering condition (11) is satisfied at time t_i, we need to get a sampled information of the output, a new control action $u(t_i) = -Ky(t_i)$ needs to be generated and applied to the plant, then $e(t)$ is reset to zero (since $e(t_i) = y(t_i) - y(t_i) = 0$), and the stabilization condition $\|e(t)\|_2 \leq \|y(t)\|_2$ is enforced again.

For event-triggered control, since the event-time is determined by the triggering condition, we need some sort of "event-detector"(an embedded hardware) to detect when the triggering condition is satisfied to schedule control task. Self-triggered control was introduced to take the advantage of the event-triggered technique without resorting to extra hardware. It does not require a dedicated hardware to monitor the state of the plant and examine when the triggering condition is satisfied, but it needs some information of the latest sampled information of the plant for scheduling of the stabilization control tasks. For self-triggered control, the intervals of time in which no attention is devoted to the plant pose a concern regarding the robustness of the closed-loop system. The main result of this paper is trying to address robustness issue of self-triggered control with respect to model structural uncertainty. We propose a robust self-triggered real-time scheduling strategy for stabilization of passive/output feedback systems, and a rigorous examination on the estimated lower bounds of the inter-sampling time and the admissible actuation update delay is provided.

IV. Main Results

In this section, we present the main results of this paper which show that under certain conditions between the output and the state of the system, our proposed self-triggered scheduling strategy will guarantee stability of the closed-loop system and the time interval between any two consecutive actuation updates will be strictly positive. Our results also considered nontrivial actuation update delays. The main results address two cases: we first present the results when the model of the plant is subject to *Output Feedback Uncertainty*; then we present the results when the model of the plant is subject to *Input Feed-forward Uncertainty*. The main results have been stated in Theorem 1 and Theorem 2 respectively followed by discussions to show that the assumptions claimed in these two theorems can be relaxed in some scenarios.

For notation convenience, we let $e(t)$ denote the output novelty error induced by the network at the actuator, and let $\tilde{e}(t)$ denote the output novelty error at the sensor. One should notice that when there is no actuation update delay, $\tilde{e}(t) = e(t),\ \forall t \geq 0$; when the actuation update delay is nontrivial, then for $t \in [t_i, t_i + \Delta_i)$, we have $e(t) = y(t) - y(t_{i-1})$ and $\tilde{e}(t) = y(t) - y(t_i)$; for $t \in [t_i + \Delta_i, t_{i+1})$, we have $e(t) = y(t) - y(t_i) = \tilde{e}(t)$; for $t \in [t_{i+1}, t_{i+1} + \Delta_{i+1})$, we have $e(t) = y(t) - y(t_i)$ while $\tilde{e}(t) = y(t) - y(t_{i+1})$. Let t_i denote the *ith event time* at which a new sampled output information is obtained by the sensor; let Δ_i denote the actuation update delay for the ith event; let $[t_{i+1} - t_i]$ denote the ith inter-sampling time; let L denote the Lipschitz constant of function f; let $\|\cdot\|_2$ denote the 2-norm of a vector.

A. Self-Triggered Real-Time Scheduling In Presence Of Output Feedback Uncertainty

Theorem 1. Consider the control system as shown in Fig.2, where the plant model is given by

$$\Sigma_o : \begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{12}$$

with $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^n$ is the control input and $y \in \mathbb{R}^n$ is the output, Σ_o is a ZSD passive system. The model uncertainty is given by

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tilde{u}) \\ \tilde{y} = \tilde{h}(\tilde{x}), \end{cases} \tag{13}$$

and we assume that the model uncertainty is a $\mathcal{L}_2$ stable dynamic system with finite $\mathcal{L}_2$ gain $\Gamma > 0$.

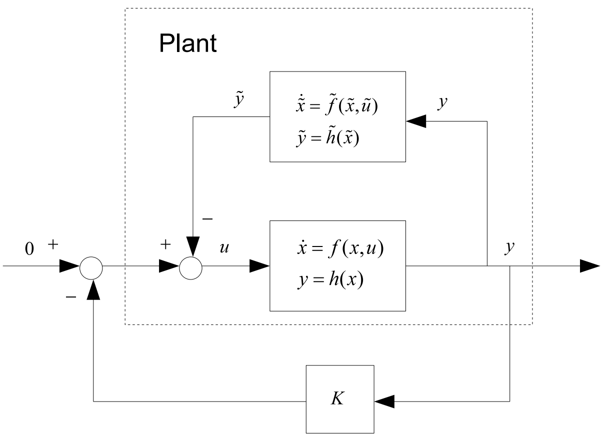

Fig. 2: Output Feedback Uncertainty

Let the following conditions be satisfied

1) $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous on compacts;
2) $h : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous on compacts and it is also a static nonlinear function of x, which belongs to a sector $[\alpha, \beta]$ such that $\alpha x^T x \leq x^T h(x) \leq \beta x^T x$, where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\alpha\beta > 0$;

3) $\tilde{h}:\mathbb{R}^p\to\mathbb{R}^q$ is Lipschitz continuous on compacts;

4) $\|\frac{\partial h(x)}{\partial x}\|_2\le\gamma$, where $0<\gamma<\infty$;

5) $\tilde{\beta}\tilde{u}^T\tilde{u}\le\tilde{u}^T\tilde{y}\le\tilde{\alpha}\tilde{u}^T\tilde{u}$, where $-\infty<\tilde{\beta}<0<\tilde{\alpha}<\infty$;

if we choose $K>-\tilde{\beta}$, then under the following scheduling strategy, the passive system under the control action (8), is asymptotically stable:

- $t_0=t_0+\Delta_0$;
- $t_1=t_0+\tau_0$, $\tau_0=\frac{1}{\gamma L(2+\zeta+\Gamma)}\ln\big(1+\frac{2+\zeta+\Gamma}{1+\zeta+\Gamma}\hat{\sigma}\big)$;
- $t_{i+1}=t_i+\Delta_i+\tau$, $i=1,2,\ldots$;

where $\zeta=\max\{\frac{1}{|\alpha|},\frac{1}{|\beta|}\}$; $\tilde{\sigma},\hat{\sigma}$ are design parameters such that $0<\tilde{\sigma}<\hat{\sigma}<\frac{K+\tilde{\beta}}{2K+\tilde{\beta}}$;

$$\tau=\frac{1}{\gamma L(2+\zeta+\Gamma)}\ln\big(\frac{\frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}+\hat{\sigma}}{\frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}+\tilde{\sigma}}\big); \tag{14}$$

Δ_i is the actuation update delay which is given by

$$\Delta_i=\min\left[\varepsilon_i^-,\varepsilon_i^+\right], \tag{15}$$

where

$$\varepsilon_i^-=\frac{1}{\gamma L(2+\zeta+\Gamma)}\ln\big(\frac{(2+\zeta+\Gamma)\tilde{\sigma}\|y(t_i)\|_2}{(1+\zeta+\Gamma)\|y(t_i)\|_2+\|y(t_i)-y(t_{i-1})\|_2}+1\big) \tag{16}$$

and

$$\varepsilon_i^+=\frac{1}{\gamma L(2+\zeta+\Gamma)}\ln\big(\frac{\frac{K+\tilde{\beta}}{2K+\tilde{\beta}}+\frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}}{\hat{\sigma}+\frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}}\big). \tag{17}$$

Proof. Since the model of the plant Σ_o is passive, we have $\dot{V}(x)\le u^Ty$, and for $t\in[t_i+\Delta_i,t_{i+1}+\Delta_{i+1})$, we have $u(t)=-Ky(t_i)-\tilde{y}(t)$, so we can obtain

$$\begin{aligned}\dot{V}(x)&\le[-Ky(t_i)-\tilde{y}(t)]^Ty(t)=-Ky(t_i)^Ty(t)-\tilde{y}(t)^Ty(t)\\&=-K[y(t)-e(t)]^Ty(t)-\tilde{y}(t)^Ty(t)\\&=-Ky(t)^Ty(t)+Ke(t)^Ty(t)-\tilde{y}(t)^Ty(t),\end{aligned} \tag{18}$$

then we can get

$$\begin{aligned}\dot{V}(x)&\le-K\|y\|_2^2+K\|e\|_2\|y\|_2-\tilde{y}^Ty\\&\le-K\|y\|_2^2+K\|e\|_2\|y\|_2-\tilde{\beta}\|y\|_2^2\\&=-(K+\tilde{\beta})\|y\|_2^2+K\|e\|_2\|y\|_2.\end{aligned} \tag{19}$$

So, if $K>-\tilde{\beta}$, and $\|e\|_2\le\frac{K+\tilde{\beta}}{K}\|y\|_2$ for $t\in[t_i+\Delta_i,t_{i+1}+\Delta_{i+1}]$, $\forall i$, then we will have $\dot{V}(x)\le0$, $\forall t\ge0$, and the closed-loop system is asymptotically stable since Σ_o is ZSD. Moreover, since $e(t)=y(t)-y(t_i)$ for $t\in[t_i+\Delta_i,t_{i+1}+\Delta_{i+1}]$, we have $\|e\|_2\ge\|y(t_i)\|_2-\|y\|_2$, and we can obtain a sufficient condition for $\|e\|_2\le\frac{K+\tilde{\beta}}{K}\|y\|_2$ to hold which is given by

$$\|e\|_2\le\frac{K+\tilde{\beta}}{2K+\tilde{\beta}}\|y(t_i)\|_2,\ \ for\ t\in[t_i+\Delta_i,t_{i+1}+\Delta_{i+1}]. \tag{20}$$

For $t \in [t_i, t_i + \Delta_i)$, we have $e(t) = y(t) - y(t_{i-1})$ and $\tilde{e}(t) = y(t) - y(t_i)$, so we can obtain

$$\begin{aligned}
\frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \|\dot{\tilde{e}}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\frac{\partial y}{\partial x}\dot{x}\|_2 \leq \|\frac{\partial y}{\partial x}\|_2\|\dot{x}\|_2 \\
&\leq \gamma\|f(x,u)\|_2 = \gamma\|f(x, -Ky(t_{i-1}) - \tilde{y})\|_2 \\
&= \gamma\|f(x, -K(y-e) - \tilde{y})\|_2 \\
&\leq \gamma L\big[\|x\|_2 + \|y\|_2 + \|e\|_2 + \|\tilde{y}\|_2\big] \\
&\leq \gamma L\big[\zeta\|y\|_2 + \|y\|_2 + \|\tilde{e} + y(t_i) - y(t_{i-1})\|_2 + \Gamma\|y\|_2\big] \\
&= \gamma L\big[(1+\zeta+\Gamma)\|y\|_2 + \|\tilde{e} + y(t_i) - y(t_{i-1})\|_2\big] \\
&= \gamma L\big[(1+\zeta+\Gamma)\|\tilde{e} + y(t_i)\|_2 + \|\tilde{e} + y(t_i) - y(t_{i-1})\|_2\big] \\
&\leq \gamma L(2+\zeta+\Gamma)\|\tilde{e}\|_2 + \gamma L(1+\zeta+\Gamma)\|y(t_i)\|_2 \\
&+ \gamma L\|y(t_i) - y(t_{i-1})\|_2,
\end{aligned} \tag{21}$$

where $\zeta = \max\{\frac{1}{|\alpha|}, \frac{1}{|\beta|}\}$. So the evolution of $\|\tilde{e}(t)\|_2$ during the time $[t_i, t_i + \Delta_i)$ is bounded by the solution of

$$\dot{\phi}(t) = C_1\phi(t) + C_2, \tag{22}$$

where $C_1 = \gamma L(2+\zeta+\Gamma)$ and $C_2 = \gamma L(1+\zeta+\Gamma)\|y(t_i)\|_2 + \gamma L\|y(t_i) - y(t_{i-1})\|_2$. With $\phi(t_i) = y(t_i) - y(t_i) = 0$, the solution to (22) for $t \in [t_i, t_i + \Delta_i)$ is given by

$$\phi(t) = \frac{C_2}{C_1}\big[e^{C_1(t-t_i)} - 1\big]. \tag{23}$$

So if we choose $0 < \tilde{\sigma} < \frac{K+\tilde{\beta}}{2K+\tilde{\beta}}$, and let $\phi(t_i + \Delta_i) = \tilde{\sigma}\|y(t_i)\|_2$, we can get an estimate of Δ_i, if we denote it by ε_i^-, then ε_i^- is given by

$$\varepsilon_i^- = \frac{1}{C_1}\ln\big[1 + \frac{C_1\tilde{\sigma}\|y(t_i)\|_2}{C_2}\big]. \tag{24}$$

Assume that the actuator updates the control action at $t = t_i + \Delta_i$, and choose $\hat{\sigma}$ such that $0 < \tilde{\sigma} < \hat{\sigma} < \frac{K+\tilde{\beta}}{2K+\tilde{\beta}}$, then for $t \in [t_i + \Delta_i, t_{i+1})$, we have

$$e(t) = \tilde{e}(t) = y(t) - y(t_i), \tag{25}$$

and we can obtain

$$\begin{aligned}\frac{d}{dt}\|e(t)\|_2 &\leq \|\dot{e}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\frac{\partial y}{\partial x}\dot{x}\|_2 \\ &\leq \|\frac{\partial y}{\partial x}\|_2\|\dot{x}\|_2 \leq \gamma\|f(x, -K(y-e)-\tilde{y})\|_2 \\ &\leq \gamma L\big[\|x\|_2 + \|y\|_2 + \|e\|_2 + \|\tilde{y}\|_2\big] \\ &\leq \gamma L\big[\zeta\|y\|_2 + \|y\|_2 + \|e\|_2 + \Gamma\|y\|_2\big] \\ &= \gamma L\big[(1+\zeta+\Gamma)\|y\|_2 + \|e\|_2\big] \\ &= \gamma L\big[(1+\zeta+\Gamma)\|e+y(t_i)\|_2 + \|e\|_2\big] \\ &\leq \gamma L(2+\zeta+\Gamma)\|e\|_2 + \gamma L(1+\zeta+\Gamma)\|y(t_i)\|_2,\end{aligned} \tag{26}$$

so the evolution of $\|e(t)\|_2$ during $[t_i+\Delta_i, t_{i+1})$ is bounded by the solution of

$$\dot{\phi}(t) = \gamma L(2+\zeta+\Gamma)\phi(t) + \gamma L(1+\zeta+\Gamma)\|y(t_i)\|_2, \tag{27}$$

with $\phi(t_i+\Delta_i) = \|y(t_i+\Delta_i) - y(t_i)\|_2 = \tilde{\sigma}\|y(t_i)\|_2$, we can get the solution to (27) which is given by

$$\phi(t) = \big(\frac{C_3}{C_1} + \tilde{\sigma}\|y(t_i)\|_2\big)e^{C_1(t-t_i-\Delta_i)} - \frac{C_3}{C_1}, \tag{28}$$

where $C_3 = \gamma L(1+\zeta+\Gamma)\|y(t_i)\|_2$.

Assume that at $t = t_{i+1}$, we have $\|e(t)\|_2 = \hat{\sigma}\|y(t_i)\|_2$, then an estimate of the time it takes for $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$ to evolve from $\tilde{\sigma}$ to $\hat{\sigma}$ is given by

$$\tau = \frac{1}{\gamma L(2+\zeta+\Gamma)}\ln\big(\frac{\hat{\sigma} + \frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}}{\tilde{\sigma} + \frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}}\big) \tag{29}$$

and notice that for any $\hat{\sigma} > \tilde{\sigma} > 0$, we have $\tau > 0$.

Next, assume that at $t = t_{i+1} + \Delta_{i+1}$, we have $\|e(t)\|_2 = \frac{K+\tilde{\beta}}{2K+\tilde{\beta}}\|y(t_i)\|_2$. Since $e(t) = y(t) - y(t_i)$ for $t \in [t_{i+1}, t_{i+1}+\Delta_{i+1})$, we can still get an estimate of the time it takes for $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$ to evolve from $\hat{\sigma}$ to $\frac{K+\tilde{\beta}}{2K+\tilde{\beta}}$ based on (27). If we denote it by ε_i^+, then we can obtain

$$\varepsilon_i^+ = \frac{1}{\gamma L(2+\zeta+\Gamma)}\ln\big(\frac{\frac{K+\tilde{\beta}}{2K+\tilde{\beta}} + \frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}}{\hat{\sigma} + \frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}}\big). \tag{30}$$

Now we could give a tight lower bound of Δ_i which is given by

$$\Delta_i = \min[\varepsilon_i^-, \varepsilon_i^+], \tag{31}$$

and the corresponding estimate of the time for the sensor to get the next new measurement is given by

$$t_{i+1} = t_i + \Delta_i + \tau. \tag{32}$$

Since we choose$0 < \tilde{\sigma} < \hat{\sigma} < \frac{K+\tilde{\beta}}{2K+\tilde{\beta}}$, for $t \in [t_i+\Delta_i, t_{i+1}+\Delta_{i+1}], \forall i$, we can guarantee that $\|e(t)\|_2 \leq \frac{K+\tilde{\beta}}{K}\|y(t)\|_2, \forall t \geq 0$(which also indicates that Δ_i is strictly positive, since $\varepsilon_i^+ > 0$, and at $t = t_i$, we have $\|e(t_i)\|_2 = \|y(t_i) - y(t_{i-1})\|_2 \leq \frac{K+\tilde{\beta}}{K}\|y(t_i)\|_2$, one can conclude that $\varepsilon_i^- \geq \frac{1}{C_1}\ln\big[1 + \frac{(2+\zeta+\Gamma)\tilde{\sigma}}{(1+\zeta+\Gamma)+\frac{\gamma L(K+\tilde{\beta})}{K}}\big] > 0$), thus $\dot{V}(x) \leq 0, \forall t \geq 0$. Since Σ_o is ZSD, the closed-loop system under the proposed self-triggered scheduling strategy is asymptotically stable. ■

Remark 2: For linear passive system, consider the model of the plant given by Σ_o

$$\Sigma_o : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx, \end{cases} \tag{33}$$

and the feedback uncertainty given by Σ_Δ

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \\ \tilde{y} = \tilde{C}\tilde{x}. \end{cases} \tag{34}$$

We assume that Σ_Δ is a $\mathcal{L}_2$ stable system with finite $\mathcal{L}_2$ gain $\Gamma > 0$. For $t \in [t_i, t_i + \Delta_i)$, we have $e(t) = y(t) - y(t_{i-1})$ and $\tilde{e}(t) = y(t) - y(t_i)$, so we can obtain

$$\begin{aligned} \frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \|\dot{\tilde{e}}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\frac{\partial y}{\partial x}\dot{x}\|_2 = \|C\dot{x}\|_2 \\ &= \|CAx + CBu\|_2 \leq \|CAx\|_2 + \|CBu\|_2, \end{aligned} \tag{35}$$

if $\frac{\|CAx\|_2}{\|y\|_2} = \frac{(x^TA^TC^TCAx)^{\frac{1}{2}}}{(x^TC^TCx)^{\frac{1}{2}}} \leq \zeta$, where $0 < \zeta < \infty$, then we can get

$$\begin{aligned} \frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \|CAx\|_2 + \|CBu\|_2 \leq \zeta\|y\|_2 + \|CBu\|_2 \\ &= \zeta\|\tilde{e} + y(t_i)\|_2 + \| - CB(Ky(t_{i-1}) + \tilde{y})\|_2 \\ &\leq \zeta\|\tilde{e}\|_2 + \zeta\|y(t_i)\|_2 + \|CBKy(t_{i-1})\|_2 + \|CB\|_2\|\tilde{y}\|_2 \\ &\leq \zeta\|\tilde{e}\|_2 + \zeta\|y(t_i)\|_2 + \|CBKy(t_{i-1})\|_2 + \|CB\|_2\Gamma\|y\|_2 \\ &\leq (\zeta + \|CB\|_2\Gamma)\|\tilde{e}\|_2 + (\zeta + \|CB\|_2\Gamma)\|y(t_i)\|_2 \\ &+ \|CBKy(t_{i-1})\|_2, \end{aligned} \tag{36}$$

so the evolution of $\|\tilde{e}(t)\|_2$ during the time $[t_i, t_i + \Delta_i)$ is bounded by the solution of

$$\dot{\phi}(t) = (\zeta + \|CB\|_2\Gamma)\phi(t) + \|CBKy(t_{i-1})\|_2 + (\zeta + \|CB\|_2\Gamma)\|y(t_i)\|_2 \tag{37}$$

with $\phi(t) = \|y(t_i) - y(t_i)\|_2 = 0$, the solution to (37) for $t \in [t_i, t_i + \Delta_i)$ is given by

$$\phi(t) = \frac{C_2}{C_1}\big[e^{C_1(t-t_i)} - 1\big], \tag{38}$$

where $C_1 = \zeta + \|CB\|_2\Gamma$ and $C_2 = \|CBKy(t_{i-1})\|_2 + (\zeta + \|CB\|_2\Gamma)\|y(t_i)\|_2$. So in this case, ε_i^- is given by

$$\varepsilon_i^- = \frac{1}{C_1}\ln\big[1 + \frac{C_1\tilde{\sigma}\|y(t_i)\|_2}{C_2}\big]. \tag{39}$$

For $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1})$, we have $e(t) = y(t) - y(t_i)$, and we can obtain

$$\begin{aligned} \frac{d}{dt}\|e\|_2 &\leq \|\dot{e}\|_2 = \|\dot{y}\|_2 = \|C\dot{x}\|_2 \leq \|CAx\|_2 + \|CBu\|_2 \\ &\leq \zeta\|y\|_2 + \| - CB(Ky(t_i) + \tilde{y})\|_2 \end{aligned} \tag{40}$$

and we can further get

$$\begin{aligned} \frac{d}{dt}\|e\|_2 &\leq \zeta\|y\|_2 + \|CBKy(t_i)\|_2 + \|CB\tilde{y}\|_2 \\ &\leq (\zeta + \|CB\|_2\Gamma)\|e\|_2 + (\zeta + \|CB\|_2\Gamma + \|CBK\|_2)\|y(t_i)\|_2, \end{aligned} \tag{41}$$

so the evolution of $\|e(t)\|_2$ during $[t_i + \Delta_i, t_{i+1} + \Delta_{i+1})$ is bounded by the solution of

$$\dot{\phi}(t) = (\zeta + \|CB\|_2\Gamma)\phi(t) + (\zeta + \|CB\|_2\Gamma + \|CBK\|_2)\|y(t_i)\|_2. \tag{42}$$

Based on this, we can get

$$\varepsilon_i^+ = \frac{1}{C_1}\ln\big[\frac{C_3 + C_1\frac{K+\tilde{\beta}}{2K+\tilde{\beta}}\|y(t_i)\|_2}{C_3 + C_1\hat{\sigma}\|y(t_i)\|_2}\big], \tag{43}$$

where

$$C_3 = (\zeta + \|CB\|_2\Gamma + \|CBK\|_2)\|y(t_i)\|_2,$$

and

$$\tau = \frac{1}{\zeta + \|CB\|_2\Gamma}\ln\big[\frac{\zeta + \|CB\|_2\Gamma + \|CBK\|_2 + (\zeta + \|CB\|_2\Gamma)\hat{\sigma}}{\zeta + \|CB\|_2\Gamma + \|CBK\|_2 + (\zeta + \|CB\|_2\Gamma)\tilde{\sigma}}\big]. \tag{44}$$

■

Remark 3: If the model of the plant is OFP(-$|\rho|$) with $\rho < 0$, then we need choose K more carefully. Since for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1})$, we have $u = -Ky(t_i) - \tilde{y}$, we can obtain

$$\dot{V}(x) \leq u^T y - \rho y^T y = [-Ky(t_i) - \tilde{y}]^T y - \rho y^T y \tag{45}$$

thus

$$\begin{aligned} \dot{V}(x) &\leq -Ky^T(t_i)y - \tilde{y}^T y - \rho y^T y = -K(y-e)^T y - \tilde{y}^T y - \rho y^T y \\ &\leq -Ky^T y + Ke^T y - \tilde{y}^T y - \rho y^T y \\ &= -(K+\rho)y^T y + Ke^T y - \tilde{y}^T y \\ &\leq -(K + \rho + \tilde{\beta})\|y\|_2^2 + K\|e\|_2\|y\|_2. \end{aligned} \tag{46}$$

So choose $K > 0$ and $K > -\rho - \tilde{\beta}$, if $\|e\|_2 \leq \frac{K+\rho+\tilde{\beta}}{K}\|y\|_2$ for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, $\forall i$, then $\dot{V}(x) \leq 0, \forall t \geq 0$. One could show that a sufficient condition for $\|e\|_2 \leq \frac{K+\rho+\tilde{\beta}}{K}\|y\|_2, \forall t \geq 0$ is given by

$$\|e\|_2 \leq \frac{K + \rho + \tilde{\beta}}{2K + \rho + \tilde{\beta}}\|y(t_i)\|_2,\ for\ t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}), \forall i. \tag{47}$$

So in this case, the design parameter $\tilde{\sigma}, \hat{\sigma}$ in the proposed self-triggered scheduling strategy in Theorem 1 should be properly chosen such that

$$0 < \tilde{\sigma} < \hat{\sigma} < \frac{K + \rho + \tilde{\beta}}{2K + \rho + \tilde{\beta}}. \tag{48}$$

■

Remark 4: One may remark that assumptions 1)-3) in Theorem 1 are very conservative, and by assuming that the output of system H belongs to a bounded sector of the state, we restrict the output to have the same dimension as the state. However, this assumption can be relaxed as long as we can get

$$\|\dot{y}\|_2 \leq p_1 \|y\|_2 + p_2 \|e\|_2 \tag{49}$$

for some constant $0 \leq p_1 < \infty$ and $0 < p_2 < \infty$, and similar self-triggered scheduling strategy can be obtained. Also in this case, the output y does not need to have the same dimension as the state x. One could check the examples shown in Section V to see how it works. ■

B. Self-Triggered Real-Time Scheduling In Presence Of Feed-forward Uncertainty

Theorem 2. Consider the control system as shown in Fig.3, where the plant model is given by

$$\Sigma_o : \begin{cases} \dot{x} = f(x, u) \\ y_0 = h(x) \end{cases} \tag{50}$$

$x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^n$ is the control input and $y_0 \in \mathbb{R}^n$ is the output; we assume Σ_o is passive and ZSD.

The plant model is subject to feed-forward model uncertainty given by

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tilde{u}) \\ \tilde{y} = \tilde{h}(\tilde{x}), \end{cases} \tag{51}$$

where $\tilde{x} \in \mathbb{R}^n$ is the state, $\tilde{u} \in \mathbb{R}^n$ is the control input and $\tilde{y} \in \mathbb{R}^n$ is the output. We assume Σ_Δ is a $\mathcal{L}_2$ stable dynamic system with finite $\mathcal{L}_2$ gain $\Gamma > 0$.

Let the following conditions be satisfied:

1) $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous on compacts;
2) $h : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous on compacts and it is also a static nonlinear function of x, which belongs to a sector $[\beta_o, \alpha_o]$ such that $\beta_o x^T x \leq x^T h(x) \leq \alpha_o x^T x$, where $0 < \alpha_o \beta_o < \infty$;
3) $\tilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous on compacts and it is also a static nonlinear function of $\tilde{x}$ which belongs to a sector $[\beta_\Delta, \alpha_\Delta]$ such that $\beta_\Delta \tilde{x}^T \tilde{x} \leq \tilde{x}^T \tilde{h}(\tilde{x}) \leq \alpha_\Delta \tilde{x}^T \tilde{x}$, where $0 < \alpha_\Delta \beta_\Delta < \infty$;
4) $\|\frac{\partial h(x)}{\partial x}\|_2 \leq \gamma_1$, where $0 < \gamma_1 < \infty$, and $\|\frac{\partial \tilde{h}(\tilde{x})}{\partial \tilde{x}}\|_2 \leq \gamma_2$, where $0 < \gamma_2 < \infty$;
5) $\tilde{\beta} \tilde{u}^T \tilde{u} \leq \tilde{u}^T \tilde{y} \leq \tilde{\alpha} \tilde{u}^T \tilde{u}$, where $-\infty < \tilde{\beta} < 0 < \tilde{\alpha} < \infty$;

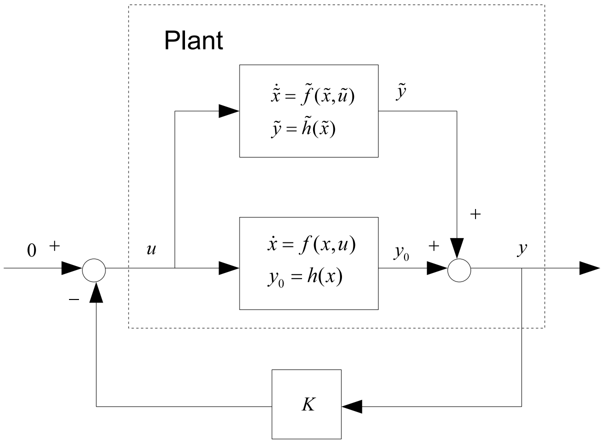

Fig. 3: Feed-forward Uncertainty

if we can select $K > 0$ and $\delta > 0$ such that $K + \tilde{\beta}K^2 - \delta > 0$ and $-\tilde{\beta}K^2 + \frac{1}{4\delta}(K + 2\tilde{\beta}K^2)^2 > 0$, then under the following scheduling strategy, the passive system under the control action (8), is asymptotically stable:

- $t_0 = t_0 + \Delta_0$;
- $t_1 = t_0 + \tau_0$, $\tau_0 = \frac{1}{C_1}\ln\big[\frac{C_3 + C_1\hat{\sigma}\|y(t_0)\|_2}{C_3}\big]$;
- $t_{i+1} = t_i + \Delta_i + \tau$, $i = 1, 2, \ldots$;

where we have

$$\sigma = \frac{\sqrt{\frac{K+\tilde{\beta}K^2-\delta}{\frac{1}{4\delta}(K+2\tilde{\beta}K^2)^2-\tilde{\beta}K^2}}}{1+\sqrt{\frac{K+\tilde{\beta}K^2-\delta}{\frac{1}{4\delta}(K+2\tilde{\beta}K^2)^2-\tilde{\beta}K^2}}}, \tag{52}$$

$\tilde{\sigma}, \hat{\sigma}$ are design parameters such that $0 < \tilde{\sigma} < \hat{\sigma} < \sigma$; $C_1 = \gamma_1 L_1 \zeta_1 + 2\gamma_1 L_1 + 2\gamma_2 L_2$, $C_2 = (\gamma_1 L_1 \zeta_1 + \gamma_1 L_1 + \gamma_2 L_2)\|y(t_i)\|_2 + (\gamma_1 L_1 \zeta_1 + \gamma_2 L_2 \zeta_2)\Gamma K\|y(t_{i-1})\|_2 + (\gamma_1 L_1 + \gamma_2 L_2)\|y(t_i) - y(t_{i-1})\|_2$, and $C_3 = \big[(\gamma_1 L_1 \zeta_1 + \gamma_1 L_1 + \gamma_2 L_2) + (\gamma_1 L_1 \zeta_1 + \gamma_2 L_2 \zeta_2)\Gamma K\big]\|y(t_i)\|_2$; $\zeta_1 = \max\{\frac{1}{|\alpha_o|}, \frac{1}{|\beta_o|}\}$, $\zeta_2 = \max\{\frac{1}{|\alpha_\Delta|}, \frac{1}{|\beta_\Delta|}\}$; L_1 is the Lipschitz constant of $f(x,u)$ and L_2 is the Lipschitz constant of $\tilde{f}(\tilde{x},\tilde{u})$;

$$\tau = \frac{1}{C_1}\ln\big[\frac{C_3 + C_1\hat{\sigma}\|y(t_i)\|_2}{C_3 + C_1\tilde{\sigma}\|y(t_i)\|_2}\big], \tag{53}$$

Δ_i is a tight bound of the actuation update delay given by

$$\Delta_i = \min\big[\varepsilon_i^-, \varepsilon_i^+\big], \tag{54}$$

where

$$\varepsilon_i^- = \frac{1}{C_1}\ln\big(1 + \frac{C_1\tilde{\sigma}\|y(t_i)\|_2}{C_2}\big), \tag{55}$$

$$\varepsilon_i^+ = \frac{1}{C_1}\ln\big(\frac{C_3 + C_1\sigma\|y(t_i)\|_2}{C_3 + C_1\hat{\sigma}\|y(t_i)\|_2}\big). \tag{56}$$

Proof. Since the model of the plant is passive, we have

$$\dot{V}(x) \leq u^T y_0, \tag{57}$$

since $y = y_0 + \tilde{y}$, we have $\dot{V}(x) \leq u^T(y - \tilde{y})$; since $u = -Ky(t_i) = -K(y - e)$ for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, we have

$$\begin{aligned}
\dot{V} &\leq u^T(y - \tilde{y}) = u^T y - u^T \tilde{y} \leq u^T y - \tilde{\beta} u^T u \\
&= -K(y-e)^T y - \tilde{\beta} K^2 (y-e)^T (y-e) \\
&= -Ky^T y + Ke^T y - \tilde{\beta} K^2 (y-e)^T (y-e) \\
&= (-K - \tilde{\beta} K^2) y^T y + (K + 2\tilde{\beta} K^2) e^T y - \tilde{\beta} K^2 e^T e \\
&= -[\delta y^T y - (K + 2\tilde{\beta} K^2) e^T y + \frac{1}{4\delta}(K + 2\tilde{\beta} K^2)^2 e^T e] \\
&+ (-K - \tilde{\beta} K^2 + \delta) y^T y + [-\tilde{\beta} K^2 + \frac{1}{4\delta}(K + 2\tilde{\beta} K^2)^2] e^T e
\end{aligned} \tag{58}$$

where $\delta > 0$ is a constant. So if we can find $K > 0$ and $\delta > 0$ such that $K + \tilde{\beta} K^2 - \delta > 0$ and $-\tilde{\beta} K^2 + \frac{1}{4\delta}(K + 2\tilde{\beta} K^2)^2 > 0$, and furthermore, if we can guarantee that

$$\|e\|_2 \leq \sqrt{\frac{K + \tilde{\beta} K^2 - \delta}{\frac{1}{4\delta}(K + 2\tilde{\beta} K^2)^2 - \tilde{\beta} K^2}} \|y\|_2, \ \forall t \geq 0 \tag{59}$$

then we can get $\dot{V}(x) \leq 0, \forall t \geq 0$. It can be shown that a sufficient condition for (59) to hold is given by

$$\|e\|_2 \leq \sigma \|y(t_i)\|_2, \ for \ t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}], \forall i, \tag{60}$$

where σ is as shown in (52). For $t \in [t_i, t_i + \Delta_i)$, we have $\tilde{e}(t) = y(t) - y(t_i)$ and $e(t) = y(t) - y(t_{i-1})$, we can get

$$\begin{aligned}
\frac{d}{dt}\|\tilde{e}(t)\|_2 \leq \|\dot{\tilde{e}}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\dot{y}_0(t) + \dot{\tilde{y}}(t)\|_2 \\
= \|\frac{\partial y_0}{\partial x}\dot{x}\|_2 + \|\frac{\partial \tilde{y}}{\partial \tilde{x}}\dot{\tilde{x}}\|_2,
\end{aligned} \tag{61}$$

thus we have

$$\begin{aligned}
\frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \gamma_1 \|\dot{x}\|_2 + \gamma_2 \|\dot{\tilde{x}}\|_2 \\
&= \gamma_1 \|f(x, u)\|_2 + \gamma_2 \|\tilde{f}(\tilde{x}, \tilde{u})\|_2 \\
&= \gamma_1 \|f\big(x, -K(y-e)\big)\|_2 + \gamma_2 \|\tilde{f}\big(\tilde{x}, -K(y-e)\big)\|_2,
\end{aligned} \tag{62}$$

if assumptions 2) and 3) are satisfied, then we have $\|x\|_2 \leq \zeta_1 \|y_0\|_2$ and $\|\tilde{x}\|_2 \leq \zeta_2 \|\tilde{y}\|_2$, where $\zeta_1 = \max\{\frac{1}{|\alpha_o|}, \frac{1}{|\beta_0|}\}$ and $\zeta_2 = \max\{\frac{1}{|\alpha_\Delta|}, \frac{1}{|\beta_\Delta|}\}$. Then we can get

$$\begin{aligned}
\frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \gamma_1 L_1\big(\|x\|_2 + \|y\|_2 + \|e\|_2\big) + \gamma_2 L_2\big(\|\tilde{x}\|_2 + \|y\|_2 + \|e\|_2\big) \\
&\leq C_1 \|\tilde{e}\|_2 + C_2,
\end{aligned} \tag{63}$$

where $C_1 = (\gamma_1 L_1 \zeta_1 + 2\gamma_1 L_1 + 2\gamma_2 L_2)$ and $C_2 = (\gamma_1 L_1 \zeta_1 + \gamma_1 L_1 + \gamma_2 L_2)\|y(t_i)\|_2 + (\gamma_1 L_1 \zeta_1 + \gamma_2 L_2 \zeta_2)\Gamma K \|y(t_{i-1})\|_2 + (\gamma_1 L_1 + \gamma_2 L_2)\|y(t_i) - y(t_{i-1})\|_2$. So the evolution of $\|\tilde{e}\|_2$ during the time $[t_i, t_i + \Delta_i)$ is bounded by the solution of

$$\dot{\phi}(t) = C_1 \phi(t) + C_2, \tag{64}$$

with initial condition $\phi(t_i) = 0$. The solution to (64) during $[t_i, t_i + \Delta_i)$ is given by

$$\phi(t) = \frac{C_2}{C_1}\big(e^{C_1(t-t_i)} - 1\big). \tag{65}$$

Assume that at $t = t_i + \Delta_i$, we have $\phi(t_i + \Delta_i) = \tilde{\sigma}\|y(t_i)\|_2$, where $0 < \tilde{\sigma} < \sigma$, then we could get an estimate of Δ_i based on (65), if we denote it by ε_i^-, then ε_i^- is given by (55).

Notice that $\varepsilon_i^- > 0$ for any $\tilde{\sigma} > 0$ and $\|y(t_i)\|_2 \neq 0$. Moreover, even as $\|y(t_i)\|_2$ goes to zero asymptotically, we will have $\varepsilon_i^- > 0$ as long as the scheduling strategy will enforce the stabilization condition (59) to be satisfied for all $t \geq 0$, since at $t = t_i$, we have $\|e(t_i)\|_2 = \|y(t_i) - y(t_{i-1})\|_2 \leq \sigma\|y(t_i)\|_2$ and this yields

$$\varepsilon_i^- \geq \frac{1}{C_1}\ln\big(1 + \frac{C_1\tilde{\sigma}}{\hat{C}_2}\big) > 0, \tag{66}$$

where $\hat{C}_2 = (\gamma_1 L_1 \zeta_1 + \gamma_1 L_1 + \gamma_2 L_2) + (\gamma_1 L_1 \zeta_1 + \gamma_2 L_2 \zeta_2)\Gamma K + (\gamma_1 L_1 + \gamma_2 L_2)\sigma$.

Assume that the actuator updates the control action at $t = t_i + \Delta_i$, for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1})$, we have $e(t) = y(t) - y(t_i)$, and

$$\begin{aligned} \frac{d}{dt}\|e(t)\|_2 &\leq \|\dot{e}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\dot{y}_0(t) + \dot{\tilde{y}}(t)\|_2 \\ &= \|\frac{\partial y_0}{\partial x}\dot{x}\|_2 + \|\frac{\partial \tilde{y}}{\partial \tilde{x}}\dot{\tilde{x}}\|_2 \leq \gamma_1\|\dot{x}\|_2 + \gamma_2\|\dot{\tilde{x}}\|_2 \\ &\leq \gamma_1\|f\big(x, -K(y-e)\big)\|_2 + \gamma_2\|\tilde{f}\big(\tilde{x}, -K(y-e)\big)\|_2 \end{aligned} \tag{67}$$

thus

$$\frac{d}{dt}\|e(t)\|_2 \leq C_1\|e\|_2 + C_3, \tag{68}$$

where $C_3 = \big[(\gamma_1 L_1 \zeta_1 + \gamma_1 L_1 + \gamma_2 L_2) + (\gamma_1 L_1 \zeta_1 + \gamma_2 L_2 \zeta_2)\Gamma K\big]\|y(t_i)\|_2$. So the evolution of $\|e(t)\|_2$ for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1})$ is bounded by the solution of

$$\dot{\phi}(t) = C_1\phi(t) + C_3 \tag{69}$$

with initial condition $\phi(t_i + \Delta_i) = \tilde{\sigma}\|y(t_i)\|_2$, the solution to (69) is given by

$$\phi(t) = \frac{C_3 + C_1\tilde{\sigma}\|y(t_i)\|_2}{C_1}e^{C_1(t-t_i-\Delta_i)} - \frac{C_3}{C_1}. \tag{70}$$

Choose $0 < \tilde{\sigma} < \hat{\sigma} < \sigma$, then we can get an estimate of the time for $\|e(t)\|_2$ to evolve from $\tilde{\sigma}\|y(t_i)\|_2$ to $\hat{\sigma}\|y(t_i)\|_2$ and from $\hat{\sigma}\|y(t_i)\|_2$ to $\sigma\|y(t_i)\|_2$, if we denote them by τ and ε_i^+ respectively, then one could verify that τ and ε_i^+ are given by (53) and (56) respectively. Notice that for any $0 < \tilde{\sigma} < \hat{\sigma} < \sigma$, we have $\tau > 0$ and $\varepsilon_i^+ > 0$.

So, a tight bound of Δ_i is given by

$$\Delta_i = \min[\varepsilon_i^-, \varepsilon_i^+], \tag{71}$$

and the corresponding estimate of the time for the sensor to get the next new measurement is given by

$$t_{i+1} = t_i + \Delta_i + \tau. \tag{72}$$

Since we choose $0 < \tilde{\sigma} < \hat{\sigma} < \sigma$, then by applying the proposed scheduling strategy in Theorem 2, we can guarantee (59) is satisfied thus $\dot{V}(x) \leq 0, \forall t \geq 0$. Then we can conclude that the closed-loop system is asymptotically stable since Σ_o is ZSD and Σ_Δ is $\mathcal{L}_2$ stable. ■

Remark 5: For linear passive system, consider the model of the plant given by Σ_o and the feed-forward uncertainty given by Σ_Δ:

$$\Sigma_o : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx, \end{cases} \qquad \Sigma_\Delta : \begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \\ \tilde{y} = \tilde{C}\tilde{x}. \end{cases} \tag{73}$$

We assume Σ_Δ is a $\mathcal{L}_2$ stable system with finite $\mathcal{L}_2$ gain Γ. For $t \in [t_i, t_i + \Delta_i)$, we have $e(t) = y(t) - y(t_{i-1})$ and $\tilde{e}(t) = y(t) - y(t_i)$, and we can obtain

$$\begin{aligned} \frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \|\dot{\tilde{e}}(t)\|_2 = \|\dot{y}(t)\|_2 \leq \|\dot{\tilde{y}}(t)\|_2 + \|\dot{y}_0(t)\|_2 \\ &= \|\tilde{C}\tilde{A}\tilde{x} + \tilde{C}\tilde{B}u\|_2 + \|CAx + CBu\|_2, \end{aligned} \tag{74}$$

if $\frac{\|CAx\|_2}{\|y_0\|_2} = \frac{(x^TA^TC^TCAx)^{\frac{1}{2}}}{(x^TC^TCx)^{\frac{1}{2}}} \leq \zeta_1$ and $\frac{\|\tilde{C}\tilde{A}\tilde{x}\|_2}{\|\tilde{y}\|_2} = \frac{(\tilde{x}^T\tilde{A}^T\tilde{C}^T\tilde{C}\tilde{A}\tilde{x})^{\frac{1}{2}}}{(\tilde{x}^T\tilde{C}^T\tilde{C}\tilde{x})^{\frac{1}{2}}} \leq \zeta_2$, then we can get for $t \in [t_i, t_i + \Delta_i)$

$$\begin{aligned} \frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \zeta_2\|\tilde{y}\|_2 + \|\tilde{C}\tilde{B}u\|_2 + \zeta_1\|y_0\|_2 + \|CBu\|_2 \\ &\leq \zeta_2\Gamma\|u\|_2 + \|\tilde{C}\tilde{B}u\|_2 + \zeta_1\|y - \tilde{y}\|_2 + \|CBu\|_2 \\ &\leq C_1\|\tilde{e}\|_2 + C_2, \end{aligned} \tag{75}$$

where $C_1 = \zeta_1$, $C_2 = K\big(\zeta_1\Gamma + \zeta_2\Gamma + \|\tilde{C}\tilde{B}\|_2 + \|CB\|_2\big)\|y(t_{i-1})\|_2 + \zeta_1\|y(t_i)\|_2$. For $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1})$, we have $e(t) = y(t) - y(t_i)$, and we can verify that

$$\frac{d}{dt}\|e(t)\|_2 \leq C_1\|\tilde{e}\|_2 + C_3. \tag{76}$$

where $C_3 = \big[K(\zeta_1\Gamma + \zeta_2\Gamma + \|\tilde{C}\tilde{B}\|_2 + \|CB\|_2) + \zeta_1\big]\|y(t_i)\|_2$. So in this case, we can still obtain τ, ε_i^+ and ε_i^- as shown in (53),(55) and (56). ■

Remark 6: One may remark that assumptions 1)-3) in Theorem 2 are conservative. However, for some cases, these assumptions can be relaxed if $\|\dot{y}_0\|_2 \leq p_{o1}\|y_0\|_2 + p_{o2}\|e\|_2$ and $\|\dot{\tilde{y}}\|_2 \leq \tilde{p}_1\|\tilde{y}\|_2 + \tilde{p}_2\|e\|_2$, for some constant $0 < p_{o1}, p_{o2} < \infty$ and $0 < \tilde{p}_1, \tilde{p}_2 < \infty$, and similar self-triggered scheduling strategy can still be obtained. Also in this case, y_0 and $\tilde{y}$ do not need to have the same dimension as x and $\tilde{x}$.■

V. EXAMPLE

Example 1. Consider the model of the plant which is a linear passive system given by

$$\Sigma_o : \begin{cases} \dot{x}_1(t) = -5x_1(t) - x_2(t) \\ \dot{x}_2(t) = -x_2(t) + u(t) \\ y(t) = x_2(t), \end{cases} \tag{77}$$

assume its feedback uncertainty is given by

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}}_1(t) = \tilde{x}_2(t) \\ \dot{\tilde{x}}_2(t) = -a\tilde{x}_1^3(t) - b\tilde{x}_2(t) + \tilde{u}(t), \ \ (a > 0, b > 0) \\ \tilde{y}(t) = \tilde{x}_2(t). \end{cases} \tag{78}$$

If we choose $V(x) = \frac{1}{2}x_2^2$ for the system Σ_o, then we have

$$\dot{V}(x) = \dot{x}_2 x_2 = (-x_2 + u)x_2 = uy - y^2, \tag{79}$$

so Σ_o is passive, also notice that it is ZSD. If we choose $\tilde{V}(\tilde{x}) = \frac{1}{4}a\tilde{x}_1^4 + \frac{1}{2}\tilde{x}_2^2$, we have

$$\dot{\tilde{V}}(\tilde{x}) = -b\tilde{y}^2 + \tilde{u}\tilde{y}, \tag{80}$$

One could verify that the finite $\mathcal{L}_2$ gain of Σ_Δ is $\frac{1}{b}$.

For $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, we have $u(t) = -Ky(t_i) - \tilde{y}(t)$, and we can obtain

$$\begin{aligned} \dot{V}(x) &\leq [-Ky(t_i) - \tilde{y}(t)]^T y(t) = -Ky(t_i)^T y(t) - \tilde{y}(t)^T y(t) \\ &= -K[y(t) - e(t)]^T y(t) - \tilde{y}(t)^T y(t) \\ &= -Ky(t)^T y(t) + Ke(t)^T y(t) - \tilde{y}(t)^T y(t) \\ &\leq -K\|y(t)\|_2^2 + K\|e(t)\|_2\|y(t)\|_2 + \|\tilde{y}(t)\|_2\|y(t)\|_2 \\ &\leq -K\|y(t)\|_2^2 + K\|e(t)\|_2\|y(t)\|_2 + \frac{1}{b}\|y(t)\|_2^2, \end{aligned} \tag{81}$$

so the stabilization condition is given by

$$\|e(t)\|_2 \leq \frac{K - \frac{1}{b}}{K}\|y(t)\|_2, \forall t \geq 0. \tag{82}$$

In this case, we can get for $t \in [t_i, t_i + \Delta_i)$,

$$\begin{aligned} \frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \|\dot{e}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\dot{x}_2\|_2 \\ &\leq \big(1 + \frac{1}{b}\big)\|\tilde{e}\|_2 + \big(1 + \frac{1}{b}\big)\|y(t_i)\|_2 + K\|y(t_{i-1})\|_2, \end{aligned} \tag{83}$$

and for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1})$, we have

$$\begin{aligned} \frac{d}{dt}\|e(t)\|_2 &\leq \|e(t)\|_2 = \|\dot{y}(t)\|_2 = \|\dot{x}_2\|_2 \\ &\leq \big(1 + \frac{1}{b}\big)\|e\|_2 + \big(1 + \frac{1}{b} + K\big)\|y(t_i)\|_2, \end{aligned} \tag{84}$$

thus we can obtain

$$\varepsilon_i^- = \frac{1}{1+\frac{1}{b}} \ln\Big[\frac{(1+\frac{1}{b})\tilde{\sigma}\|y(t_i)\|_2}{(1+\frac{1}{b})\|y(t_i)\|_2 + K\|y(t_{i-1})\|_2} + 1\Big], \tag{85}$$

$$\varepsilon_i^+ = \frac{1}{1+\frac{1}{b}} \ln\Big[\frac{(1+\frac{1}{b})\frac{K-\frac{1}{b}}{2K-\frac{1}{b}} + 1 + \frac{1}{b} + K}{(1+\frac{1}{b})\hat{\sigma} + 1 + \frac{1}{b} + K}\Big], \tag{86}$$

and

$$\tau = \frac{1}{1+\frac{1}{b}} \ln\Big[\frac{(1+\frac{1}{b})\hat{\sigma} + 1 + \frac{1}{b} + K}{(1+\frac{1}{b})\tilde{\sigma} + 1 + \frac{1}{b} + K}\Big]. \tag{87}$$

(Notice that in this example, the output y dose not belongs to a bounded sector of the full-state, y belongs to a bounded sector of x_2, while the unobservable state x_1 is ZSD. However, we can get $\|\dot{y}\|_2 \leq p_1\|y\|_2 + p_2\|e\|_2$ for some constant $0 \leq p_1 < \infty$ and $0 < p_2 < \infty$, as what we have mentioned in Remark 4, so assumptions 1)- 3) in Theorem 1 are relaxed in this case.)

The simulation result for $b = 10$, $K = 3$, $\tilde{\sigma} = 0.05$, $\hat{\sigma} = 0.2324$ and $\sigma = 0.2824$ is shown in Fig.4, where $\sigma(t)$ shows the evolution of $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$, $[t_{k+1}^i - t_k^i]$ shows the evolution of the inter-sampling time, $|e(t)|$ shows the evolution of $\|y(t) - y(t_i)\|_2$. Based on (85), (86) and (87), we conclude that the inter-sampling time should be larger than 0.0908s with admissible actuation update delay $\Delta_i \geq 0.0121s$(in simulation, actuation update delay is generated from the uniform distribution on the interval [0, 0.0121]), stability of the closed-loop system is verified from the simulation results.

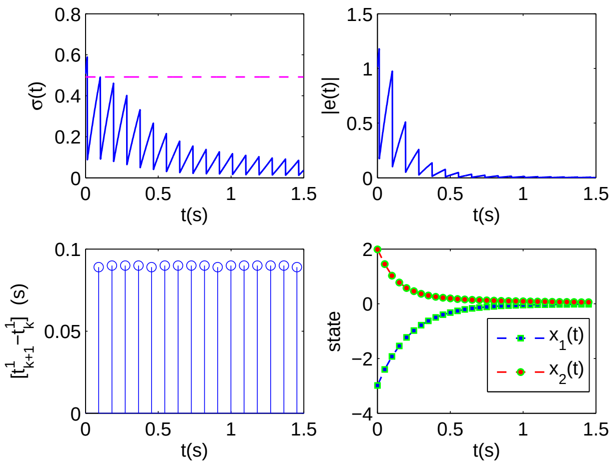

Fig. 4: simulation result of Example 1

Example 2. Consider the model of the plant which is given by

$$\Sigma_o : \begin{cases} \dot{x}_1(t) = -3x_1^3(t) + x_1(t)x_2(t) \\ \dot{x}_2(t) = 3x_2(t) + 2u(t) \\ \ y(t) = x_2(t), \end{cases} \tag{88}$$

assume its feedback uncertainty is given by

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}}_1(t) = \tilde{x}_2(t) \\ \dot{\tilde{x}}_2(t) = -a\tilde{x}_1^3(t) - b\tilde{x}_2(t) + \tilde{u}(t), \ (a > 0, b > 0) \\ \ \tilde{y}(t) = \tilde{x}_2(t). \end{cases} \tag{89}$$

We can see that Σ_o is ZSD but unstable. If we choose the storage function $V(x) = \frac{1}{4}x_2^2$ for Σ_o, we can get

$$\dot{V}(x) = uy + 1.5y^2. \tag{90}$$

From example 1, we know that if we choose the storage function $\tilde{V}(\tilde{x}) = \frac{1}{4}a\tilde{x}_1^4 + \frac{1}{2}\tilde{x}_2^2$ for Σ_Δ, then we could verify that Σ_Δ has finite $\mathcal{L}_2$ gain $\frac{1}{b}$. Based on Remark 3 , we need to choose $K > 1.5 + \frac{1}{b}$, and $0 < \tilde{\sigma} < \hat{\sigma} < \frac{K-1.5-\frac{1}{b}}{2K-1.5-\frac{1}{b}}$.

In this case, we can get for $t \in [t_i, t_i + \Delta_i)$,

$$\begin{aligned} \frac{d}{dt}\|\tilde{e}(t)\|_2 \le \|\dot{e}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\dot{x}_2\|_2 \\ \le \big(3 + \frac{2}{b}\big)\|\tilde{e}\|_2 + \big(3 + \frac{2}{b}\big)\|y(t_i)\|_2 + 2K\|y(t_{i-1})\|_2, \end{aligned} \tag{91}$$

and for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1})$, we have

$$\begin{aligned} \frac{d}{dt}\|e(t)\|_2 \le \|e(t)\|_2 = \|\dot{y}(t)\|_2 = \|\dot{x}_2\|_2 \\ \le \big(3 + \frac{2}{b}\big)\|e\|_2 + \big(3 + \frac{2}{b} + 2K\big)\|y(t_i)\|_2, \end{aligned} \tag{92}$$

thus we can obtain

$$\varepsilon_i^- = \frac{1}{3 + \frac{2}{b}} \ln\big[\frac{(3 + \frac{2}{b})\tilde{\sigma}\|y(t_i)\|_2}{(3 + \frac{2}{b})\|y(t_i)\|_2 + 2K\|y(t_{i-1})\|_2} + 1\big], \tag{93}$$

$$\varepsilon_i^+ = \frac{1}{3 + \frac{2}{b}} \ln\big[\frac{(3 + \frac{2}{b})\frac{K-1.5-\frac{1}{b}}{2K-1.5-\frac{1}{b}} + 3 + \frac{2}{b} + 2K}{(3 + \frac{2}{b})\hat{\sigma} + 3 + \frac{2}{b} + 2K}\big], \tag{94}$$

and

$$\tau = \frac{1}{3 + \frac{2}{b}} \ln\big[\frac{(3 + \frac{2}{b})\hat{\sigma} + 3 + \frac{2}{b} + 2K}{(3 + \frac{2}{b})\tilde{\sigma} + 3 + \frac{2}{b} + 2K}\big]. \tag{95}$$

Choose $a = 3$, $b = 10$, $K = 2.6$, $\tilde{\sigma} = 0.05$, $\hat{\sigma} = 0.2278$ and $\sigma = 0.2778$ is shown in Fig.5. Based on (93), (94) and (95), we conclude that the inter-sampling time should be larger than 0.0201s with actuation update delay $\Delta_i \ge 0.0054s$(in simulation, actuation update delay is generated from the uniform distribution on the interval [0, 0.0054]), stability of the closed-loop system is verified from the simulation results.

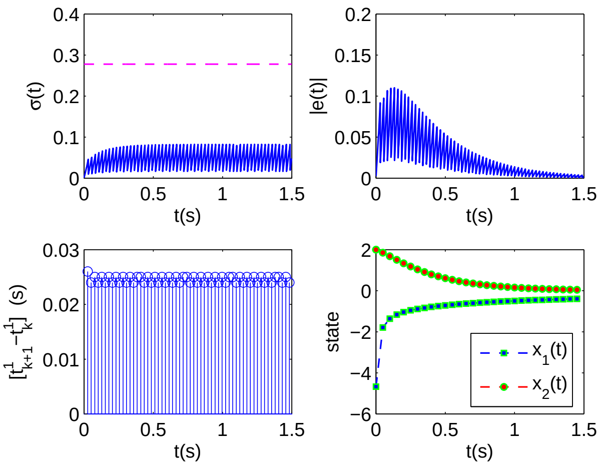

Fig. 5: simulation result of Example 2

VI. Conclusion

In this paper, we propose a robust self-triggered real-time scheduling strategy for stabilization of passive/output feedback passive systems. We assume that the model of the plant is passive or output feedback passive, and we assume that the structure uncertainty is a $\mathcal{L}_2$ stable dynamic system in a feedback/feedforward interconnection with the model of the plant. We derived the self-triggered real-time scheduling strategies for both cases and we have also shown that the inter-sampling time under the proposed scheduling strategy is strictly positive and the admissible actuation update delay is nontrivial. Simulation results are also provided.

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