CONTROL SYSTEMS TECHNICAL REPORT

#52

"Existence and Characterization of Two Degrees of Freedom Stabilizing Controllers"

O. R. González
and
P. J. Antsaklis

Revised November 1986

The work was supported in part by the National Science Foundation under Grant ECS 84-05714.

O. R. Gonzalez and P. J. Antsaklis, "Existence and Characterization of Two Degree of Freedom Stabilizing Controllers," Control Systems Technical Report #52, Dept. of Electrical and Computer Engr., Univ. of Notre Dame, May 1986.

ABSTRACT

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"Existence and Characterization of Two Degrees of Freedom Stabilizing Controllers"[1]

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ABSTRACT

A comprehensive study of internal stability for a class of general multivariable systems is presented in this paper. These systems consist of a general plant model where the measured and the controlled variables are not necessarily the same, and a two degrees of freedom controller. In this paper we elucidate and extend the necessary and sufficient conditions for the existence of internally stabilizing controllers for the plant, and introduce two novel theorems to determine internal stability of the compensated systems which clarify the relation between the stability conditions of two degrees of freedom and single degree of freedom compensated systems. These theorems lead naturally to parametric characterizations of all internally stabilizing two degrees of freedom controllers. Furthermore, these results are extended and a general internal stability criterion over a desirable region of the complex plane is derived using factorizations of transfer function matrices over an appropriate ring.

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1. INTRODUCTION

1.1 Introduction

In this paper we study the stability of the linear, time-invariant, finite dimensional, multivariable control systems represented in Figure 3 (in Section 2). These systems consist of a general plant S_p where the controlled and measured variables are not necessarily the same, and a general linear controller S_C. We are interested in the two degrees of freedom controller which has received renewed interest in the literature because of its usefulness to address control problems with multiple objectives [1-11]. In some of these papers, the relation between the stability conditions of a two degrees of freedom compensated system and a well studied single degree of freedom system is not clear. We show that the internal stability of the compensated systems $\Sigma(S_p,S_C)$ depicted in Figure 3 is an extension of a single degree of freedom stability condition, and that if the plant S_p and controller S_C are "admissible" then the compensated system is internally stable if and only if a single degree of freedom stability condition is satisfied. The admissibility conditions extend the usual stabilizability and detectability conditions to the more general plant and controller models used here. It is shown that admissibility means that S_p and S_C are stabilizable and detectable from a specific input and output, respectively. In particular the plant S_p (controller S_C) is admisible if it is stabilizable from u (u') and detectable from y_m (y_c').

The internal stability issue is made clear by using internal descriptions [12,13,3] and by maintaining a direct relation to the internal descriptions in the analysis. To satisfy the latter requirement factorizations of transfer function matrices [13-16] are used. In Sections 2 independently the system sensitivity (to the plant) and the input-output transfer function matrix. By introducing an auxiliary output z, two additional degrees of freedom are provided that can be used to specify the controller structure. These additional degrees of freedom are expected to help in the design of reliable/fault-tolerant controller structures. The theory developed in this paper does not address the controller structure problem. Nevertheless, the basic internal stability result still holds, that is, if the plant S_p and the general controller $\underline{S}_C$ are admissible as described above then the general compensated system $\Sigma(S_p,\underline{S}_C)$ is internally stable if and only if a single degree of freedom stability condition is satisfied (see [24, Corollary 20]).

The paper is organized as follows. In Section 2 we derive the necessary and sufficient conditions for the existence of a stabilizing controller S_C for the plant S_p. The results in this section extend and complement previous results [27,28]. The analysis of the internal stability of the compensated system $\Sigma(S_p,S_C)$ is given in Section 3. In this section we also derive several parametric characterizations of all stabilizing controllers. The results are extended in Section 4 to solve the usual problem in control of placing the eigenvalues of $\Sigma(S_p,S_C)$ in a desirable region of the complex plane $\mathbb{C}$.

The background needed can be obtained from [5,13-16]. Some of the notation used is defined as follows. The set of all rational transfer functions with real coefficients is denoted by $\mathbb{R}(s)$; some of its important subsets are : $\mathbb{R}_p(s)$, $\mathbb{R}_{p,s}(s)$ the sets of proper, and proper, stable transfer functions, respectively. $M(\mathbb{R}_{p,s}(s))$ denotes the set of all matrices with entries in $\mathbb{R}_{p,s}(s)$

Example 1. Consider the following system interconnection

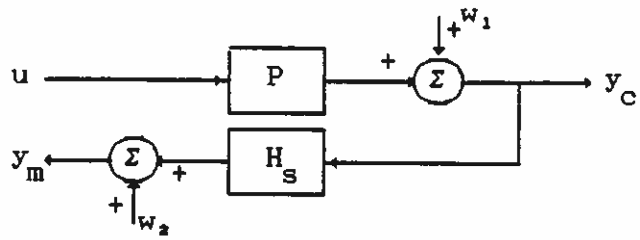

Figure 2 A classical plant-sensor configuration.

where P and H_s are transfer function matrices which completely describe the plant and sensor, respectively, and w_1, w_2 are exogenous signals. The input-output equations describing the system represented in Figure 2 are $y_c = Pu + w_1$, $y_m = H_s Pu + H_s w_1 + w_2$. For this system we have that the exogenous signal $w = [w_1^t, w_2^t]^t$, and the four transfer in (1) are given by $P_{11} = H_s P$, $P_{12} = [H_s, I]$, $P_{21} = P$ and $P_{22} = [I, 0]$.

This simple example shows that the four transfer matrices in (1) do not correspond to separate parts of the plant but rather they are a useful mathematical way to represent an input-output description of a plant. This general plant model is useful in the formulation and analysis of multi-objective control problems [37], and the internal stability theory to be developed here will assist in the solution of these problems.

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Let

$$C = [C_y \quad C_r] = \tilde{D}_c^{-1}[\tilde{N}_y \quad \tilde{N}_r] \tag{4}$$

with $(\tilde{D}_c, [\tilde{N}_y \quad \tilde{N}_r])$ a l.c. factorization in $\mathbb{R}[s]$, that is, $\tilde{D}_c, \tilde{N}_y, \tilde{N}_r \in M(\mathbb{R}[s])$ and $\tilde{D}_c$ square and nonsingular; so that an irreducible realization of the controller is:

$$\begin{aligned} \tilde{D}_c z_c &= -\tilde{N}_y y_m + \tilde{N}_r r + \tilde{N}_y \eta_2 \\ y_c' &= z_c \,. \end{aligned} \tag{5}$$

With the above definitions an internal description of the compensated system is

$$\begin{aligned} \underline{D}_o \begin{bmatrix} z \\ z_c \end{bmatrix} &= \begin{bmatrix} 0 \\ \tilde{N}_r \end{bmatrix} r + \begin{bmatrix} U_2 \\ 0 \end{bmatrix} w + \begin{bmatrix} U_1 \\ 0 \end{bmatrix} \eta_1 + \begin{bmatrix} 0 \\ \tilde{N}_y \end{bmatrix} \eta_2 \\ \begin{bmatrix} y_m \\ y_c \end{bmatrix} &= \begin{bmatrix} V_1 & 0 \\ V_2 & 0 \end{bmatrix} \begin{bmatrix} z \\ z_c \end{bmatrix} \end{aligned} \tag{6}$$

where

$$\underline{D}_o := \begin{bmatrix} T & -U_1 \\ \tilde{N}_y V_1 & \tilde{D}_c \end{bmatrix}. \tag{7}$$

Assume that the compensated system is well-defined, that is, $(I + C_y P_{11})^{-1}$ exists, then we say that <u>the compensated system is internally stable</u> if $\underline{D}_o^{-1}$ is stable or if the zeros of

$$|\underline{D}_o| = |T| \; |\tilde{D}_c - \tilde{N}_y V_1 T^{-1} U_1| \tag{8}$$

are in the "stable or good" region of the complex plane. Without loss of generality consider $\mathbb{C}^-$, the open left half of the complex plane, to be the "stable" region. In this setting, internal stability corresponds to the usual definition, that is, when all the exogenous inputs (w, r, η_1, η_2) are set to zero and for all initial conditions the states of the compensated system $\Sigma(S_p, S_c)$ go to zero asymptotically. In Section 4 the theory is extended to consider an appropriately defined "good" region of the complex plane.

relating u to y_m when w=0 and is described by the triple $\{V_1,T,U_1\}$ is stabilizable and detectable, that is, the transfer matrix $P_{11} = V_1T^{-1}U_1$ must contain all the poles of the plant S_p in $\mathbb{C}^+$ with the same McMillan degree and some given structure. These comments are clear since T is the denominator matrix of the plant description in (3) and since by (A1) and (A2) there are no $\mathbb{C}^+$ "cancellations" [25,26] in $V_1T^{-1}U_1$. It is important to notice that even though the plant S_p is assumed to be controllable from $[w^t, u^t]^t$ and observable from $[y_m{}^t, y_c{}^t]^t$ that it might not even be stabilizable or detectable from certain inputs or outputs, respectively. Theorem 1 shows that the existence of a stabilizing controller which uses the measured variables y_m to come up with the control variables u requires that S_p be stabilizable from u and detectable from y_m which is the usual condition in single degree of freedom systems. Otherwise, the controller S_C would not have available all the information about the unstable modes of the plant S_p and internal stability would not be possible.

From conditions (B1) and conditions (A1) and (A2) it can be seen that the existence of a stabilizing controller C can also be determined from the denominators of the plant transfer function matrix P (T) and of P_{11} (D_1 or $\tilde{D}_1$) :

<u>Corollary 1.1</u> : There exists an internally stabilizing controller C for the plant S_p if and only if

$$\text{(B1.1)} \quad |D_1|^{-1}\ |T| \text{ is a Hurwitz polynomial}$$

where T and D_1 are defined in (3) and (9), respectively.

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Sufficiency of (ii): Suppose (A1) and (A2) are satisfied and we want to show that there exists a stabilizing controller C. In this case the subsystem $S_{P_{11}}$ described by the triple $\{V_1, T, U_1\}$ is stabilizable and detectable [12,13] so that if $V_1T^{-1}U_1 = N_1D_1^{-1} = P_{11}$ with (N_1, D_1) a r.c. factorization in $\mathbb{R}[s]$ then the zeros of $|T|$ and $|D_1|$ in $\mathbb{C}^+$ are the same; furthermore, $|D_1|$ divides $|T|$.

A stabilizing controller S_C exists if there exists a C that can place all the compensated system's eigenvalues in $\mathbb{C}^-$. Consequently, consider the compensated system's characteristic polynomial :

$$
\begin{aligned}
|\underline{D}_o| &= |T|\ |\tilde{D}_c - \tilde{N}_y V_1 T^{-1} U_1| \\
&= |T|\ |\tilde{D}_c - \tilde{N}_y N_1 D_1^{-1}| \\
&= |T|\ |D_1|^{-1}\ |\tilde{D}_c D_1 - \tilde{N}_y N_1| \ . \qquad (12)
\end{aligned}
$$

Clearly from (12) and the discussion above it is observed that all the zeros of $|T|$ in $\mathbb{C}^+$ cancel in $|T|\ |D_1|^{-1}$, so that any C that sets the zeros of $|\tilde{D}_c D_1 - \tilde{N}_y N_1|$ in $\mathbb{C}^-$ will be a stabilizing controller. Since $S_{P_{11}}$ is stabilizable there exists a C_y that will stabilize it, and C_r can be chosen to be stable, making $|\tilde{D}_c D_1 - \tilde{N}_y N_1|$ a Hurwitz polynomial. Therefore, there exists a C that will make $\underline{D}_o^{-1}$ stable and hence the compensated system will be internally stable.

Necessity : Suppose there exists a stabilizing controller C and suppose that (A1) or (A2) is not satisfied, then at least one of the zeros of $|T|$ in $\mathbb{C}^+$ will not be a zero of $|D_1|$. In this case there is no controller C that can make $|\underline{D}_o|$ Hurwitz, since $|\underline{D}_o| = |T|\ |D_1|^{-1}\ |\tilde{D}_c D_1 - \tilde{N}_y N_1|$ will have at least one of the zeros of $|T|$ in $\mathbb{C}^+$. Therefore, (A1) and (A2) are necessary conditions too.•

Before showing that (i) and (iv) in Theorem 1 are equivalent consider another way to determine when the compensated system will be internally

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$$H_{y\eta} = \begin{bmatrix} D_{C_y} D_k^{-1} \tilde{N}_1 & N_1 D_k^{-1} \tilde{N}_{C_y} & D_{C_y} D_k^{-1} \tilde{D}_1 P_{12} & D_{C_y} D_k^{-1} \tilde{N}_1 C_r \\ P_{21} D_1 \tilde{D}_k^{-1} \tilde{D}_{C_y} & P_{21} D_1 \tilde{D}_k^{-1} \tilde{N}_{C_y} & P_{22} - P_{21} D_1 \tilde{D}_k^{-1} \tilde{N}_{C_y} P_{12} & P_{21} D_1 \tilde{D}_k^{-1} \tilde{D}_{C_y} C_r \\ -N_{C_y} D_k^{-1} \tilde{N}_1 & N_{C_y} D_k^{-1} & -N_{C_y} D_k^{-1} \tilde{D}_1 P_{12} & D_1 \tilde{D}_k^{-1} \tilde{D}_{C_y} C_r \end{bmatrix} \tag{17}$$

<u>Proof of Theorem 1</u>: Sufficiency of (iv) : Assume $P_{21}D_1$, $\tilde{D}_1P_{12}$, $P_{22} - P_{21}D_1x_2P_{12} \in M(\mathbb{R}_{p,s}(s))$ then we want to show that there exists a stabilizing controller $C=[C_y \;\; C_r]$. Note that if C_y stabilizes $S_{P_{11}}$, then $\tilde{D}_k^{-1}$ and D_k^{-1} are stable, and that if C_r is chosen appropriately, eg. stable, then every block matrix of $H_{y\eta}$ is stable except for the 2,3-block matrix. It helps to rewrite the 2,3-block matrix using C_y from the set of stabilizing controllers of $S_{P_{11}}$ so that

$$C_y = (x_1 - K\tilde{N}_1)^{-1} (x_2 + K\tilde{D}_1) \tag{18}$$

where $\tilde{N}_1$, $\tilde{D}_1$, x_1, $x_2 \in M(\mathbb{R}[s])$ satisfy (9)-(11), and K is a stable rational transfer function matrix with poles the zeros of $|D_k|=\alpha|\tilde{D}_k|$, $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} P_{22} - P_{21}(I + C_yP_{11})^{-1}C_yP_{12} &= P_{22} - P_{21}D_1(x_2 + K\tilde{D}_1)P_{12} \\ &= (P_{22} - P_{21}D_1x_2P_{12}) - P_{21}D_1K\tilde{D}_1P_{12} . \end{aligned} \tag{19}$$

The first term in parenthesis on the right hand side is stable by hypothesis; the last term on the right hand side is also stable since it is the product of three stable transfer function matrices: $P_{21}D_1$, K, $\tilde{D}_1P_{12}$. Therefore, the 2,3-block matrix is stable, showing that $H_{y\eta}$ is stable and thus the existence of a stabilizing controller has been established.

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The following two examples illustrate the use of some of the conditions developed in this section.

Example 2. Consider the system used in Example 1 (see Figure 2). An internally stabilizing controller for this system exists if and only if $\nu^+(P_{11})=\nu^+(H_sP)=\nu^+(H_s)+\nu^+(P)$ (Theorem 1, condition B1) which is the usual assumption made when analyzing such systems.

The next example clarifies another aspect of the general plant: how the exogenous signal w affects the measured and controlled variables. Since S_c uses the measured variables y_m of the plant to come up with the control variables u, it is of interest to examine the way how the exogenous signal w can affect the controlled variables y_c. This is done by comparing P_{12} and P_{22}, since P_{12} and P_{22} are the transfer functions from the where the exogenous signals affect the plant model to the measured and controlled variables, respectively (as an illustration see Example 1).

Example 3. Consider the following input-output representation of a plant:

$$y_m = \frac{1}{(s+\alpha)(s+\beta)}u + \frac{c_1}{s-\beta}w$$
$$y_c = \frac{c_2}{s-\beta}u + P_{22}(s)w$$

where $\alpha,\beta > 0$ and $\alpha\neq-\beta$. The problem is to characterize the set of transfer functions $P_{22}(s)$ so that the plant is stabilizable.

For this problem the conditions (C1),(C2) and (C3) of Theorem 1 will be most useful. Let $P_{11}=n_1/d_1=1/[(s+\alpha)(s-\beta)]$ then $x_1=0$, $x_2=1$ satisfy the Diophantine equation $x_1d_1+x_2n_1=1$. Clearly, $P_{21}d_1=c_2(s+\alpha)$ and $d_1P_{12}=c_1(s+\alpha)$ are stable transfer functions. Hence the plant is internally stabilizable if and only if (C3) is satisfied. (C3) is given by $P_{22}-P_{21}d_1x_2P_{12}$ is stable description of the plant (1). Whereas the third set, (C1), (C2) and (C3), combines the input-output description of the plant (1) with an internal description of the subsystem $S_{P_{11}}$ ($y_m = P_{11}u$). So that depending upon the plant's characteristics one of the sets of conditions in Theorem 1 could be more useful. For example, if $P_{12} = 0$ and P_{11} and P_{22} are square then from (B1) and the conditions (C1), (C2) and (C3) it is clear that P_{22} must be stable for the existence of a stabilizing controller. This fact is not as evident from (A1) and (A2).

3) The necessity of conditions (A1),(A2) and (C1),(C2)is well known in the regulation problem literature [29-31] where (A1),(A2) and (C1),(C2) are stated as assumptions needed to guarantee internal stability.

4) The transfer matrix in (C3) can be rewritten in the following form

$$P_{22} - P_{21}D_1x_2P_{12} = P_{22} - P_{21}\,\tilde{x}_2\tilde{D}_1P_{12}\,, \tag{23}$$

since $D_1x_2 = \tilde{x}_2\tilde{D}_1$ as seen from (10) and (11). The transfer matrix on the left hand side in (23) has to be stable for a particular factorization of P_{11} and polynomial matrix x_2 satisfying (11). The transfer matrix will remain stable for all other coprime factorizations of P_{11} and polynomial matrices x_2 that satisfy (11).

5) The transfer matrix in (C3):

$$P_{22} - P_{21}D_1x_2P_{12} \tag{24}$$

was obtained by using a particular parameterization of all stabilizing controllers C_y of $S_{P_{11}}$. The parameterization uses the stable rational matrix K introduced in [17]; but the condition is independent of the choice of K and holds for any coprime parameterization of P_{11}. A question that arises is: what is the form of (C3) when other parameters (see Section 3) are used to parameterize the set of stabilizing controllers of $S_{P_{11}}$? It can be shown that when other parameterizations are used to derive the transfer matrix in

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<u>**Theorem 2.**</u> If the plant S_p is admissible then the compensated system $\Sigma(S_p,S_c)$ is internally stable if and only if

(a) the control law $u = -C_y y_m$ stabilizes the subsystem $S_{P_{11}}$ $(y_m=P_{11}u)$, and

(b) C_r is such that $M = (I + C_yP_{11})^{-1}C_r$ satisfies $D_1^{-1}M = X$, a stable transfer matrix, where C_y satisfies (a) and $P_{11}=N_1D_1^{-1}$ is a coprime polynomial matrix factorization.

Theorem 2 shows that stability in a two degrees of freedom configuration is based on the stability of a well studied single degree of freedom configuration - condition (a), while condition (b) represents an extension of previous results because of using a two degrees of freedom controller. Observe that for a single degree of freedom controller condition (b) follows immediately from (a).

<u>Proof of Theorem 2</u>: Internal stability is not affected if the input signals $w=0$, $\eta_1=0$, $\eta_2=0$ or if we let

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} T^{-1}U_1 = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} D^{-1} \tag{25}$$

where $([N_{11}{}^t, N_{21}{}^t]^t, D)$ is a r.c. factorization in $\mathbb{R}[s]$. Using (25), an irreducible realization of the plant is given by $Dz=u$, $[y_m{}^t, y_c{}^t]=[N_{11}{}^t, N_{21}{}^t]^tz$ and an irreducible realization of the controller C was given in (5), combining these two realizations gives an internal description of the compensated system $\Sigma(S_p,S_c)$:

$$\begin{aligned} (\tilde{D}_cD + \tilde{N}_yN_{11})\, z &= \tilde{N}_r r \\ \begin{bmatrix} y_m \\ y_c \end{bmatrix} &= \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} z \end{aligned} \tag{26}$$

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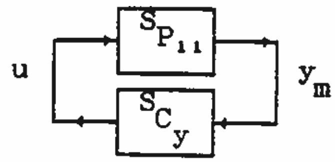
Sufficiency: Let $C=[C_y \ \ C_r]$ satisfy (a) and (b) of Theorem 2. If $C=\tilde{D}_c^{-1}[\tilde{N}_y \ \ \tilde{N}_r]$ is a l.c. factorization in $\mathbb{R}[s]$ and G_ℓ is a g.c.l.d. of $(\tilde{D}_c, \tilde{N}_y)$ then (28b) is true for some left coprime polynomial matrices $\tilde{D}_{C_y}$ and $\tilde{N}_{C_y}$ $(C_y=\tilde{D}_{C_y}^{-1}\tilde{N}_{C_y})$. Similarly, if $P_{11}=N_{11}D^{-1}$ and G_r is a g.c.r.d. of N_{11} and D then (28a) is true for some coprime polynomial matrices N_1 and D_1 $(P_{11}=N_1D_1^{-1})$. Because (a) is satisfied, $\tilde{D}_{C_y}D_1+\tilde{N}_{C_y}N_1=\tilde{D}_k$ where $\tilde{D}_k \in M(\mathbb{R}[s])$ is such that $\tilde{D}_k^{-1}$ is stable. The expression for $\tilde{D}_k$ can be premultiplied and postmultiplied by G_ℓ and G_r, respectively, to obtain

$$\tilde{D}_cD + \tilde{N}_yN_{11} = G_\ell\tilde{D}_kG_r \ . \tag{31}$$

In view of (27) and $\tilde{D}_k^{-1}$ stable, for internal stability it suffices to show that G_ℓ^{-1} and G_r^{-1} are stable. First observe that G_r is a common right divisor of V_1 and T defined in (3), since $V_1T^{-1}U_1=N_{11}D^{-1}$. Then, since (A2) must be satisfied, G_r^{-1} is stable. This in turn implies that G_ℓ^{-1} must be stable, since (31) can be written as $G_\ell^{-1}D_o=\tilde{D}_kG_r$ or as $G_\ell^{-1}\tilde{N}_r=\tilde{D}_kX$ with $(G_\ell,\tilde{N}_r)$ l.c. in $\mathbb{R}[s]$ (if they were not prime, $C=\tilde{D}_c^{-1}[\tilde{N}_y \ \ \tilde{N}_r]$ would not have been a l.c. factorization in $\mathbb{R}[s]$). Therefore, $D_o=G_\ell\tilde{D}_kG_r$ in (31) satisfies D_o^{-1} stable, that is, the compensated system $\Sigma(S_p,S_c)$ is internally stable.•

Q.E.D.

The proof of Theorem 2 shows that a necessary condition for the compensated system $\Sigma(S_p,S_c)$ to be internally stable is that G_ℓ^{-1} must be stable. This condition can be written as $\tilde{D}_{C_y}\tilde{D}_c^{-1}$ is stable or as $|\tilde{D}_{C_y}|^{-1}\,|\tilde{D}_c|$ is a Hurwitz polynomial. In this form the similarity between this condition and the admissibility of S_p is evident (see Corollary 1.1). This leads to a similar definition of admissibility for the two degrees of freedom controller S_c.



$y_m = P_{11}u$

$u = -C_y y_m$

Figure 4. A single degree of freedom system.

These comments are formalized in a theorem.

<u>Theorem 2.1</u> : If the plant S_P and the controller S_C are admissible then the compensated system $\Sigma(S_P, S_C)$ is internally stable if and only if the control law $u = -C_y y_m$ stabilizes the subsystem $S_{P_{11}}$ $(y_m = P_{11}u)$.

Theorem 2 shows a way to parameterize all stabilizing compensators $C=[C_y \quad C_r]$. First, use any of the known characterizations of C_y that stabilize the subsystem $S_{P_{11}}$, then use condition (b) of the theorem to parameterize C_r. This fact is used in the following proposition and corollaries to give several controller parameterizations. The choices in parametrically characterizing all feedback stabilizing controllers C_y are extensively discussed by Antsaklis and Sain in [10].

<u>Proposition 3.</u> All stabilizing controllers C are given by any of the following characterizations:

(*1*) $C = (x_1 - K\tilde{N}_1)^{-1}\,[(x_2 + K\tilde{D}_1) \quad X]$

where K and X are stable transfer matrices so that $|x_1 - K\tilde{N}_1| \neq 0$. For C proper need $D_1(x_1 - K\tilde{N}_1)$ biproper (that is, $D_1(x_1 - K\tilde{N}_1)$ and its inverse are proper) and $D_1(x_2 + K\tilde{D}_1)$ proper, or $K = (\tilde{L}_1 - x_2)\tilde{D}_1^{-1} = (D_1^{-1}Q - x_2)\tilde{D}_1^{-1}$ is such that $\tilde{L}_1$ or Q satisfy the causality conditions.

$$\Rightarrow C_r = (X_1 - K\tilde{N}_1)^{-1} X$$

since $M = D_1X$, with X a stable rational matrix. Therefore,

$$C = (X_1 - K\tilde{N}_1)^{-1} [(X_2 + K\tilde{D}_1) \quad X].$$

The other cases can be proven in a similar manner, starting with the parameterizations of C_y which can be found in [19,21].

The additional conditions given to restrict the characterizations to proper controllers C can be obtained as follows. First, observe that C is proper if and only if C_y and C_r are proper. Consider the parameterization in (4), if Q proper and $(I-QP_{11})$ biproper, then C_y is proper. With these additional conditions C_r is proper too, since $C_r=(I-QP_{11})^{-1}D_1X$ where $(I-QP_{11})$ is biproper and $D_1X=M$ is proper (one of the matrices in $H_{u\eta}$). Therefore, (4) is a characterization of proper stabilizing controllers C if in addition Q is proper and $(I-QP_{11})$ is biproper. The causality conditions on Q are easily extended for the other parameters by considering the relations between parameters given below in (32).• **Q.E.D.**

Additional parameterizations of the controller C can be found for special cases of the plant S_p. The following two corollaries consider two special cases of S_p, but are not meant to consider every possible case. In Corollary 3.1 S_p is assumed to be stable, that is, P_{11} is stable [19,21].

<u>Corollary 3.1.</u> Assume that the plant S_p is stable then all stabilizing controllers C are given by any of the following characterizations :

(6) $C = ((I - \tilde{L}_1N_1)\ D_1^{-1})^{-1}[\tilde{L}_1 \quad X]$

where $\tilde{L}_1$ and X are stable transfer matrices with $|I-\tilde{L}_1N_1|\neq 0$.

(7) $C = (I - QP_{11})^{-1}\ [Q \quad D_1X]$

where Q and X are stable transfer matrices with $|I-QP_{11}|\neq 0$.

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$$G_\ell \tilde{D}_k G_r z = \tilde{N}_r r$$
$$\begin{bmatrix} y_m \\ y_c \end{bmatrix} = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} z \tag{33}$$

which is an alternate way to write the internal description in (26). In this way the effect of the particular choice of the parameter on the compensated system's eigenvalues and, in general in the overall system description can be determined. It turns out that with some of the parameterizations it is easier to place the compensated system's poles and to minimize the number of hidden modes. This is true if the parameterization in (*1*) or the polynomial parameterization in (*2*) is used.

A very useful internal stability result was given in Theorem 2. Other ways to analyze internal stability of $\Sigma(S_p,S_C)$ which help understand the properties of $\Sigma(S_p,S_C)$ are given next. Let ψ_Σ, ψ_p and ψ_C denote the characteristic polynomials of $\Sigma(S_p,S_C)$, S_p and S_C, respectively and define

$$S_1 = (I + P_{11}C_y)^{-1}, \qquad S_2 = (I + C_yP_{11})^{-1}, \tag{34}$$

$$\begin{bmatrix} u \\ u' \end{bmatrix} = H_{u\eta} \begin{bmatrix} \eta_1 \\ \eta_2 \\ w \\ r \end{bmatrix} \tag{35}$$

where

$$H_{u\eta} = \begin{bmatrix} -C_yS_1P_{11} & C_yS_1 & -C_yS_1P_{12} & S_1C_r \\ -S_1P_{11} & -S_1P_{11}C_y & -S_1P_{12} & -P_{11}S_2C_r \end{bmatrix}. \tag{36}$$

When S_P and S_C are admissible only a subset of the block matrices in $H_{y\eta}$ or $H_{\dot{u}\eta}$ needs to be checked to determine internal stability of $\Sigma(S_P,S_C)$. In particular only the matrices related to a single degree of freedom stability need to be tested. This and other well known single degree of freedom stability conditions are presented in the following corollary.

<u>Corollary 4.1.</u> If S_P and S_C are admissible, then the following statements are equivalent.

(i) $\Sigma(S_P,S_C)$ is internally stable.

(ii) The zero polynomial of $\begin{bmatrix} I & C_y \\ -P_{11} & I \end{bmatrix}$ is Hurwitz.

(iii) $\begin{bmatrix} S_2 & -Q \\ P_{11}S_2 & S_1 \end{bmatrix}$ is stable.

(iv) $D^{-1}[S_2C_y \quad S_2]$ is stable.

<u>Proof</u>: Under the assumption of admissibility of S_P and S_C the proofs are the same as in [19].•

<u>Remarks</u>

1) The study of the stability of a two degrees of freedom compensated system $\Sigma(S_P,S_C)$ has been broken down into two conditions via Theorem 2. The first condition reduces the problem to the known case of the stability of a single degree of freedom configuration. The second step is peculiar to the two degrees of freedom configuration, showing how the additional degree of freedom affects internal stability. Theorem 2 is also useful to parameterize all stabilizing controllers C.

2) The study of the stability of a two degrees of freedom compensated system was simplified in Theorem 2.1 where it is shown that the admissibility

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4. $\mathbb{R}_g$-STABILITY

In this section the internal stability theory developed so far is extended to solve the usual problem in control of placing the compensated system's eigenvalues in a desirable region of the complex plane $\mathbb{C}$. Let S_g denote this region which corresponds to the "good" portion of the complex plane so that S_g is symmetric with respect to the real axis and contains at least one real point.

Let $\mathbb{R}_g(s)$ be a nonempty subset of $\mathbb{R}_p(s)$, the ring of proper rational functions with real coefficients, consisting of the proper rational functions which have all their poles in S_g. Then it can be shown that $\mathbb{R}_g(s)$ is a proper Euclidean domain [33,34]. In particular $\mathbb{R}_g(s)$ is a principal ideal domain [35], giving $\mathbb{R}_g(s)$ the same nice algebraic properties of the polynomial ring $\mathbb{R}[s]$. Furthermore, the theory of polynomial matrix factorizations [13] is easily extended to matrix factorizations over $\mathbb{R}_g(s)$ by using [5-Chapter 4, 15, 16]. A fundamental concept is that a given transfer function can be represented as the ratio of two rational functions in $\mathbb{R}_g(s)$.

The problem is to place all the compensated system's eigenvalues in S_g, using the two degrees of freedom control law defined in (2).

<u>Definition 3.</u> (i) A system is said to be $\mathbb{R}_g$-stable if all its eigenvalues are in S_g.

(ii) S_C is said to be an $\mathbb{R}_g$-stabilizing controller if the compensated system $\Sigma(S_P,S_C)$ is $\mathbb{R}_g$-stable.

So the problem is to find an $\mathbb{R}_g$-stabilizing controller S_C for the plant S_P. The first step is to find the necessary and sufficient conditions for the existence of an $\mathbb{R}_g$-stabilizing controller S_C for the plant S_P. The second

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where

$$\underline{D}_o' = \begin{bmatrix} T' & -U_1' \\ \tilde{N}_y'V_1' & \tilde{D}_c' \end{bmatrix} \tag{41}$$

where $\underline{D}_o' \in M(\mathbb{R}_g(s))$ is square, nonsingular and biproper. A relation between $|\underline{D}_o'|$, the characteristic determinant of $\Sigma(S_p,S_C)$ [5], and $|\underline{D}_o|$ from (6), the characteristic polynomial, can be obtained from the Appendix and [36]. It can be shown that $|\underline{D}_o'|$ and $|\underline{D}_o|$ are associates, that is, $|\underline{D}_o'|$ and $|\underline{D}_o|$ differ by a unit in $\mathbb{R}_g(s)$, a miniphase function in $\mathbb{R}_g(s)$. So, all the compensated system's eigenvalues in $\Omega = \mathbb{C}\backslash S_g$, if any, will be zeros of $|\underline{D}_o'|$. Therefore, $\Sigma(S_p,S_C)$ is said to be $\mathbb{R}_g$-stable if and only if $\underline{D}_o'$ is a unimodular matrix in $\mathbb{R}_g(s)$, that is, $\underline{D}_o'$, $\underline{D}_o'^{-1} \in M(\mathbb{R}_g(s))$.

The necessary and sufficient conditions for the existence of an $\mathbb{R}_g$-stabilizing controller C are similar in form to the ones given in Theorem 1. The following definitions are needed. Let

$$P_{11} = N_1'D_1'^{-1} = \tilde{D}_1'^{-1}\tilde{N}_1' \tag{42}$$

be coprime $\mathbb{R}_g$-factorizations, that is, (N_1',D_1') and $(\tilde{D}_1',\tilde{N}_1')$ are r.c. and l.c. in $\mathbb{R}_g(s)$, respectively, with D_1' and $\tilde{D}_1'$ nonsingular and biproper, and let

$$U' = \begin{bmatrix} X_1' & X_2' \\ -\tilde{N}_1' & \tilde{D}_1' \end{bmatrix}, \qquad U'^{-1} = \begin{bmatrix} D_1' & -\tilde{X}_2' \\ N_1' & \tilde{X}_1' \end{bmatrix} \tag{43}$$

be two unimodular matrices in $\mathbb{R}_g(s)$ satisfying

$$U'\,U'^{-1} = I \quad \text{and} \quad U'^{-1}\,U' = I. \tag{44}$$

Consider the partial fraction expansion $P_{ij} = P_{ij}^{+} + P_{ij}^{-}$ $i,j=1,2$ where all the poles of P_{ij}^{-} (P_{ij}^{+}) are in S_g ($\Omega=\mathbb{C}\backslash S_g$), and let $\nu^{+}[P_{ij}]$ denote the McMillan degree of P_{ij}^{+}, that is, $\nu[P_{ij}^{+}]$.

Proof of Theorem 5: The proofs of the above conditions are very similar to the proofs of the conditions in Theorem 1 so only the sufficiency proof of (ii) is given.

Sufficiency of (ii): Suppose (A1') and (A2') are satisfied and we want to show that there exists an $\mathbb{R}_g$-stabilizing controller C. Note that $P_{11} = V_1'T'^{-1}U_1'$ where $\{V_1',T',U_1',0\}$ is a bicoprime factorization of P_{11} in $\mathbb{R}_g(s)$, and $P_{11} = N_1'D_1'^{-1}$ is a coprime $\mathbb{R}_g$-factorization. Then by Theorem 4.3.22 of [5] $|T'|$ and $|D_1'|$ are associates, so that the zeros of $|T'|$ and $|D_1'|$ in Ω are the same.

Consider the compensated system's characteristic determinant :

$$\begin{aligned} |\underline{D}_o'| &= |T'|\ |\tilde{D}_c' - \tilde{N}_y'V_1'T'^{-1}U_1'| \\ &= |T'|\ |D_1'|^{-1}\ |\tilde{D}_c'D_1' - \tilde{N}_y'N_1'|\ . \end{aligned} \tag{45}$$

Now, from (5.8) and the discussion above it is clear that all the zeros of $|T'|$ in Ω cancel in $|T'|\ |D_1'|^{-1}$, so that any C that sets the zeros of $|\tilde{D}_c'D_1' - \tilde{N}_y'N_1'|$ in S_g will do. Since conditions (A1') and (A2') are satisfied, the subsystem $S_{P_{11}}$ of S_P relating u to y_m when w=0 and is described by the triple $\{V_1,T,U_1\}$ is stabilizable and detectable in S_g, then it can be shown that there exists an $\mathbb{R}_g$-stabilizing controller C_y of $P_{11} = V_1'T'^{-1}U_1'$, and C_r can be picked to be stable. Therefore, there exists a C that will make $\underline{D}_o'$ unimodular in $\mathbb{R}_g(s)$.• Q.E.D.

The study of $\mathbb{R}_g$-stability is simplified by writing

$$\begin{bmatrix} V_1' \\ V_2' \end{bmatrix} T'^{-1}U_1' = \begin{bmatrix} N_{11}' \\ N_{21}' \end{bmatrix} D'^{-1} \tag{46}$$

where $([N_{11}'^{\,t}\ \ N_{21}'^{\,t}]^t, D')$ is r.c. in $\mathbb{R}_g(s)$, and by setting the input signals w=0, η_1=0, and η_2=0. Note that $\{[V_1'^{\,t}\ \ V_2'^{\,t}]^t, T', U_1', 0\}$ and $\{[N_{11}'^{\,t}\ \ N_{21}'^{\,t}]^t, D', 0, 0\}$ are bicoprime factorizations in $\mathbb{R}_g(s)$ of the same transfer matrix, Theorem 6.

To show (b) consider

$$M = (I + C_y P_{11})^{-1} C_r = [D_1'(\tilde{D}_{C_y}' D_1' + \tilde{N}_{C_y}' N_1')^{-1} \tilde{D}_{C_y}'](\tilde{D}_c'^{-1} \tilde{N}_r')$$
$$= D_1' \tilde{D}_k'^{-1} G_\ell'^{-1} \tilde{N}_r'$$
$$= D_1' X \tag{50}$$

with $X = (G_\ell' \tilde{D}_k')^{-1} \tilde{N}_r' \in M(\mathbb{R}_g(s))$, since $G_\ell' \tilde{D}_k'$ is unimodular in $\mathbb{R}_g(s)$ and $\tilde{N}_r' \in M(\mathbb{R}_g(s))$. This shows that (b) is necessary too.

Sufficiency : Let $C = [C_y \quad C_r]$ satisfy (a) and (b) of Theorem (iii). If $C = \tilde{D}_c'^{-1} [\tilde{N}_y' \quad \tilde{N}_r']$ is a l.c. $\mathbb{R}_g$-factorization and G_ℓ' is a g.c.l.d. of $(\tilde{D}_c', \tilde{N}_y')$ then (48) is satisfied for some l.c. pair in $\mathbb{R}_g(s)$ $(\tilde{D}_{C_y}', \tilde{N}_{C_y}')$. Because (a) is satisfied we have that $\tilde{D}_k' = \tilde{D}_{C_y}' D' + \tilde{N}_y' N_{11}'$ is a unimodular matrix in $\mathbb{R}_g(s)$.

Premultiply the last equation by G_ℓ' to obtain

$$\tilde{D}_c' D' + \tilde{N}_y' N_{11}' = G_\ell' \tilde{D}_k' = D_o' . \tag{51}$$

In view of (47) and $\tilde{D}_k'$ being a unimodular matrix in $\mathbb{R}_g(s)$, it suffices to show that G_ℓ' is a unimodular matrix in $\mathbb{R}_g(s)$ to prove that $\Sigma(S_P, S_C)$ is $\mathbb{R}_g$-stable.

Since (b) is satisfied, $X = D_1'^{-1} M = \tilde{D}_k'^{-1} G_\ell'^{-1} \tilde{N}_r' \in M(\mathbb{R}_g(s))$. This implies that $G_\ell'^{-1} \tilde{N}_r' = \tilde{D}_k' X \in M(\mathbb{R}_g(s))$ or that $G_\ell'^{-1}$ is a unimodular matrix in $\mathbb{R}_g(s)$, since G_ℓ' and $\tilde{N}_r'$ are l.c. in $\mathbb{R}_g(s)$ (if they were not l.c., $C = \tilde{D}_c'^{-1} [\tilde{N}_y' \quad \tilde{N}_r']$ would not have been a l.c. factorization in $\mathbb{R}_g(s)$). Therefore, $D_o' = G_\ell' \tilde{D}_k'$ in (51) is a unimodular matrix in $\mathbb{R}_g(s)$.• **Q.E.D.**

The proof of Theorem 6 shows that a necessary condition for internal stability in this case is that G_ℓ' be unimodular in $M(\mathbb{R}_g(s))$. This condition implies that $|\tilde{D}_{C_y}'|$ and $|\tilde{D}_c'|$ are associates which is a generalization of controller admissibility as described in Definition 2.

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(3) $C = (I - Q'P_{11})^{-1} [Q' \quad D_1'X']$

where $Q', X' \in M(\mathbb{R}_g(s))$ are such that $[I - Q'P_{11} \quad Q']\tilde{D}_1'^{-1} \in M(\mathbb{R}_g(s))$ with $|I - Q'P_{11}| \neq 0$ and $(I - Q'P_{11})$ biproper.

(4) $C = ((I - \tilde{L}_1'N_1')D_1'^{-1})^{-1} [\tilde{L}_1' \quad X']$

where $\tilde{L}_1', X' \in M(\mathbb{R}_g(s))$ such that $(I - \tilde{L}_1'N_1')D_1'^{-1} \in M(\mathbb{R}_g(s))$ with $|I - \tilde{L}_1'N_1'| \neq 0$ and $(I - \tilde{L}_1'N_1')$ biproper.

Additional characterizations for special cases of the plant S_p and the relations between the parameters can be found as in Section 3 in a similar way.

Remarks

1) An extension of the results in Sections 2 and 3 has been made using matrix factorizations over $\mathbb{R}_g(s)$. It was shown that all the results developed in Sections 2 and 3 carry over to this more general setting and have a similar form. For example, the characterizations in Proposition 7 are similar in form to the ones given in Proposition 3 except that in this case they parameterize the $\mathbb{R}_g$-stabilizing controllers C. A disadvantage of using this method in control systems design is that there is no "tight control" of the compensated system's eigenvalues since not all of the compensated system's eigenvalues are zeros of $|\underline{D}_o'|$, the characteristic determinant of $\Sigma(S_p, S_c)$. There could be additional eigenvalues which are not zeros of $|\underline{D}_o'|$ although they would be in S_g. The presence of these additional eigenvalues could lead to unnecessarily high order controllers.

2) The theory developed in this section can be easily extended to consider matrix factorizations over other rings. In this way the results presented here can be used to analyze the stability of continuous and discrete systems,

Let

$$N_r' D'^{-1} N_\ell' = N_1' D_1'^{-1}$$

with (N_1', D_1') r.c. in $\mathbb{R}_g(s)$ and with D_1' square, nonsingular and biproper. Then $P = N_1' D_1'^{-1}$, and $|D_1'|$, $|D'|$ are associates [5, Theorem 4.3.22] meaning that the zeros of $|D_1'|$ and $|D'|$ in Ω are the same. The relation between a bicoprime factorization of P in $\mathbb{R}_g(s)$ and internal descriptions can be obtained from the following lemma:

<u>Lemma</u> [36]. The pair (N_1', D_1') with $N_1', D_1' \in M(\mathbb{R}_g(s))$ defines a right coprime factorization of P in $\mathbb{R}_g(s)$ if and only if

$$\begin{bmatrix} D_1' \\ N_1' \end{bmatrix} = \begin{bmatrix} D \\ N \end{bmatrix} \Pi$$

where $P = ND^{-1}$, a r.c. factorization in $\mathbb{R}[s]$ and $\Pi, \Pi^{-1} \in M(\mathbb{R}_g(s))$ with $D\Pi$ biproper.

Notice that a bicoprime factorization of P in $\mathbb{R}_g(s)$ can be used to described the system S:

$$D'z' = N_\ell' u$$
$$y = N_r' z'$$

since by the above Lemma the relation between the triple $\{N_r', D', N_\ell'\}$ and an irreducible differential operator description of S is given by $Dz=u$, $y=Nz$.

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