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Passivity Indices for Symmetrically Interconnected Distributed Systems

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Abstract—**In this paper, the passivity indices for both linear and nonlinear multi-agent systems with feedforward and feedback interconnections are derived. For linear systems, the passivity indices are explicitly characterized, while the passivity indices in the nonlinear case are characterized by a set of matrix inequalities. We also focus on symmetric interconnections and specialize the passivity indices results to this case. An illustrative example is also given.**

I. Introduction

Passive systems can be thought of as systems that do not generate energy, but only store or release the energy which was provided. The notion of energy here is a generalization from the traditional notion of energy in physics and is characterized by a storage function. Passivity is a special case of dissipativity, which can be applied to linear and nonlinear systems. The benefit of passivity is that when two passive systems are interconnected in parallel or in feedback, the overall system is still passive. Thus passivity is preserved when large-scales systems are combined from components of passive subsystems. Passivity indices are used to measure the excess and shortage of passivity by rendering the system passive with feedback and feedforward, and describe the performance of passive systems[1],

The concepts of passivity and dissipativity are introduced in [2], [3], [4]. There have been early studies on interconnected systems[5]. Recent papers [6], [7] study the stability conditions in interconnected passive systems, and the close relationships between output feedback passivity and L_2 stability. [8] designs a passivity-based controller for asymptotic stabilization of interconnected port-Hamiltonian systems. Passivity indices of cascade systems are measured in [9].

Symmetry, as one basic feature of shapes and graphs, has been exhibited in many real-world networks, such as the Internet and power grid, resulting from the process of tree-like or cyclic growing. Since symmetry is related to the concept of a high degree of repetitions or regularities, the study of symmetry has been appealing in many scientific areas, such as Lie groups in quantum mechanics and crystallography in chemistry.

In the classical theory of dynamical systems, symmetry has also been extensively studied. For example, to simplify the analysis and synthesis of large-scale dynamic systems, it is always of interest to reduce the dynamics of a system into smaller symmetric subsystems, which potentially simplifies control, planning or estimation tasks. When dealing with multi-agent systems with various information constraints and protocols, under certain conditions such systems can be expressed as or decomposed into interconnections of lower dimensional systems, which may lead to better understanding of system properties such as stability and controllability. Then the existence of symmetry here means that the system dynamics are invariant under transformations of coordinates.

Early research on symmetry in dynamical systems could be found in [10], [11], [12]. Symmetry in the sense of distributed systems containing multiple instances of identical subsystems are studied in [13], [14], where the controllability of the entire class of systems can be determined by reducing the model and examining a lower order member of the equivalence class. Different forms of symmetry, such as star-shaped or cyclic symmetry[15] give different stability conditions for interconnected systems. We will also show that in this paper.

The current paper is motivated by the interest of sufficient stability conditions in [15] and passivity as a binary system characterization. Passivity indices used in [1] can measure the level of passivity thus imply the degree of stability. With the distributed setup in this paper, the passivity indices for both linear and nonlinear multi-agent systems with feedforward and feedback interconnections are derived. For linear systems, the passivity indices are explicitly characterized, while the passivity indices in the nonlinear case are characterized by a set of matrix inequalities. We also focus on symmetric interconnections and specialize the passivity indices results to this case. An illustrative example is also given.

The paper is organized as follows. In Section II, we introduce some background on passivity indices and symmetry in dynamical systems. In Section III, the passivity indices for interconnected systems are derived for both linear and nonlinear systems. Section IV deals with symmetric interconnections. Section V contains simulation results, followed by concluding remarks in Section VI.

II. Preliminaries and Background

A. Passive and Dissipative Systems

Consider the nonlinear system

$$\dot{x} = f(x, u), \quad y = h(x, u) \tag{1}$$

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where $x \in \mathcal{X} \subset \mathbb{R}^n$, $y \in \mathcal{Y} \subset \mathbb{R}^m$, $u \in \mathcal{U} \subset \mathbb{R}^m$, $m \leq n$. Let $\mathbb{U}$ be an inner product space whose elements are functions $u : \mathbb{R} \to \mathbb{R}$. Also let $\mathbb{U}^m$ be the space of n-tuples over $\mathbb{U}$, with inner product $\langle y, u \rangle = \sum_{i=1}^{m} \langle y_i, u_i \rangle$. Then for any $y, u \in \mathbb{U}^m$ and any $T \in \mathbb{R}$, a truncation u_T can be defined via

$$u_T(t) = \begin{cases} u(t), & for\ t < T \\ 0, & otherwise \end{cases}$$

A truncated inner product is defined as $\langle u, v \rangle_T = \langle u_T, v_T \rangle$in an extended space $\mathbb{U}_e^m = \{u | u_t \in \mathbb{U}^m, \forall T \in \mathbb{R}\}$. The inner product over the interval $[0, T]$ for continuous time is denoted as $\langle y, u \rangle_T = \int_0^T y^\mathsf{T}(t) u(t) dt$.

A system with m inputs and m outputs may now be formally defined as a relation on $\mathbb{U}_e^m \times \mathbb{U}_e^m$, that is a set of pairs ($u \in \mathbb{U}_e^m$, $y \in \mathbb{U}_e^m$), where *u* is an input and *y* the corresponding output.

Definition 1. [2] A system is *dissipative* if there exists a positive definite storage function $V(x)$ such that for some supply rate $\omega(u, y)$ and for all $t_1 < t_2$

$$\int_{t_1}^{t_2} \omega(u, y) dt \geq V(x_2(t)) - V(x_1(t))$$

[3] A system is $(Q, S, R) - dissipative$ if the system is dissipative with respect to the supply rate

$$\begin{aligned} \omega(u, y) &= \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \\ &= y^T Q y + 2 y^T S u + u^T R u \end{aligned}$$

where $Q \in \mathbb{R}^{p \times p}$, $S \in \mathbb{R}^{p \times m}$, and $R \in \mathbb{R}^{m \times m}$ be constant matrices, with *Q* and *R* symmetric.

[1] A system is *passive* and simultaneous input feedforward passive(IFP) and output feedback passive(OFP) if the system is dissipative with respect to the supply rate

$$\omega(u, y) = (1 + \rho\nu) y^T u - \nu u^T u - \rho y^T y$$

where $\rho \in \mathbb{R}$ is the OFP index and $\nu \in \mathbb{R}$ is the IFP index.

1. When $\rho = \nu = 0$, $\omega(u, y) = y^T u$. The system is said to be passive.

2. When $\rho = 0$, $\nu \neq 0$, $\omega(u, y) = y^T u - \nu u^T u$. The system is said to be input feedforward passive(IFP).

3. When $\rho \neq 0$, $\nu = 0$, $\omega(u, y) = y^T u - \rho y^T y$. The system is said to be output feedback passive(OFP).

In the case when $\nu > 0$ or $\rho > 0$, the system is said to be input strictly passive(ISP) or output strictly passive(OSP) respectively. Obviously passivity is a special case of dissipativity. Passivity indices ρ and ν can be generalized to be $\rho(y)$ and $\nu(u)$ if the feedforward and the feedback is not static.

B. Symmetry in Dynamical Systems

Intuitively, symmetry in dynamical systems means that the system dynamics are invariant under transformations. We formalize this in terms of the concepts of diffeomorphism. Consider the nonlinear system 1 and the local group of transformations on $\mathcal{X} \times \mathcal{U}$ defined by $(X, U) = (\varphi_g(x), \psi_g(u))$ where $g \in G$, φ_g acting on $\mathcal{X}$ and ψ_g acting on $\mathcal{U}$ are local diffeomorphisms.

Definition 2. [16]The system 1 has a *G-symmetry* or is *G-invariant* if $f(\varphi_g(x), \psi_g(u)) = D\varphi_g(x) \cdot f(x, u)$ for all g, x, u.

$$\begin{tikzcd} T\mathcal{X} \arrow[r, "D\varphi_g"] & T\mathcal{X} \\ \mathcal{X} \times \mathcal{U} \arrow[u, "f"] \arrow[r, "\varphi_g \times \psi_g"] \arrow[d, "h"] & \mathcal{X} \times \mathcal{U} \arrow[u, "f"] \arrow[d, "h"] \\ \mathcal{Y} \arrow[r, "\varrho_g"] & \mathcal{Y} \end{tikzcd}$$

Figure 1. Symmetry in geometric control

Definition 3. [16]The output $y = h(x, u)$ is *G-equivalent* if there exists a transformation group$(\rho_g)_{g \in G}$ on $\mathcal{Y}$ such that $f(\varphi_g(x), \psi_g(u)) = \rho_g h(x, u)$ for all g, x, u.

With $(X, U) = (\varphi_g(x), \psi_g(u))$ and $Y = \rho_g(y)$, the properties can be rewritten as $\dot{X} = f(X, U)$, $Y = h(X, U)$. It means the system remains unchanged under the transformation. The two previous definitions can be illustrated by the commutative diagram in Fig. 1.

C. Symmetric Distributed System

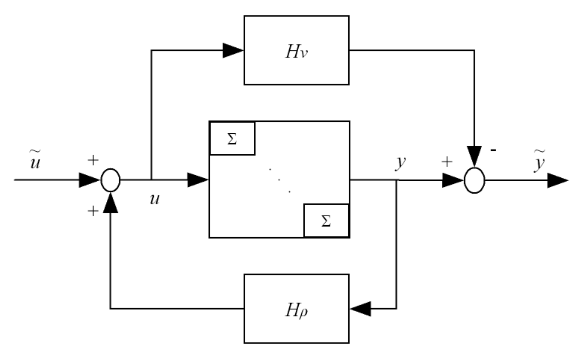

Figure 2. Interconnected multi-agent system

Consider a multi-agent dynamical system consisting of subsystems Σ on the diagonal as in Fig. 2. Σ is described by

$$\begin{aligned} \dot{x}_i &= f(x_i, u_i) \\ y_i &= h(x_i, u_i) \end{aligned} \tag{2}$$

with storage function $V_i(x)$, supply rate $\omega_i(u_i, y_i)$, where $i = 1, \ldots, m$. H_ρ and H_ν are constant feedback and feedforward interconnection matrices, the system inputs and outputs are stacked $u = [u_1^T, u_2^T, \ldots, u_m^T]^T$, $y = [y_1^T, y_2^T, \ldots, y_m^T]^T$. $(\tilde{u}, \tilde{y})$ is the input and output for the interconnected system. To describe the relationships between multiple agents, we

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have a digraph G to model the interaction topology for a network control system [17].

Here, for a symmetric group S_p consisted of p subsystems, we consider three types of symmetries, namely star-shaped symmetry, cyclic symmetry and chain symmetry. Intuitively, in a star-shaped symmetric group, subsystems do not have interconnections with each other. In cyclic symmetric group, subsystems contribute to a close related automorphism group. In chain symmetry, each subsystem has interconnections with its two neighbors, except the leading and ending agents. See Fig. 3, 4, 5.

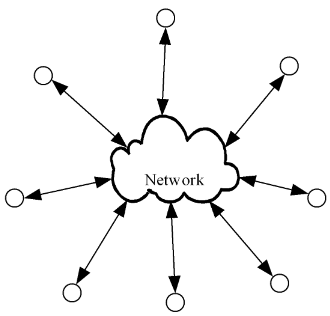

Fig.3 Star-shaped symmetry

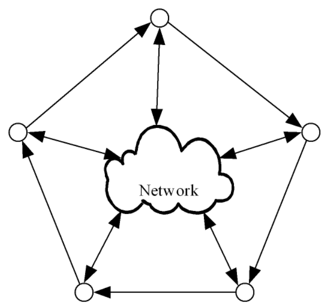

Fig.4 Cyclic symmetry

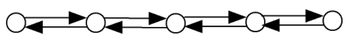

Fig.5 Chain symmetry

III. Passivity Indices for Interconnected Systems

Let subsystem Σ be a passive system with OFP index ρ and IFP index ν. It can be easily shown that the stacked system is also passive with indices (ρ, ν). If we define the storage function as $V(x) = \sum_{i=1}^{m} V_i(x)$, then $\dot{V}(x)$ is bounded because

$$\dot{V}(x) = \sum_{i=1}^{m} \dot{V}_i(x) \leq \sum_{i=1}^{m} \omega_i(u_i, y_i)$$

$$= (1+\rho\nu) \sum_{i=1}^{m} y_i^T u_i - \nu \sum_{i=1}^{m} u_i^T u_i - \rho \sum_{i=1}^{m} y_i^T y_i$$

$$= (1+\rho\nu) y^T u - \nu u^T u - \rho y^T y \triangleq \omega(u, y)$$

So $\omega(u, y)$ is the supply rate for the multi-agent system. With feedback interconnection matrix H_ρ and feedforward interconnection matrix H_ν (see Fig. 2), the new input and output is given by

$$\begin{cases} \tilde{y} = y - H_\nu u \\ \tilde{u} = u - H_\rho y \end{cases} \tag{3}$$

Next we are deriving the passivity indices of the interconnected distributed system with respect to input-output pair $(\tilde{u}, \tilde{y})$.

A. Linear Passive Systems

Theorem 1 For a minimum phase linear system $G(s)$ consisting of scalar passive subsystems, with only the output feedback loop, i.e. $H_\rho \neq 0$, $H_\nu = 0$, the output feedback passivity index is given by

$$\tilde{\rho} = \rho - \bar{\lambda}\left(\frac{H_\rho + H_\rho^T}{2}\right) \tag{4}$$

Proof: According to [1], the OFP index is defined as

$$\rho = \rho\left(G(s)\right) = \frac{1}{2} \min_{\omega \in \mathbb{R}} \underline{\lambda}\left(G^{-1}(j\omega) + \left[G^{-1}(j\omega)\right]^*\right) \tag{5}$$

where $\underline{\lambda}(\cdot)$ is the minimum eigenvalue of a matrix. For the closed loop system which is also minimum phase

$$G_{cl}(s) = (I - GH_\rho)^{-1} G = (G^{-1} - H_\rho)^{-1}$$

the new OFP index is given by

$$\tilde{\rho} = \rho\left(G_{cl}(s)\right) = \frac{1}{2} \min_{\omega \in \mathbb{R}} \underline{\lambda}\left(G_{cl}^{-1}(j\omega) + \left[G_{cl}^{-1}(j\omega)\right]^*\right)$$

$$= \frac{1}{2} \min_{\omega \in \mathbb{R}} \underline{\lambda}\left(G^{-1}(j\omega) + \left[G^{-1}(j\omega)\right]^* - H_\rho - H_\rho^*\right)$$

It is known [18] that if two matrices A and B commute, so that $AB = BA$, $A, B \in \mathbb{R}^n$, then the two sets of eigenvalues $\{\lambda_i(A+B)\}$ and $\{\lambda_i(A) + \lambda_i(B)\}$ are equal; also $\underline{\lambda}(A+B) = \underline{\lambda}(A) + \underline{\lambda}(B)$. Since all scalar subsystems have the same dynamic, $G^{-1}(j\omega) + \left[G^{-1}(j\omega)\right]^*$ is a diagonal matrix with identical diagonal entries, which commutes with any matrix. Thus

$$\tilde{\rho} = \frac{1}{2} \min_{\omega \in \mathbb{R}} \underline{\lambda}\left(G^{-1}(j\omega) + \left[G^{-1}(j\omega)\right]^*\right) + \frac{1}{2}\underline{\lambda}\left(-H_\rho - H_\rho^*\right)$$

$$= \rho - \bar{\lambda}\left(\frac{H_\rho + H_\rho^T}{2}\right)$$

■

When $G(s)$ is not diagonal, we can derive the boundary of $\tilde{\rho}$. From Fact 5.12.2[18], if A and B are Hermitian matrices, $\underline{\lambda}(A) + \underline{\lambda}(-B) \leq \underline{\lambda}(A - B) \leq \underline{\lambda}(A) - \underline{\lambda}(B)$. Since here $G^{-1}(j\omega) + \left[G^{-1}(j\omega)\right]^*$ and $-H_\rho - H_\rho^*$ are both Hermitian matrices,

$$\tilde{\rho} \leq \frac{1}{2} \min_{\omega \in \mathbb{R}} \left(\underline{\lambda}\left(G^{-1}(j\omega) + \left[G^{-1}(j\omega)\right]^*\right) - \underline{\lambda}\left(H_\rho + H_\rho^*\right)\right)$$

$$= \rho - \underline{\lambda}\left(\frac{H_\rho + H_\rho^T}{2}\right)$$

$$\tilde{\rho} \geq \frac{1}{2} \min_{\omega \in \mathbb{R}} \left(\underline{\lambda}\left(G^{-1}(j\omega) + \left[G^{-1}(j\omega)\right]^*\right)\right.$$

$$\left.+ \underline{\lambda}\left(-H_\rho - H_\rho^*\right)\right) = \rho - \bar{\lambda}\left(\frac{H_\rho + H_\rho^T}{2}\right)$$

i.e. for any G and H_ρ,

$$\rho - \bar{\lambda}\left(\frac{H_\rho + H_\rho^T}{2}\right) \leq \tilde{\rho} \leq \rho - \underline{\lambda}\left(\frac{H_\rho + H_\rho^T}{2}\right)$$

Theorem 2 For a linear stable system $G(s)$ consisting of scalar passive subsystems, with only the input feedforward loop, i.e. $H_\rho = 0$, $H_\nu \neq 0$. The input feedforward passivity index is

$$\tilde{\nu} = \nu - \overline{\lambda}\left(\frac{H_\nu + H_\nu^T}{2}\right) \tag{6}$$

Proof: According to [1], for a linear stable system $G(s)$, the IFP index can be calculated as

$$\nu = \nu\left(G(s)\right) = \frac{1}{2}\min_{\omega\in\mathbb{R}} \underline{\lambda}\left(G(j\omega) + G^*(j\omega)\right)$$

for the closed loop system which is also stable

$$G_{cl}(s) = G - H_\nu$$

the new IFP index is

$$\tilde{\nu} = \frac{1}{2}\min_{\omega\in\mathbb{R}} \underline{\lambda}\left(G_{cl}(j\omega) + G_{cl}^*(j\omega)\right)$$

$$= \frac{1}{2}\min_{\omega\in\mathbb{R}} \underline{\lambda}\left(G(j\omega) + G^*(j\omega) - H_\nu - H_\nu^*\right)$$

$$= \nu - \overline{\lambda}\left(\frac{H_\nu + H_\nu^T}{2}\right)$$

■

A similar approach can be found in [19] as a passivation method. Also, if G is not diagonal, the following inequality holds

$$\nu - \bar{\lambda}\left(\frac{H_\nu + H_\nu^T}{2}\right) \leq \tilde{\nu} \leq \nu - \underline{\lambda}\left(\frac{H_\nu + H_\nu^T}{2}\right)$$

B. Nonlinear Systems

Theorem 3 Consider a nonlinear multi-agent system with the input feedforward interconnection matrix H_ν and output feedback interconnection matrix H_ρ, where $I - H_\nu H_\rho$ and $I - H_\rho H_\nu$ are nonsingular. If the subsystems have the same passivity indices (ρ, ν), then the passivity indices $(\tilde{\rho}, \tilde{\nu})$ for the whole system satisfy

$$\tilde{P} - \tilde{H}^T P \tilde{H} \geq 0 \tag{7}$$

where

$$P = \begin{bmatrix} -\rho I & \frac{1+\rho\nu}{2} I + S \\ \frac{1+\rho\nu}{2} I - S^T & -\nu I \end{bmatrix}$$

$$\tilde{P} = \begin{bmatrix} -\tilde{\rho} I & \frac{1+\tilde{\rho}\tilde{\nu}}{2} I + \tilde{S} \\ \frac{1+\tilde{\rho}\tilde{\nu}}{2} I - \tilde{S}^T & -\tilde{\nu} I \end{bmatrix}$$

$$\tilde{H} = \begin{bmatrix} (I - H_\nu H_\rho)^{-1} & (I - H_\nu H_\rho)^{-1} H_\nu \\ (I - H_\rho H_\nu)^{-1} H_\rho & (I - H_\rho H_\nu)^{-1} \end{bmatrix}$$

S and $\tilde{S}$ can be any matrix.

Proof: Write (3) in matrix form

$$\begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix} = H \begin{bmatrix} y \\ u \end{bmatrix}$$

where

$$H = \begin{bmatrix} I & -H_\nu \\ -H_\rho & I \end{bmatrix}$$

Let $\tilde{H} = H^{-1}$, then

$$\begin{bmatrix} y \\ u \end{bmatrix} = \tilde{H} \begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix}$$

where $\tilde{H}$ can be calculated as

$$\tilde{H} = H^{-1} = \begin{bmatrix} (I - H_\nu H_\rho)^{-1} & (I - H_\nu H_\rho)^{-1} H_\nu \\ (I - H_\rho H_\nu)^{-1} H_\rho & (I - H_\rho H_\nu)^{-1} \end{bmatrix}$$

From the definition of passivity index [1]

$$\dot{V}(x) \leq (1 + \rho\nu) u^T y - \nu u^T u - \rho y^T y = \begin{bmatrix} y \\ u \end{bmatrix}^T P \begin{bmatrix} y \\ u \end{bmatrix}$$

where S can be any matrix. Then

$$\dot{V}(x) \leq \begin{bmatrix} y \\ u \end{bmatrix}^T P \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix}^T \tilde{H}^T P \tilde{H} \begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix}$$

For the new passivity index pair $(\tilde{\rho}, \tilde{\nu})$, we are requiring a more loose boundary for $\dot{V}(x)$

$$\dot{V}(x) \leq \tilde{u}^T \tilde{y} - \tilde{\nu} \tilde{u}^T \tilde{u} - \tilde{\rho} \tilde{y}^T \tilde{y} = \begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix}^T \tilde{P} \begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix}$$

where $\tilde{\rho}$ and $\tilde{\nu}$ are the largest value to make the inequality holds. Thus we are requiring

$$\tilde{P} - \tilde{H}^T P \tilde{H} \geq 0$$

■

Note that in the nonlinear case, passive subsystems are no longer required to be scalar systems. Normally there is no analytic way to derive $\tilde{\rho}$ and $\tilde{\nu}$ separately from the matrix inequality (7), but there are special cases:

1) OFP case

Only consider a feedback loop and ρ. In this case, $H_\nu = 0$, $\nu = \tilde{\nu} = 0$, $H = \begin{bmatrix} I & 0 \\ -H_\rho & I \end{bmatrix}$, $\tilde{H} = H^{-1} = \begin{bmatrix} I & 0 \\ H_\rho & I \end{bmatrix}$. For any S, let $\tilde{S} = S$, then

$$\tilde{P} - \tilde{H}^T P \tilde{H} = \begin{bmatrix} -\tilde{\rho} I & \frac{1}{2} I + S \\ \frac{1}{2} I - S^T & 0 \end{bmatrix}$$

$$- \begin{bmatrix} I & H_\rho^T \\ 0 & I \end{bmatrix} \begin{bmatrix} -\rho I & \frac{1}{2} I + S \\ \frac{1}{2} I - S^T & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ H_\rho & I \end{bmatrix}$$

$$= \begin{bmatrix} \rho I - \tilde{\rho} I - \frac{1}{2}(H_\rho + H_\rho^T) - S H_\rho + H_\rho^T S^T & 0 \\ 0 & 0 \end{bmatrix}$$

Since $-S H_\rho + H_\rho^T S^T$ is a skew symmetric matrix, $x^T(-S H_\rho + H_\rho^T S^T) x = 0$ for all $x \in \mathbb{R}^n$, which means we are requiring

$$\rho I - \tilde{\rho} I - \frac{1}{2}(H_\rho + H_\rho^T) \geq 0$$

or

$$\tilde{\rho} \leq \rho - \overline{\lambda}\left(\frac{H_\rho + H_\rho^T}{2}\right)$$

Because passivity index is defined as the maximal possible value, the new OFP index $\tilde{\rho}$ of the interconnected distributed system with respect to input-output pair $(\tilde{u}, \tilde{y})$ is

$$\tilde{\rho} = \tilde{\rho}_{\max} = \rho - \bar{\lambda}\left(\frac{H_\rho + H_\rho^T}{2}\right) \quad (8)$$

which is the same as (4). If we let $\rho > 0$, $E = -H_\rho$, $S = 0$, then $\tilde{\rho}$ can be positive if E is diagonally stable with $D = \frac{1}{2}I$, that is, $DE + E^T D < 0$. Since output passivity implies L_2 stability, this result shows that a family of L_2 stable systems with a feedback connection can also be L_2 stable, given E diagonally stable. This conclusion can also be found in [6] and [7], where each subsystem has a distinct passivity index γ_i instead of a uniformed passivity index ρ as here.

2) IFP case

Only consider feedforward loop and ν. In this case, $H_\rho = 0$, $\rho = \tilde{\rho} = 0$, $H = \begin{bmatrix} I & -H_\nu \\ 0 & I \end{bmatrix}$, $\tilde{H} = H^{-1} = \begin{bmatrix} I & H_\nu \\ 0 & I \end{bmatrix}$. For any S, let $\tilde{S} = S$, then

$$\tilde{P} - \tilde{H}^T P \tilde{H} = \begin{bmatrix} 0 & \frac{1}{2}I + S \\ \frac{1}{2}I - S^T & -\tilde{\nu}I \end{bmatrix}$$

$$- \begin{bmatrix} I & 0 \\ H_\nu^T & I \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2}I + S \\ \frac{1}{2}I - S^T & -\nu I \end{bmatrix} \begin{bmatrix} I & H_\nu \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \nu I - \tilde{\nu}I - \frac{1}{2}(H_\nu + H_\nu^T) - H_\nu^T S + S^T H_\nu \end{bmatrix}$$

where $H_\nu^T S - S^T H_\nu$ is a skew symmetric matrix. Similarly, we are requiring

$$\nu I - \tilde{\nu} I - \frac{1}{2}(H_\nu + H_\nu^T) \geq 0$$

i.e.

$$\tilde{\nu} \leq \nu - \bar{\lambda}\left(\frac{H_\nu + H_\nu^T}{2}\right)$$

thus the new IFP indices of the interconnected distributed system with respect to input-output pair $(\tilde{u}, \tilde{y})$ is

$$\tilde{\nu} = \tilde{\nu}_{\max} = \nu - \bar{\lambda}\left(\frac{H_\nu + H_\nu^T}{2}\right) \quad (9)$$

which is the same as (6).

IV. Symmetric Interconnections

Symmetry existing in the structure of interconnection may reduce the complexity of system analysis. For instance, if the symmetry is linear symmetry, i.e. a diffeomorphism $\varphi_g(x)$ is on a sub-group of general linear group $GL(n, \mathbb{R}^n)$, then the properties of matrices can be applied instantly.

A permutation of a set $X = \{1, \ldots, p\}$ is a one-to-one mapping of X onto itself. Such a permutation is written,

$$\begin{pmatrix} 1 & 2 & \cdots & p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix}$$

which represents that 1 is mapped to k_1, 2 is mapped to k_2, etc. Such permutation can be represented by $\varphi_g(x) = Px$, where P is a orthogonal matrix with only one entry equal to 1 in each row and column, and other entries 0. If all subsystems have the same dynamic and the interconnection matrix is invariant under the permutation, then a symmetry exists in the multi-agent system.

Since the measurement of passivity indices only relies on the eigenvalues of interconnection matrices, we can examine the matrix properties and give a more explicit result.

A. Cyclic Symmetry

Cyclic interconnection matrix $H = circ([v_0\, v_1 \cdots v_{m-1}])$ is invariant under cyclic permutation. We can calculate the eigenvalue of $\frac{H+H^T}{2}$ to derive explicit values for the passivity indices.

$$\lambda(H) = \sum_{j=0}^{m-1} v_j \lambda_i^j \,, i = 0, 1, \ldots, m-1$$

where

$$\lambda_i = e^{\frac{2\pi i}{m}}$$

The sum of two cyclic matrices is still a cyclic matrix.

$$\frac{H + H^T}{2} = circ([v_0\, \frac{v_1 + v_{m-1}}{2} \cdots \frac{v_{m-1} + v_1}{2}])$$

$$\lambda\left(\frac{H + H^T}{2}\right) = v_0 + \sum_{j=0}^{m-1} \frac{v_j + v_{m-j}}{2} e^{\frac{2\pi i j}{m}} \,, i = 0, \ldots, m-1$$

Then we can derive (8) and (9) more explicitly.

B. Star-shaped Symmetry

Intuitively, in a star-shaped symmetric group, subsystems do not have interconnections with each other, but commute with the center agent with different connection weights.

$$H = \begin{bmatrix} h & b_1 & \ldots & b_m \\ c_1 & h & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_m & 0 & \ldots & h \end{bmatrix}$$

$$\frac{H + H^T}{2} = \begin{bmatrix} h & \frac{b_1 + c_1}{2} & \ldots & \frac{b_m + c_m}{2} \\ \frac{b_1 + c_1}{2} & h & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b_m + c_m}{2} & 0 & \ldots & h \end{bmatrix}$$

Then

$$\lambda\left(\frac{H + H^T}{2}\right) = h \pm \frac{1}{2}\sqrt{\sum_{i=1}^{m} (b_i + c_i)^2},\ 0 \quad (10)$$

which will be shown as an example in Section V.

C. Chain Symmetry

In chain symmetry, each subsystem has interconnections with its two neighbors, except the leading and ending agents.

$$H = Tr(a,b,c) = \begin{bmatrix} b & c & & 0 \\ a & b & \ddots & \\ & \ddots & \ddots & c \\ 0 & & a & b \end{bmatrix}$$

$$\lambda(H) = b + 2c\sqrt{\frac{a}{c}}\cos\frac{i\pi}{m+1}\ , i = 1,\ldots,m$$

$$\frac{H+H^T}{2} = Tr(\frac{a+c}{2}, b, \frac{a+c}{2})$$

$$\lambda(\frac{H+H^T}{2}) = b + (a+c)\cos\frac{i\pi}{m+1}\ , i = 1,\ldots,m$$

V. Example

Given a minimun phase linear multi-agent system G with only output feedback interconnection matrix H_ρ,

$$G(s) = \begin{bmatrix} \frac{s+2}{s+4} & 0 & 0 & 0 \\ 0 & \frac{s+2}{s+4} & 0 & 0 \\ 0 & 0 & \frac{s+2}{s+4} & 0 \\ 0 & 0 & 0 & \frac{s+2}{s+4} \end{bmatrix}$$

$$H_\rho = \begin{bmatrix} 1 & -1 & 4 & 3 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

From (5) and (10)

$$\bar{\lambda}(\frac{H_\rho + H_\rho^T}{2}) = 4.7417$$

From Nyquist plots of $G^{-1}(j\omega)$ and $G_{cl}^{-1}(j\omega)$,

$$\rho = \frac{1}{2}\min_{\omega\in\mathbb{R}} \underline{\lambda}\left(G^{-1}(j\omega) + \left[G^{-1}(j\omega)\right]^*\right) = 1$$

$$\tilde{\rho} = \frac{1}{2}\min_{\omega\in\mathbb{R}} \underline{\lambda}\left(G_{cl}^{-1}(j\omega) + \left[G_{cl}^{-1}(j\omega)\right]^*\right) = -3.7417$$

thus it can be verified that

$$\tilde{\rho} = \rho - \bar{\lambda}(\frac{H_\rho + H_\rho^T}{2})$$

Here $\tilde{\rho} = -3.7417 < 0$ means the multi-agent system G is short of passivity after the feedback loop is closed. But if

$$H_\rho = \begin{bmatrix} -3 & -1 & 4 & 3 \\ 3 & -3 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ 1 & 0 & 0 & -3 \end{bmatrix}$$

then passivity is preserved because

$$\tilde{\rho} = \rho - \bar{\lambda}(\frac{H_\rho + H_\rho^T}{2}) = 1 - 0.7417 = 0.2583 > 0$$

VI. Conclusions

In this paper, the passivity indices for both linear and nonlinear multi-agent systems with feedforward and feedback interconnections are derived. For linear systems, the passivity indices are explicitly characterized, while the passivity indices in the nonlinear case are characterized by a set of matrix inequalities. We also focus on symmetric interconnections and specialize the passivity indices results to this case. An illustrative example is also given.

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