Output Synchronization of Multi-Agent Systems with Event-Driven Communication: Communication Delay and Signal Quantization

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Abstract

In this paper, we study the output synchronization problem of multi-agent systems with event-driven communication, in which the data transmissions among neighboring agents are event-based rather than pre-scheduled periodically. We propose a set-up for the coupled agents to achieve output synchronization with event-driven communication in the presence of constant communication delays by using scattering transformation. Thus, whenever the agent satisfies its triggering condition, a scattering variable which contains the current output information of the agent will be sent to its corresponding neighbors, and the neighbors will extract reference information from its received scattering variables for its own control action update. Quantization effects on output synchronization with event-driven communication have also been studied. The result presented in the current paper is an important extension of applying event-driven communication to control of multi-agent systems, especially when it is difficult to derive a common upper bound on the admissible network induced delays based on the event-triggering condition or when the network induced delays between coupled agents are larger than the inter-event time implicitly determined by the event triggering condition.

Index Terms

output synchronization, event-driven communication, quantization effects, communication delay, passivity, graph theory, control of multi-agent systems

I. Introduction

Recently, several researchers have suggested the idea of event-based control as a promising technique to reduce communication and computation load for the purpose of control in many control applications. In a typical event-based implementation, the control signals are kept constant until the violation of a "event triggering condition" on certain signals triggers the re-computation of the control actions. The possibility of reducing the number of re-computations, and thus of transmissions, while guaranteeing desired levels of performance makes event-based control very appealing in networked control systems(NCSs). A comparison of time-driven and event-driven control for stochastic systems favoring the latter can be found in [18]; a deterministic event-triggered strategy was introduced in [19]; similar results on deterministic self-triggered feedback control have been reported in [20], [21], [22]; an event-triggered real-time scheduling approach for stabilization of passive and output feedback passive (OFP) systems has been proposed in [25].

On the other hand, control of multi-agent systems is facilitated by recent technological advances on computing and communication resources. Several results concerning multi-agent cooperative control have appeared in recent literature involving agreement or consensus algorithms [5], [6], [7] and [10], formation control and group coordination [8], [9], and distributed estimation [11], to name a few. Important aspects in the implementation of distributed algorithms for control of multi-agent systems are communication transmissions and actuation update schemes. Most of the work in the literature assumes that the execution of the distributed controller and the scheduling of the communication transmission are implemented in a conservative way, where a tight bound is selected as the maximal allowable inter-transmission time to guarantee the performance of the interconnected systems for all possible operating points. This traditional methodology may lead to inefficient implementation of distributed control algorithms in terms of processor usage or available communication bandwidth. Thus, event-driven communication in control of multi-agent systems is of interest because of the potential of reducing communication load and implementation cost. While most of the work on event-triggered control focus on sensor-actuator NCSs, there is not many work on applying event-triggered control in control of multi-agent systems, although a recent work on event-triggered control for consensus problem has been reported in [12].

There are two important problems among others needed to be addressed by applying event-driven communication in control of multi-agent systems. First, triggering condition which assures that the coupled agents to achieve a mutual objective (as requested by many applications for control of multi-agent systems) has to be derived. The implementation of the triggering condition should only requires the local information of the corresponding agent and is easy to check. The second problem is that the proposed event-driven communication strategy has to be embedded with some sort of "robustness" with respect to the imperfections of the communication networks. We try to address event-driven communication for control of multi-agent systems by focusing on the two problems just mentioned.

In this paper, output synchronization problem of multi-agent systems with event-driven communication has been studied. We assume all the agents in the network are lossless and we propose a set-up to achieve output synchronization of coupled agents with event-driven communication in the presence of arbitrary constant network induced delays. Triggering condition to achieve output synchronization is derived based on the rectified scattering transformation (see [14], [15], [16] for details on scattering transformation) applied in our proposed set-up. Whenever the agent satisfies its triggering condition, a scattering variable which contains the current output information of the agent will be sent to its corresponding neighbors, and the neighbors will extract reference information from its received scattering variables for its own control action update. The proposed set-up in the current paper is an important extension of applying event-driven communication to control of multi-agent systems, especially when it is difficult to derive a common upper bound on the admissible network induced delays based on the triggering condition or when the network induced delays between coupled agents are larger than the inter-event time implicitly determined by the event-triggering condition.

Quantization effects on output synchronization of multi-agent system with event-driven communication has also been investigated in this paper. We first study the quantization effects when there are no data transmission delays in the networks. Event-driven consensus problem with quantization is singled out as a case study. Then we further study the quantization effects when there are arbitrary constant data transmission delays in the networks and we have shown that with the event-driven communication set-up applied in this paper, output synchronization error of the studied multi-agent system is essentially bounded by the quantization errors of the signals transmitted in the networks. The rest of this paper is organized as follows: we first introduce some background on passive system and graph theory in section II; the problem is stated in section III; we first derive the triggering condition for output synchronization without considering network induced delays in section IV, and we also obtain an analysis of the inter-event time based on the triggering condition, which is provided in section V; in section VI, the continuous consensus problem is re-formulated with event-driven communication as a case study; the results for achieving output synchronization with event-driven communication in the presence of constant network induced delays are presented in section VII; quantization effect on output synchronization is studied in section VIII and section X; finally, the conclusion in provided in section XI.

II. Background Material

A. Passivity

Consider the following dynamic system which can be used to describe both linear and nonlinear systems:

$$H: \begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases} \tag{1}$$

where $x \in \mathbb{X} \subset \mathbb{R}^n$, $u \in \mathbb{U} \subset \mathbb{R}^m$ and $y \in \mathbb{Y} \subset \mathbb{R}^m$ are the state, input and output variables, respectively, and $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{Y}$ are the state, input and output spaces, respectively. The representation $\phi(t, t_0, x_0, u)$ is used to denote the state at time t reached from the initial state x_0 at t_0.

Definition 1(supply rate)[4]: The supply rate $\omega(t) = \omega(u(t), y(t))$ is a real valued function defined on $\mathbb{U} \times \mathbb{Y}$, such that for any $u(t) \in \mathbb{U}$ and $x_0 \in \mathbb{X}$ and $y(t) = h(\phi(t, t_0, x_0, u))$, $\omega(t)$ satisfies

$$\int_{t_0}^{t_1} |\omega(\tau)| d\tau < \infty. \tag{2}$$

Definition 2(Dissipative System)[4]: System H with supply rate $\omega(t)$ is said to be dissipative if there exists a nonnegative real function $V(x) : \mathbb{X} \to \mathbb{R}^+$ ($\mathbb{R}^+$ is the set of nonnegative real numbers), called the storage function, such that, for all $t_1 \geq t_0 \geq 0$, $x_0 \in \mathbb{X}$ and $u \in \mathbb{U}$,

$$V(x_1) - V(x_0) \leq \int_{t_0}^{t_1} \omega(\tau) d\tau \tag{3}$$

where $x_1 = \phi(t_1, t_0, x_0, u)$.

Definition 3(Passive System)[4]: System H is said to be **passive** if there exists a storage function $V(x)$ such that

$$V(x_1) - V(x_0) \leq \int_{t_0}^{t_1} u(\tau)^T y(\tau) d\tau, \tag{4}$$

if $V(x)$ is $\mathcal{C}^1$, then we have

$$\dot{V}(x) \leq u(t)^T y(t), \ \forall t \geq 0. \tag{5}$$

One can see that passive system is a special case of dissipative system with supply rate $\omega(t) = u(t)^T y(t)$. If $V(x_1) - V(x_0) = \int_{t_0}^{t_1} u(\tau)^T y(\tau) d\tau$, then we say the system is *lossless*.

B. Graph Theory

Information exchange between agents can be modeled as a graph. In the following, we give some basic terminologies and definitions from graph theory [23].

We consider finite weighted directed graphs $G := (V, E)$ with no self-loops and *adjacency matrix A*, where V denotes the set of all vertices, E denotes the set of all edges, and $A := [a_{ij}]$ with $a_{ij} > 0$ if there is a directed edge from vertex i into vertex j, and $a_{ij} = 0$ otherwise. The *in-degree* and *out-degree* of vertex k are given by $d_i(k) = \sum_j a_{jk}$ and $d_o(k) = \sum_j a_{kj}$ respectively.

The *Laplacian* matrix of a directed graph is defined as $L = D - A$, where D is the diagonal matrix of vertex out-degrees.

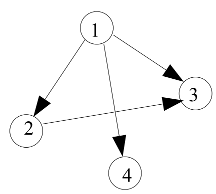

Fig. 1: example on graph Laplacian

Example 1: Consider a graph as shown in Fig.1, where we define

$$a_{ij} = \begin{cases} a, & \text{if vertex } i \text{ sends information to vertex } j; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

and $a > 0$ represents the coupling strength between coupled agents. Then we can get the *adjacency matrix* A and the *degree matrix* D

$$A = \begin{bmatrix} 0 & a & a & a \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{7}$$

and the graph Laplacian is given by

$$L = \begin{bmatrix} 3a & -a & -a & -a \\ 0 & a & -a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

Definition 4(strongly connected graph)[23]: A directed graph is strongly connected if for any pair of distinct vertices ν_i and ν_j, there is a directed path from ν_i to ν_j.

Definition 5(balanced graph)[23]: A vertex is balanced if its in-degree is equal to its out-degree. A directed graph is balanced if every vertex is balanced.
Definition 6(weakly connected)[10]: A *path* of length r in a directed graph is a sequence $\nu_0, \ldots, \nu_r$ of $r+1$ distinct vertices such that for every $i \in \{0, \ldots, r-1\}$, (ν_i, ν_{i+1}) is an edge. A *weak path* is a sequence $\nu_0, \ldots, \nu_r$ of $r+1$ distinct vertices such that for each $i \in \{0, \ldots, r-1\}$, either (ν_i, ν_{i+1}) or (ν_{i+1}, ν_i) is an edge. A directed graph is *weakly connected* if any two vertices can be joined by a weak path.
Lemma 1 [23]: Let G be a directed graph and suppose it is balanced. Then G is strongly connected if and only if it is weakly connected.

III. Problem Statement And Assumptions

The evolution of multi-agent NCSs depends fundamentally on their interconnection topology. We list below several assumptions regarding the interconnection topology that we will make in the sequel. The specific assumption(s) used will be made clear in the statement of a given result.
A1. The topology of the underlying communication graph is weakly connected point-wise in time and form a directed balanced graph with respect to information exchange.
A2. The topology of the underlying communication graph is weakly connected point-wise in time, bidirectional and balanced.
Definition 7(Output Synchronization)[10]: Suppose we have a network of N agents, the agents are said to output synchronize if

$$y_j(t) - y_i(t) \to 0 \text{ as } t \to \infty, \ \forall i, j = 1, \ldots, N.$$

It has been shown in [10] that for a group of N networked passive systems, suppose that the agents are coupled together using the control

$$u_i(t) = \sum_{j \in \mathcal{N}_i} K\big[y_j(t) - y_i(t)\big], \quad i = 1, 2, \ldots, N \tag{9}$$

where K is a positive constant and $\mathcal{N}_i$ denotes the set of agents transmitting their outputs to the i^{th} agent. Then under assumption **A1**, the networked passive systems are globally stable and the agents output synchronize.

The output synchronization results in [10] require that each agent communicates with its neighboring agents continuously. In this paper, we reformulate the above control problem and take event-driven communication into consideration. Consider a networked control system which consists of N lossless agents each denoted by H_i, for $i = 1, 2, \ldots, N$. Agent H_i transmits its current output information to its corresponding neighbors $\mathcal{Z}_i$ ($\mathcal{Z}_i$ denotes the set of agents receiving output information from H_i) whenever its event triggering condition is satisfied. The time sequence of data transmission (event time) for H_i is denoted by $\{t_{k_i}\}$, for $k = 0, 1, 2, \ldots$. We summarize the problem we try to solve in this paper as follows: What is the triggering condition and the control law for the coupled agents to achieve output synchronization with event-driven communication? How frequent the data transmission is under the triggering condition? Moreover(which should be more interesting), when the data transmission between each coupled agents is subject to communication delay, and the delay could be much larger than the inter-event time obtained based on the triggering condition for the no delay case, can we still achieve output synchronization with event-driven communication? Further more, if we also consider quantization of the transmitted signals in the networks, what is the quantization effect on output synchronization of the multi-agent system with event-driven communication?

IV. Triggering Condition for Output Synchronization Without Communication Delays

Assume that the control input to agent H_i is given by

$$u_i(t) = \sum_{j \in \mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i}), \text{ for } t \in [t_{k_i}, t_{k_i+1}), \quad i = 1, 2, \ldots, N \tag{10}$$

where a is a positive constant represents the coupling between agent H_j and agent H_i as defined in the adjacency matrix of the underlying communication graph; $\widehat{y}_{k_j}$ represents the latest output information received by H_i from H_j by the time t, for $j \in \mathcal{N}_i$; $\widehat{y}_{k_i} = y_i(t_{k_i})$ represents the latest transmitted output information of H_i at the latest event time t_{k_i}.

We first assume there is no data transmission delay in the communication network and the topology of the underlying communication graph is fixed. The triggering condition for output synchronization is shown in the theorem below.

Theorem 1. Consider a network of N lossless agents with control (10). Under assumption **A1**, if each agent H_i transmits its current output information to its neighbors whenever the following triggering condition is satisfied

$$\|e_i(t)\|_2 = \frac{\delta_1 \sum_{j\in\mathcal{N}_i} \|\widehat{y}_{k_j} - \widehat{y}_{k_i}\|_2^2}{\|\sum_{j\in\mathcal{N}_i}(\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2},\ \forall t \geq 0 \tag{11}$$

where $\delta_1 \in (0, 0.5]$, $e_i(t) = y_i(t) - \widehat{y}_{k_i}$, for $t \in [t_{k_i}, t_{k_i+1})$, $k_i = 0, 1, 2, \ldots$, then the agents output synchronize asymptotically.

Proof: Since each agent is lossless, we have $\dot{V}_i(t) = u_i^T(t)y_i(t),\ \forall t \geq 0$, where $V_i(t)$ is the storage function for agent H_i. Consider a storage function for the multi-agent system as $V = \sum_{i=1}^N V_i$, then we have

$$\begin{aligned}\dot{V} &= \sum_{i=1}^N \dot{V}_i = \sum_{i=1}^N u_i^T y_i = \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})^T y_i \\ &= \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})^T (e_i + \widehat{y}_{k_i}), \quad \forall t \geq 0\end{aligned} \tag{12}$$

and we can further get

$$\begin{aligned}\dot{V} &= \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})^T e_i + \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})^T \widehat{y}_{k_i} \\ &= \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})^T e_i + \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a\widehat{y}_{k_j}^T \widehat{y}_{k_i} - \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a\widehat{y}_{k_i}^T \widehat{y}_{k_i}.\end{aligned} \tag{13}$$

As the information exchange graph is balanced, we have

$$\sum_{i=1}^n \sum_{j\in\mathcal{N}_i} \widehat{y}_{k_i}^T \widehat{y}_{k_i} = \frac{1}{2}\sum_{i=1}^N \sum_{j\in\mathcal{N}_i} \widehat{y}_{k_i}^T \widehat{y}_{k_i} + \frac{1}{2}\sum_{i=1}^N \sum_{j\in\mathcal{N}_i} \widehat{y}_{k_j}^T \widehat{y}_{k_j}, \tag{14}$$

and therefore it follows that

$$\begin{aligned}\dot{V} &= \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})^T e_i - \frac{a}{2}\sum_{i=1}^N \sum_{j\in\mathcal{N}_i} (\widehat{y}_{k_i} - \widehat{y}_{k_j})^T (\widehat{y}_{k_i} - \widehat{y}_{k_j}) \\ &\leq a\sum_{i=1}^N \|e_i\|_2 \|\sum_{j\in\mathcal{N}_i}(\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2 - \frac{a}{2}\sum_{i=1}^N \sum_{j\in\mathcal{N}_i} \|\widehat{y}_{k_i} - \widehat{y}_{k_j}\|_2^2,\end{aligned} \tag{15}$$

so if

$$\|e_i\|_2 \leq \frac{\sum_{j\in\mathcal{N}_i} \|\widehat{y}_{k_i} - \widehat{y}_{k_j}\|_2^2}{2\|\sum_{j\in\mathcal{N}_i}(\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2}, \quad \forall t \geq 0, \tag{16}$$

then $\dot{V} \leq 0$. Note that the triggering condition (11) actually guarantees that (16) is satisfied.

Moreover, since we can rewrite $\dot{V}$ as

$$\begin{aligned}\dot{V} &= \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})^T y_i = \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a\big[(y_j - e_j) - (y_i - e_i)\big]^T y_i \\ &= \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a\big[(y_j - y_i) - (e_j - e_i)\big]^T y_i \\ &= \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a(y_j - y_i)^T y_i - \sum_{i=1}^N \sum_{j\in\mathcal{N}_i} a(e_j - e_i)^T y_i = -Y^T LY + E^T LY,\end{aligned} \tag{17}$$

where $Y = [y_1^T, y_2^T, \ldots, y_N^T]^T$ is the output vector of the multi-agent system. Thus based on Lasalle's Invariance Principle [1] and strong connectivity of the underlying communication graph, $\dot{V} = -Y^T LY + E^T LY \leq 0$ implies output synchronization of those coupled agents. ■

V. Analysis of Inter-event Time Based on The Triggering Condition

The triggering condition shown in Theorem 1 explicitly determines the time instants at which each agent should transmit its current output information to its neighbors in order to achieve output synchronization. Another problem needs to be answered is how often the event-driven data transmission is needed under the derived triggering condition? In general, it is not easy to get a common lower bound on the inter-event time since we are dealing with heterogeneous multi-agent systems, and in many situations, zeno inter-event time may not be avoided unless a specified lower bound on the inter-event time is imposed. In the following proposition, we give an analysis of the inter-event time based on the triggering condition provided in Theorem 1.

Proposition 1. Consider the dynamics of H_i given by

$$H_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i), \end{cases} \tag{18}$$

let the following assumptions be satisfied

1) $f_i(x_i, u_i) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is locally Lipschitz continuous in x_i on a compact set $S_{x_i} \subset \mathbb{R}^m$ with Lipschitz constant L_{x_i};
2) $\|f_i(x_i, u_i) - f_i(x_i, 0)\|_2 \leq L_{u_i}\|u_i\|_2$ for all $x_i \in S_{x_i}$ with some nonnegative constant L_{u_i};
3) $h_i(x_i) : \mathbb{R}^m \to \mathbb{R}^m$ belongs to a sector (K_{i1}, K_{i2}), with $K_{i1} x_i^T x_i \leq x_i^T h_i(x_i) \leq K_{i2} x_i^T x_i$, where $K_{i1} \in \mathbb{R}$, $K_{i2} \in \mathbb{R}$ and $0 < K_{i1}K_{i2} < \infty$;
4) $\left\|\frac{\partial h_i}{\partial x_i}\right\|_2 \leq \gamma_i$, where $0 < \gamma_i < \infty$;

then with the control (10), the inter-event time $[t_{k_i+1} - t_{k_i}]$ implicitly determined by the triggering condition (11) is strictly positive.

Proof: Since $e_i(t) = y_i(t) - \widehat{y}_{k_i}$ for $t \in [t_{k_i}, t_{k_i+1})$, we can get for $t \in [t_{k_i}, t_{k_i+1})$

$$\begin{aligned} \frac{d}{dt}\|e_i\|_2 &\leq \|\dot{e}_i\|_2 = \|\dot{y}_i\|_2 = \|\dot{h}_i(x_i)\|_2 \\ &= \Big\| \frac{\partial h_i}{\partial x_i} f_i(x_i, 0) + \frac{\partial h_i}{\partial x_i}\big[f_i(x_i, u_i) - f_i(x_i, 0)\big] \Big\|_2 \\ &\leq \gamma_i L_{x_i}\big\|x_i\big\|_2 + \gamma_i L_{u_i}\big\|u_i\big\|_2 \\ &= \gamma_i L_{x_i}\big\|x_i\big\|_2 + \gamma_i L_{u_i}\Big\| \sum_{j \in \mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})\Big\|_2. \end{aligned} \tag{19}$$

Since $h_i(x_i)$ belongs to the sector (K_{i1}, K_{i2}), one can verify that $\|x_i\|_2 \leq \zeta_i \|y_i\|_2$, where

$$\zeta_i = \max\Big\{\frac{1}{|K_{i1}|}, \frac{1}{|K_{i2}|}\Big\}. \tag{20}$$

Therefore, we have

$$\begin{aligned} \frac{d}{dt}\|e_i\|_2 &\leq \gamma_i L_{x_i}\zeta_i\|y_i\|_2 + \gamma_i L_{u_i}\| \sum_{j \in \mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2 \\ &= \gamma_i L_{x_i}\zeta_i\|e_i + \widehat{y}_{k_i}\|_2 + \gamma_i L_{u_i}\| \sum_{j \in \mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2 \\ &\leq \gamma_i L_{x_i}\zeta_i\|e_i\|_2 + \gamma_i L_{x_i}\zeta_i\|\widehat{y}_{k_i}\|_2 + \gamma_i L_{u_i}\| \sum_{j \in \mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2, \end{aligned} \tag{21}$$

so the evolution of $\|e_i\|_2$ during the time interval $[t_{k_i}, t_{k_i+1})$ is bounded by the solution to

$$\frac{d}{dt}\phi(t) = \gamma_i L_{x_i}\zeta_i\phi(t) + \gamma_i L_{x_i}\zeta_i\|\widehat{y}_{k_i}\|_2 + \gamma_i L_{u_i}\| \sum_{j \in \mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2, \tag{22}$$

with initial condition $\phi(t_{k_i}) = 0$. Thus the time for $\|e_i\|_2$ to evolve from 0 to $\frac{\delta_1 \sum_{j\in\mathcal{N}_i} \|\widehat{y}_{k_i} - \widehat{y}_{k_j}\|_2^2}{\|\sum_{j\in\mathcal{N}_i} (\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2}$ is lower bounded by the solution to $\phi(t_{k_i} + \tau_{k_i}) = \frac{\delta_1 \sum_{j\in\mathcal{N}_i} \|\widehat{y}_{k_i} - \widehat{y}_{k_j}\|_2^2}{\|\sum_{j\in\mathcal{N}_i} (\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2}$. Let

$$\sigma_o = \frac{\delta_1 \sum_{j\in\mathcal{N}_i} \|\widehat{y}_{k_i} - \widehat{y}_{k_j}\|_2^2}{\|\sum_{j\in\mathcal{N}_i} (\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2}, \tag{23}$$

then we can get

$$\tau_{k_i} = \frac{1}{\gamma_i L_{x_i} \zeta_i} \ln\Big(1 + \frac{L_{x_i}\zeta_i\sigma_o}{L_{x_i}\zeta_i\|\widehat{y}_{k_i}\|_2 + L_{u_i}\|\sum_{j\in\mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2}\Big). \tag{24}$$

So before agent H_i output synchronizes with its neighbors, we will have $\tau_{k_i} > 0$. Moreover, when H_i output synchronizes with its neighbors, then there is no need for data transmission any more, thus $\tau_{k_i} = \infty$. The proof is completed. ■

Remark 1. As shown in (24), when we are dealing with multi-agent system with heterogeneous dynamics, it is usually difficult to get a common lower bound on $\{\tau_{k_i}\}_s$. Thus, it is not very practical to impose an common upper bound on the admissible network induced delays based on the inter-event time implicitly determined by the triggering condition.

VI. Case Study: Event-Driven Consensus Problem

In this section, we apply the results obtained in the previous sections to study the first order consensus problem. Since data transmissions among those coupled agents are event-based rather than synchronized, one could consider the control problem studied in this section as "asynchronous consensus" problem reported in [2], [3].

The system considered consists of N agents, with $x_i \in \mathbb{R}$ denoting the state of agent H_i. Note that the results derived in this section are extendable to arbitrary dimensions by using Kronecker algebra. We assume that agent's motion obeys a single integrator model

$$\begin{aligned} \dot{x}_i &= u_i \\ y_i &= x_i \end{aligned} \tag{25}$$

with control

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a(\widehat{y}_{k_j} - \widehat{y}_{k_i}) \tag{26}$$

for $t \in [t_{k_i}, t_{k_i+1})$, where $a > 0$ is some positive scalar.

Theorem 2. Consider a network of N agents with each agent's dynamics described by (25)-(26). Assume there is no data transmission delay in the network. Under assumption **A1**, if each agent H_i transmits its current output information to its coupled neighbors whenever the triggering condition (11) is satisfied, then those coupled agents output synchronize to their initial average asymptotically, i.e.,

$$\lim_{t\to\infty} x_i(t) = \overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i(0), \tag{27}$$

for $i = 1, 2, \ldots, N$.

Proof: The proof to show output synchronization under the triggering condition (11) is identical to the proof shown in Theorem 1 since single integrator model is lossless. Thus, we have $\lim_{t\to\infty}(x_j - x_i) = 0,\ \forall i, j$, it remains to show agents output synchronize to their initial average. Let

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i, \tag{28}$$

then we have

$$\begin{aligned} \dot{\overline{x}} &= \frac{1}{N}\sum_{i=1}^{N}\dot{x}_i = \frac{1}{N}\sum_{i=1}^{N} u_i = \frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(\widehat{x}_{k_j} - \widehat{x}_{k_i}) \\ &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(x_j - x_i) - \frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(e_j - e_i), \end{aligned} \tag{29}$$

under assumption A1, we have $\dot{\overline{x}} = 0$, $\forall t \geq 0$, thus

$$\overline{x} = \overline{x}(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0). \tag{30}$$

Since $\lim_{t\to\infty}(x_j - x_i) = 0,\ \forall i, j$, this implies that $\lim_{t\to\infty} x_i = \frac{1}{N}\sum_{i=1}^{N} x_i(0),\ \forall i$, and the proof is completed. ■

Proposition 2. Consider a network of N agents with each agent's dynamics described by (25)-(26). Assume there is no data transmission delay in the network. Under assumption **A1**, the inter-event time $[t_{k_i+1} - t_{k_i}]$ implicitly determined by the triggering condition (11) is lower bounded by

$$t_{k_i+1} - t_{k_i} \geq \tau_{k_i} = \frac{\delta_1 \sum_{j\in\mathcal{N}_i} \|\widehat{x}_{k_j} - \widehat{x}_{k_i}\|_2^2}{a\|\sum_{j\in\mathcal{N}_i}(\widehat{x}_{k_j} - \widehat{x}_{k_i})\|_2^2} \tag{31}$$

for $k = 0, 1, 2, \ldots$, with $\delta_1 \in (0, 0.5]$.

Proof: For $t \in [t_{k_i}, t_{k_i+1})$,

$$\frac{d}{dt}\|e_i\|_2 \leq \|\dot{e}_i\|_2 = \|\dot{x}_i\|_2 = \|u_i\|_2 = \|a\sum_{j\in\mathcal{N}_i}(\widehat{x}_{k_j} - \widehat{x}_{k_i})\|_2, \tag{32}$$

so the evolution of $\|e_i\|_2$ during the time interval $[t_{k_i}, t_{k_i+1})$ is bounded by the solution to

$$\frac{d}{dt}\phi(t) = \|a\sum_{j\in\mathcal{N}_i}(\widehat{x}_{k_j} - \widehat{x}_{k_i})\|_2 \tag{33}$$

with initial condition $\phi(t_{k_i}) = 0$. Thus the time for $\|e_i\|_2$ to evolve from 0 to $\frac{\delta_1 \sum_{j\in\mathcal{N}_i}\|\widehat{y}_{k_i} - \widehat{y}_{k_j}\|_2^2}{\|\sum_{j\in\mathcal{N}_i}(\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2}$ is lower bounded by the solution to $\phi(t_{k_i} + \tau_{k_i}) = \frac{\delta_1 \sum_{j\in\mathcal{N}_i}\|\widehat{y}_{k_i} - \widehat{y}_{k_j}\|_2^2}{\|\sum_{j\in\mathcal{N}_i}(\widehat{y}_{k_j} - \widehat{y}_{k_i})\|_2}$, and we can get

$$\tau_{k_i} = \frac{\delta_1 \sum_{j\in\mathcal{N}_i} \|\widehat{x}_{k_j} - \widehat{x}_{k_i}\|_2^2}{a\|\sum_{j\in\mathcal{N}_i}(\widehat{x}_{k_j} - \widehat{x}_{k_i})\|_2^2}. \tag{34}$$

The proof is completed. ■

Example 2. We consider the "asynchronous consensus" problem as discussed above, the topology of the underlying communication graph is given by

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{35}$$

The simulation results are shown in Fig.2 and Fig.3.

In Fig.2, the x-axis shows the time instants of events while the y-axis shows the length of inter-event time of each agent. Fig.3 shows the evolution of agent's state. With initial state $x_1(0) = 20, x_2(0) = 4, x_3(0) = 100, x_4(0) = -60, x_5(0) = -15$, and $\frac{1}{N}\sum_{i=1}^{N} x_i(0) = 9.8$, the agent's state converges to their initial average.

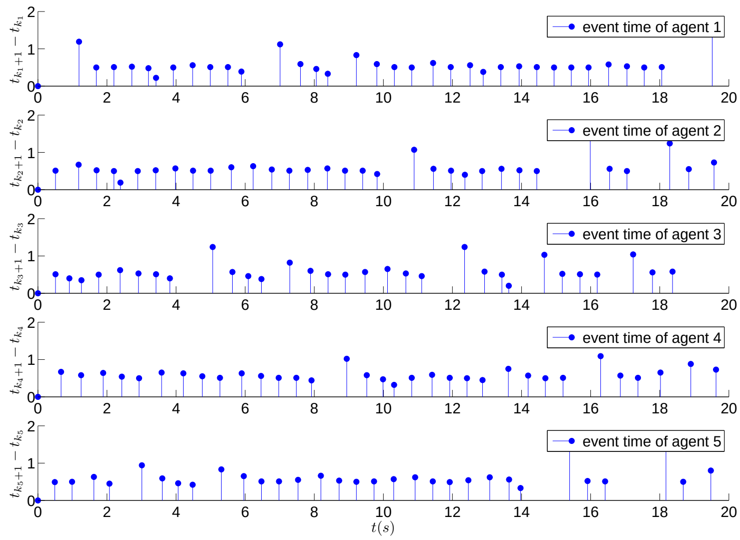

Fig. 2: simulation result of example 2: event time

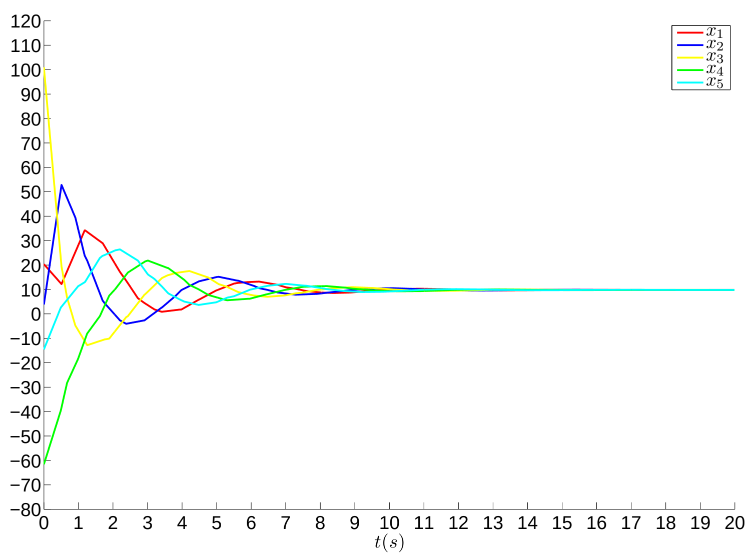

Fig. 3: simulation result of example 2: consensus

VII. Output Synchronization with Event-Driven Communication and Constant Communication Delays: A Scattering Transformation Approach

In this section, we propose a set-up to achieve output synchronization of multi-agent systems in the presence of constant network induced delays by using event-driven communication and scattering transformation. Scattering transformation has been used earlier in the problem of bilateral teleoperation and NCSs to guarantee delay independent stability, see [10], [13], [16] and [17]. In our setting for event-driven communication with scattering transformation, the agents transmit the so called "scattering variables" instead of their outputs to the neighbors, and the data transmissions are event-based. The set-up for the event-driven communication strategy with scattering transformation is illustrated schematically in Fig.4.

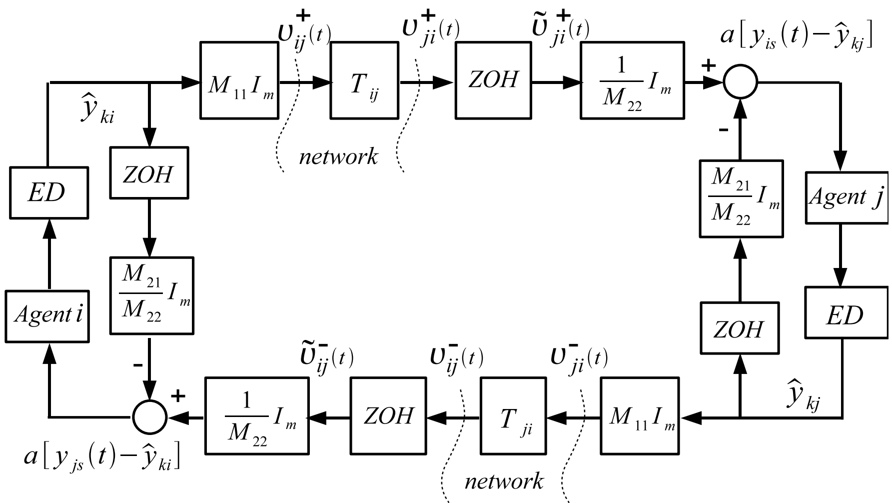

Fig. 4: event-driven communication with scattering transformation

In Fig.4, the "ED" block represents the "event-detector" , and whenever the event-detector detects that the corresponding agent satisfies its specific triggering condition, an updated scattering variables ($\upsilon^{+}_{ij}(t)$ or $\upsilon^{-}_{ji}(t)$ as shown in Fig.4) will be obtained and sent to the neighboring agents. The event time of agent i is defined by the time sequence $\{t_{k_i}\}, k_i = 0, 1, 2, \ldots$ and the event time of agent j is defined by the time sequence $\{t_{k_j}\}, k_j = 0, 1, 2, \ldots$. The "ZOH" block represents the zero-order hold, thus $\widetilde{\upsilon}^{+}_{ji}(t)$ holds the last sample of $\upsilon^{+}_{ji}(t)$ and $\widetilde{\upsilon}^{-}_{ij}(t)$ holds the last sample of $\upsilon^{-}_{ij}(t)$. T_{ji} represents the communication delay form agent j to agent i while T_{ij} represents the communication delay form agent i to agent j. T_{ij} and T_{ji} are not necessarily equal to each other. As the scattering variables are transmitted over networks, we have

$$\upsilon^{+}_{ji}(t) = \upsilon^{+}_{ij}\big(t - T_{ij}\big) \;\text{ and }\; \upsilon^{-}_{ij}(t) = \upsilon^{-}_{ji}\big(t - T_{ji}\big),\; \forall (i,j) \in E(G). \tag{36}$$

Let the agents be coupled together using the control

$$u_i(t) = a\big[y_{js}(t) - \widehat{y}_{k_i}\big],\; \text{for } t \in [t_{k_i}, t_{k_i+1}),\; k_i = 0, 1, 2, \ldots, \tag{37}$$

with $\widehat{y}_{k_i} = y_i(t_{k_i})$, and

$$u_j(t) = a\big[y_{is}(t) - \widehat{y}_{k_j}\big],\; \text{for } t \in [t_{k_j}, t_{k_j+1}),\; k_j = 0, 1, 2, \ldots, \tag{38}$$

with $\widehat{y}_{k_j} = y_j(t_{k_j})$, $\forall (i,j) \in E(G)$. $a > 0$ is a constant representing the coupling among agents as shown in the adjacency matrix of the underlying communication graph. We assume $y_{is}(t)$, $y_{js}(t)$, $\widehat{y}_{k_i}$ and $\widehat{y}_{k_j}$ are signals belonging to $\mathcal{L}_{2e}$. The variables $y_{js}(t)$ and $y_{is}(t)$ are derived out of the scattering transformation which is given by

$$\begin{cases} \text{For } t \in [t_{k_i}, t_{k_i+1}), \;\; \dfrac{1}{M_{22}}\widetilde{\upsilon}^{-}_{ij}(t) - \dfrac{M_{21}}{M_{22}}\widehat{y}_{k_i} = a\big[y_{js}(t) - \widehat{y}_{k_i}\big] \\ \qquad\qquad \text{At } t = t_{k_i}, \;\; M_{11}\widehat{y}_{k_i} = \upsilon^{+}_{ij}(t) \end{cases} \tag{39}$$

and

$$\begin{cases} \text{For } t \in [t_{k_j}, t_{k_j+1}), \quad \frac{1}{M_{22}} \widetilde{\upsilon}_{ji}^{+}(t) - \frac{M_{21}}{M_{22}} \widehat{y}_{k_j} = a\big[y_{is}(t) - \widehat{y}_{k_j}\big] \\ \qquad \text{At } t = t_{k_j}, \quad M_{11} \widehat{y}_{k_j} = \upsilon_{ji}^{-}(t), \end{cases} \tag{40}$$

$\forall (i,j) \in E(G)$. Positive constants M_{11}, M_{21}, M_{22} are the parameters of the scattering transformation. The superscript $+,-$ for the scattering variables is a convention for the direction of the power flow.

As shown in Fig.4, agent i transmits the scattering variables $\upsilon_{ij}^{+}(t)$ to agent j who receives it as the scattering variables $\upsilon_{ji}^{+}(t)$. Agent j then uses the control $a\big[y_{is}(t) - \widehat{y}_{kj}\big]$ to extract the variables $y_{is}(t)$ output of the variable $\widetilde{\upsilon}_{ji}^{+}(t)$. A similar procedure is used to obtain the variables $y_{js}(t)$ by agent i. One should notice that agent i is participating in $|\mathcal{N}_i|$ closed-loops as the one demonstrated in Fig.4, where $|\mathcal{N}_i|$ is the number of neighbors of agent i.

Theorem 3. Consider the set-up of event-driven communication with scattering transformation between any coupled lossless agent i and agent j with m inputs and m outputs as shown in Fig.4, $\forall (i,j) \in E(G)$. Choose $M_{11} = M_{21} = \frac{\sqrt{a}}{2}$, $M_{22} = \frac{1}{\sqrt{a}}$. Assume that the communication delays between agent i and agent j are constant and finite. Then if agent i transmits its current output information to its neighbors whenever the following triggering condition is satisfied

$$\|e_i(t)\|_2 = \frac{\delta_3 \sum_{j \in \mathcal{N}_i} \big\| y_{js}(t) - \widehat{y}_{k_i} \big\|_2^2}{\big\| \sum_{j \in \mathcal{N}_i} \big[y_{js}(t) - \widehat{y}_{k_i} \big] \big\|_2}, \ \forall t \geq 0 \tag{41}$$

where $e_i(t) = y_i(t) - \widehat{y}_{k_i}$, for $t \in [t_{k_i}, t_{k_{i+1}})$, and $\delta_3 \in (0,1]$, then under **A2**, those coupled agents will output synchronize asymptotically.

Proof: The proof is provided in the Appendix A. ■

Remark 2. If $T_{ij} = T_{ji} = 0$, since

$$\widetilde{\upsilon}_{ji}^{+} = M_{11} \widehat{y}_{k_i} = M_{21} \widehat{y}_{k_j} + M_{22} a \big[y_{is}(t) - \widehat{y}_{k_j} \big], \text{ for } t \in [t_{k_j}, t_{k_j+1}], \tag{42}$$

we can get

$$y_{is}(t) = \frac{1}{2} \big(\widehat{y}_{k_i} + \widehat{y}_{k_j} \big), \text{ for } t \in [t_{k_j}, t_{k_j+1}]. \tag{43}$$

Similarly, we can obtain

$$y_{js}(t) = \frac{1}{2} \big(\widehat{y}_{k_i} + \widehat{y}_{k_j} \big), \text{ for } t \in [t_{k_i}, t_{k_i+1}]. \tag{44}$$

where $\widehat{y}_{k_i}$ and $\widehat{y}_{k_j}$ are the latest output information sent by agent i and agent j respectively. And one can verify that, in this case, the triggering condition (41) becomes

$$\|e_i(t)\|_2 = \frac{\delta_1 \sum_{j \in \mathcal{N}_i} \big\| \widehat{y}_{k_j} - \widehat{y}_{k_i} \big\|_2^2}{\big\| \sum_{j \in \mathcal{N}_i} (\widehat{y}_{k_j} - \widehat{y}_{k_i}) \big\|_2}, \tag{45}$$

with $\delta_1 \in (0, 0.5]$, $\forall t \geq 0$, which is the same as the triggering condition derived in Theorem 1 for no data transmission delays case.

Example 3. We consider again the "asynchronous consensus" problem studied in section VI, the underlying communication graph is given by

$$L = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}, \tag{46}$$

thus the topology of the underlying communication graph satisfies **A2**. Let the communication delay between each coupled agents be randomly generated from the interval $[1,2]s$, and we use the set-up of event-driven communication with scattering transformation for each coupled agents, the simulation results with initial condition $x_1(0) = -20, x_2(0) = 15, x_3(0) = 32, x_4(0) = 68, x_5(0) = 0$ are shown in Fig.5-Fig.6, the states of agents finally converge to a value around 10.4 while their initial average is 19. Thus in this case, the event-driven consensus problem cannot guarantees agreement around the initial average if we consider arbitrary constant communication delays among the coupled agents. Also observing from simulations, with different communication delays, the final agreement is also different. Fig.7 and Fig.8 show the simulation results by randomly generating communication delays from the interval $[0,1]s$ and $[2,3]s$ while the initial conditions of agents are kept the same. However, if there is no communication delays between any coupled agents, then the states of agents will still converge to their initial average with the scattering transformation set-up, this is shown in Fig.9.

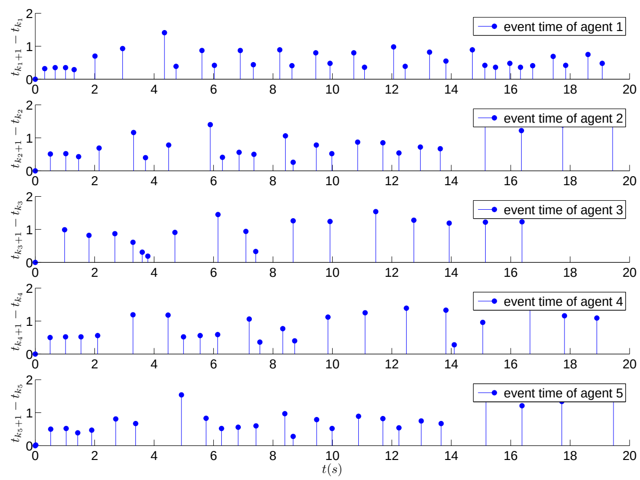

Fig. 5: simulation result of example 3: event time with communication delays in [1,2]s

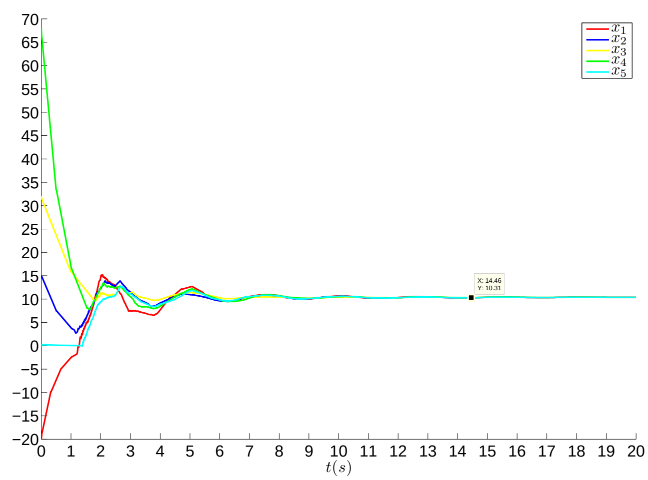

Fig. 6: simulation result of example 3: consensus with communication delays in [1,2]s reaches agreement at 10.3

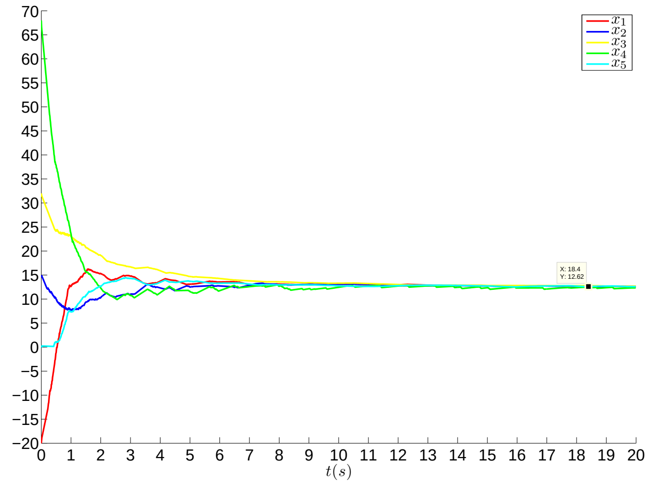

Fig. 7: simulation result of example 3: consensus with communication delays in [0,1]s reaches agreement around 12.6

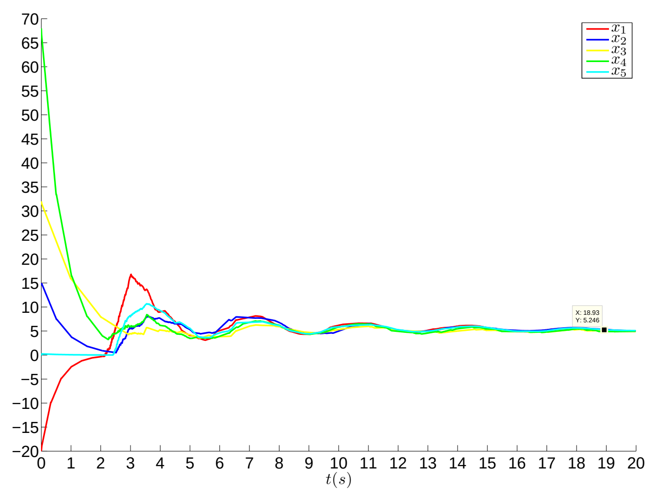

Fig. 8: simulation result of example 3: consensus with communication delays in [2,3]s reaches agreement around 5.6

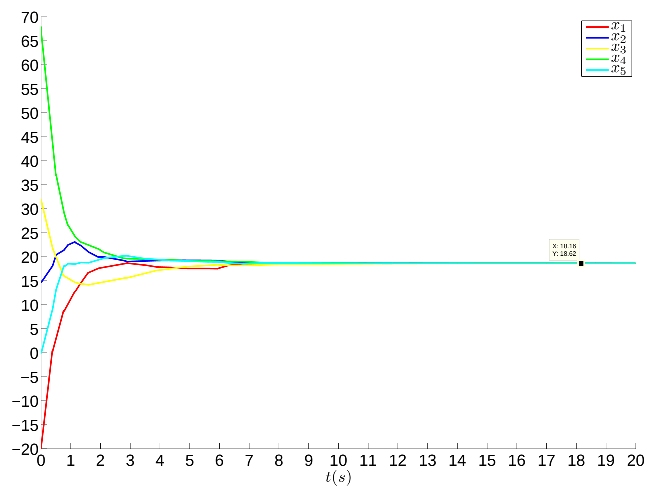

Fig. 9: simulation result of example 3: consensus with no communication delays reaches agreement around 19

VIII. Triggering Condition For Output Synchronization Without Communication Delays: Quantization Effects

It was assumed in the previous sections that each agent is equipped with an "event-detector" which is able to measure the output of the agent with infinite precision, and the event-detector uses that measurement to examine the corresponding triggering condition of the agent and transmit that measurement through the network whenever the triggering condition is satisfied. In reality, however, the transmitted measurement first has to be quantized in order to be represented by a finite number of bits and to be used in processor operations and carried over a digital communication network. Thus, it becomes necessary to study the effects of quantization error on output synchronization of the networked multi-agent system with event-driven communication.

Assume that the control input to agent H_i is given by

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big],\ \text{for } t\in[t_{k_i}, t_{k_i+1}), k_i = 0,1,2,\dots, \tag{47}$$

$q(\widehat{y}_{k_j})$ is the quantized latest transmitted output information of agent H_j by the time t and $q(\widehat{y}_{k_i})$ is the quantized latest transmitted output information of agent H_i by the time t. We first assume there is no communication delay in the networks.

Each agent is equipped with an "event detector" and a "quantizer". The event detector can continuously (or with adequately small sampling period) monitor the output of the agent, and whenever it detects the "triggering condition" associated with the corresponding agent is satisfied, it will get a sample of the agent's current output information denoted by $y_i(t_{k_i}) = \widehat{y}_{k_i}$(with t_{k_i} denoting the event time of agent i) and sends this sampled output information to the quantizer. The quantizer then processes the received data and the quantized output information $q(\widehat{y}_{k_i})$ will be sent to the neighboring agents of agent H_i. We assume the data processing time in the quantizer is negligible.

For $t\in[t_{k_i}, t_{k_i+1})$, let $e_i(t) = y_i(t) - \widehat{y}_{k_i}$ denote the output novelty error with respect to the sampled output information; let $\varepsilon_{k_i} = \widehat{y}_{k_i} - q(\widehat{y}_{k_i})$ denote the quantization error with respect to the sampled output information; let $\widetilde{e}_i(t) = y_i(t) - q(\widehat{y}_{k_i})$ denote the output novelty error with respect to the quantized sampled output information. One can verify that $\widetilde{e}_i(t) = y_i(t) - \widehat{y}_{k_i} + \varepsilon_{k_i}$. With event-driven communication and quantized sampled output information transmitted between coupled agents, we have the following theorem.

Theorem 4. Consider a network of N lossless agents with control (47). Under assumption **A1**, if each agent H_i transmits its current output information to its neighbors whenever the following triggering condition is satisfied

$$\|e_i(t)\|_2 = \delta_4\big(\frac{1-\kappa}{2} - \frac{1}{2\beta}\big)\frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2,\ \forall t\geq 0, \tag{48}$$

where $\delta_4\in(0,1]$, $0<\kappa<1$ and $1<\frac{1}{1-\kappa}<\beta$, then the output synchronization error of the studied multi-agent system is bounded by the quantization errors of agents' latest transmitted output information by the time t.

Proof: The proof is provided in the Appendix B. ■

IX. Special Case: Event-Driven Consensus Problem With Quantization

In this section, we study the consensus problem with event-driven communication and quantization as a special case for the problems investigated in the previous section. We assume that agent's motion obeys a single integrator model as shown in (25) with control

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big] \tag{49}$$

for $t\in[t_{k_i}, t_{k_i+1})$, where $q(\widehat{y}_{k_i})$ is the quantized value of agent $H_i{'}s$ latest transmitted output information, $q(\widehat{y}_{k_j})$ is the quantized value of agent $H_j{'}s$ latest transmitted output information.

Lemma 2. The cascade connection of an integrator and a passive memoryless function h as shown in Fig.10, is still lossless from u to $h(x)$.

Proof: Passivity of h guarantees that $\int_0^x h(\sigma)d\sigma \geq 0$ for all x. With $V(x) = \int_0^x h(\sigma)d\sigma$ as the storage function, we have $\dot{V} = h(x)\dot{x} = yu$. Hence the system is lossless. ■

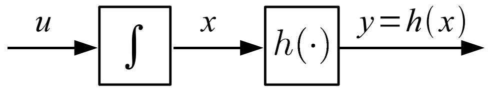

Fig. 10: cascade connection of an integrator and a passive memoryless function

Remark 3. Lemma 2 indicates that the cascade connection of an integrator and a passive memoryless quantizer can be studied as a lossless system with the quantized output as the new output of the system. This result enables us to derive the triggering condition for the event-triggered consensus problem with quantization.

Assume that each agent is equipped with a passive memoryless quantizer $q(\cdot)$ and an event detector which is denoted by “ED”as shown in Fig.11. The event detector continuously (or sampling with an adequately fast sampling rate) monitors the output of the quantizer connected with the agent, and whenever it detects that the triggering condition associated with the agent is satisfied, a quantized output information $q(\widehat{y}_{k_i})$ at that event time (t_{k_i}) will be transmitted to the agent's corresponding neighbors. The theorem below provides a triggering condition to achieve consensus among the coupled agents.

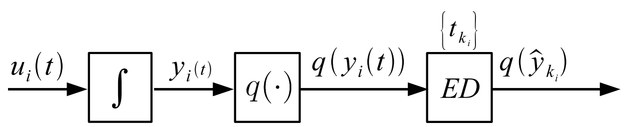

Fig. 11: cascade connection of an integrator and a passive memoryless quantizer

Theorem 5. Consider a network of N agents with each agent's dynamics described by (25) and (49). Assume there is no data transmission delay in the network. Under assumption **A1**, if each agent H_i transmits its current output information to its coupled agents whenever the following triggering condition is satisfied

$$\varepsilon_i(t) = \frac{\delta_5 \sum_{j\in\mathcal{N}_i} \left\| q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i}) \right\|_2}{\left\| \sum_{j\in\mathcal{N}_i} \left[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i}) \right] \right\|_2}, \quad \forall t \geq 0, \tag{50}$$

for some $\delta_5 \in (0, 0.5]$, where $\varepsilon_i(t) = q(y_i(t)) - q(\widehat{y}_{k_i})$, then those coupled agents will converge to a value around their initial average asymptotically, i.e.,

$$\lim_{t\to\infty} x_i(t) \approx \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i(0), \quad \forall i.$$

Proof: The proof is provided in Appendix C. ■

Example 4. We consider the “asynchronous consensus” problem as discussed above, the underlying information exchange graph is given by (35), which satisfies assumption **A1**. Assume that each agent is equipped with a uniform mid-tread quantizer with quantization level 0.5 (one can verify that a uniform mid-tread quantizer is passive since $y_i(t)q(y_i(t)) \geq 0$). The simulation results are shown in Fig.12-Fig.15. In Fig.12, the x-axis shows the time instants of events while the y-axis shows the length of inter-event time of each agent. Fig.13 shows the evolution of quantized output of each agent, Fig.14 shows the evolution of the state of each agent and Fig.15 shows the evolution of average of the agents' state. With initial state $x_1(0) = 20, x_2(0) = 4, x_3(0) = 100, x_4(0) = -60, x_5(0) = -15$, we have $\frac{1}{N}\sum_{i=1}^{N} x_i(0) = 9.8$. And one can see from Fig.13-Fig.15 that while the quantized output of each agent converges to 10, the average of the agents' state keeps constant at their initial average 9.8 along with time, and the state of each agent finally converges to a value around 9.8.

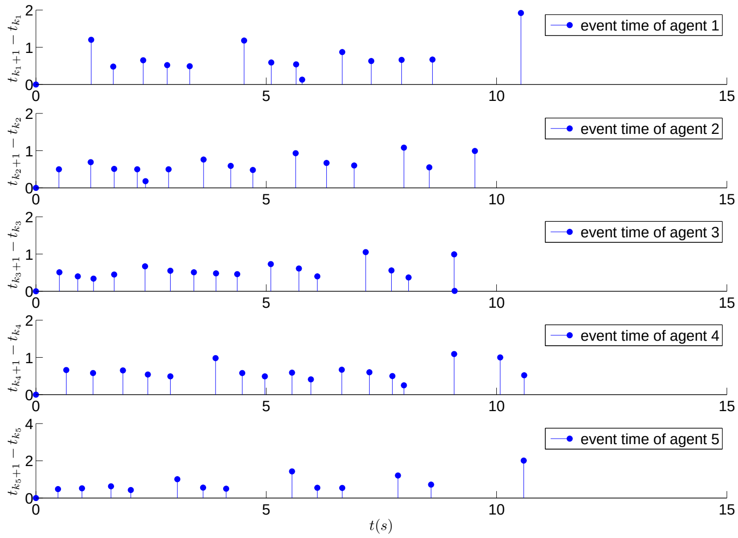

Fig. 12: example 4: event time

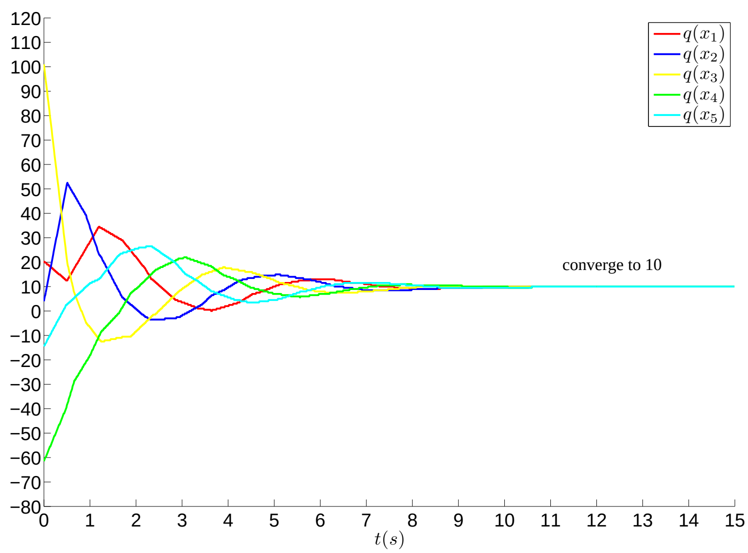

Fig. 13: example 4: evolution of quantized output of each agent

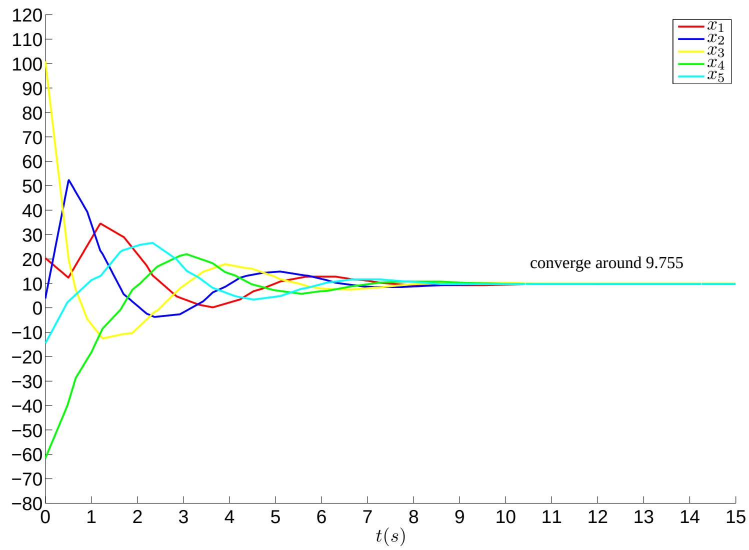

Fig. 14: example 4: evolution of the state of each agent

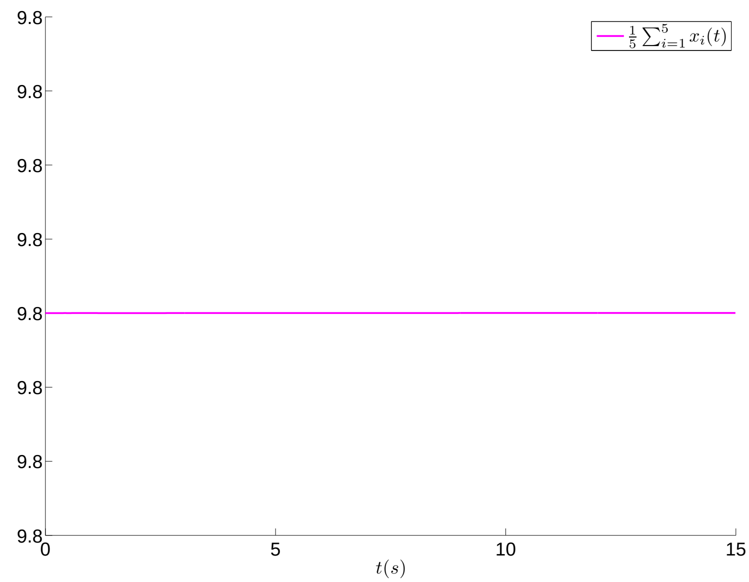

Fig. 15: example 4: evolution of average of the states

X. Triggering Condition For Output Synchronization With Constant Communication Delays: Quantization Effects

In section VII, we proposed a set-up to achieve output synchronization among coupled agents with event-driven communication in the presence of constant network induced delays by using scattering transformation. In this section, in addition to communication delays, we take quantization into consideration and further study the quantization effects on the output synchronization based on the set-up shown in section VII.

Let the agents be coupled together using the control

$$\begin{aligned} u_i(t) &= \sum_{j\in\mathcal{N}_i} a\big[y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big], \text{ for } t\in[t_{k_i}, t_{k_i+1}),\ k_i = 0,1,2,\dots, \text{ and} \\ u_j(t) &= \sum_{i\in\mathcal{N}_j} a\big[y_{is}(t) - q(M_{21}\widehat{y}_{k_j})\big], \text{ for } t\in[t_{k_j}, t_{k_j+1}),\ k_j = 0,1,2,\dots, \end{aligned} \tag{51}$$

$\forall (i,j)\in E(G)$, where $q(M_{21}\widehat{y}_{k_i})$ and $q(M_{21}\widehat{y}_{k_j})$ denote the quantized values of $M_{21}\widehat{y}_{k_i}$ and $M_{21}\widehat{y}_{k_j}$ respectively. The variables $y_{is}(t)$ and $y_{js}(t)$ are derived out of the scattering transformation which are given by

$$\begin{cases} \text{for } t\in[t_{k_i}, t_{k_i+1}), & \dfrac{1}{M_{22}}\widetilde{\upsilon}^-_{ij}(t) - \dfrac{1}{M_{22}}q(M_{21}\widehat{y}_{k_i}) = a\big[y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big] \\ \text{at } t = t_{k_i}, & \upsilon^+_{ij}(t) = q(M_{11}\widehat{y}_{k_i}), \quad \text{and} \end{cases} \tag{52}$$

$$\begin{cases} \text{for } t\in[t_{k_j}, t_{k_j+1}), & \dfrac{1}{M_{22}}\widetilde{\upsilon}^+_{ji}(t) - \dfrac{1}{M_{22}}q(M_{21}\widehat{y}_{k_j}) = a\big[y_{is}(t) - q(M_{21}\widehat{y}_{k_j})\big] \\ \text{at } t = t_{k_j}, & \upsilon^-_{ji}(t) = q(M_{11}\widehat{y}_{k_j}) \end{cases} \tag{53}$$

$\forall (i,j)\in E(G)$.

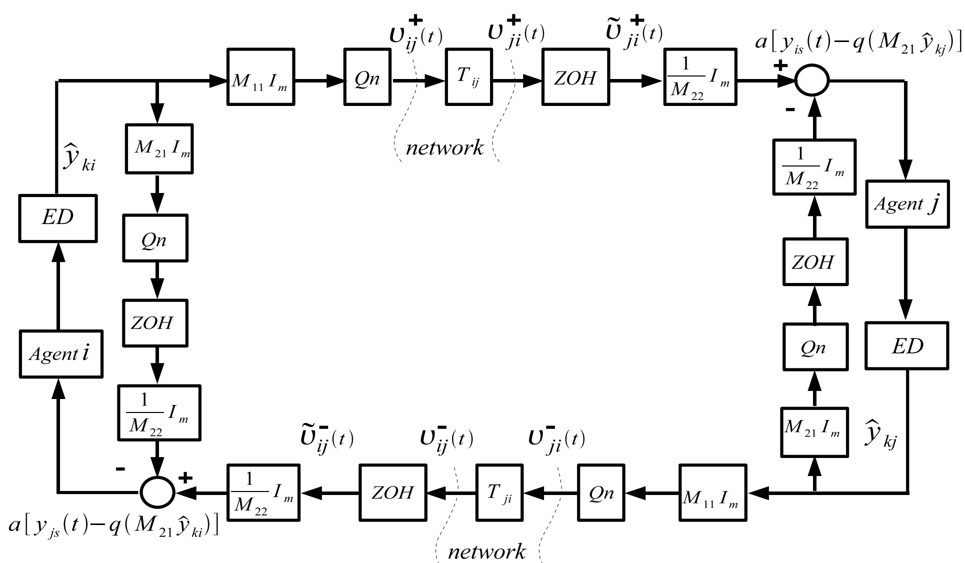

Fig. 16: event-driven communication with scattering transformation and quantization effect

Positive constants M_{11}, M_{21}, M_{22} are the parameters of the scattering transformation. The superscript $+,-$ for the scattering variables is a convention for the direction of the power flow. The set-up for the event-driven communication strategy with scattering transformation is illustrated schematically in Fig.16. The “ED” block represents the “event-detector” , and whenever the event-detector detects that the corresponding agent satisfies its specific triggering condition, a newly sampled output information will be sent to the quantizer (denoted by “Qn”) and updated scattering variables ($\upsilon^+_{ij}(t)$ or $\upsilon^-_{ji}(t)$ as shown in Fig.8) will be obtained and sent to the coupled agents. The event time of agent i is defined by the time sequence $\{t_{k_i}\}, k_i = 0,1,2,\dots$ and the event time of agent j is defined by the time sequence $\{t_{k_j}\}, k_j = 0,1,2,\dots$. The “ZOH” block represents the zero-order hold, thus $\widetilde{\upsilon}^+_{ji}(t)$ holds the last sample of $\upsilon^+_{ji}(t)$ and $\widetilde{\upsilon}^-_{ij}(t)$ holds the last sample of $\upsilon^-_{ij}(t)$. T_{ji} represents the communication delay form agent j to agent

i while T_{ij} represents the communication delay form agent i to agent j. T_{ij} and T_{ji} are not necessarily equal to each other. As the scattering variables are transmitted over networks, we have

$$\upsilon_{ji}^{+}(t) = \upsilon_{ij}^{+}\big(t - T_{ij}\big) \quad \text{and} \quad \upsilon_{ij}^{-}(t) = \upsilon_{ji}^{-}\big(t - T_{ji}\big),\ \forall (i,j) \in E(G). \tag{54}$$

Agent i transmits the scattering variables $\upsilon_{ij}^{+}(t)$ to agent j who receives it as the scattering variables $\upsilon_{ji}^{+}(t)$. Agent j uses the control $a\big[y_{is}(t) - q(M_{21}\widehat{y}_{kj})\big]$ to extract the variables $y_{is}(t)$ output of the variable $\widetilde{\upsilon}_{ji}^{+}(t)$. A similar procedure is used to obtain the variables $y_{js}(t)$ by agent i. One should notice that agent i is participating in $|\mathcal{N}_i|$ closed-loops as the one demonstrated in Fig.16, where $|\mathcal{N}_i|$ is the number of agents that send output information to agent i.

Theorem 6. Assume that the underlying information exchange graph satisfies assumption **A2** and the data transmission delays between each coupled agents are constant and finite. Consider the set-up of event-driven communication with scattering transformation and quantization between any coupled lossless agent i and agent j (with m inputs and m outputs) as shown in Fig.16. The control action for each agent is given by (51). The parameters of the scattering transformation are chosen such that $M_{21} = M_{11} > 0$ and $aM_{22} = 2$. Define $\widehat{e}_i(t) = M_{11}\big[y_i(t) - \widehat{y}_{k_i}\big]$ as the output novelty error of agent i, $i = 1, 2, \dots, N$. If agent i transmits its current output information to its neighbors whenever the following triggering condition is satisfied

$$\|\widehat{e}_i(t)\|_2 = \frac{\delta_6(1-\frac{\beta}{2})\gamma}{|\mathcal{N}_i|} \sum_{j\in\mathcal{N}_i} \big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2,\ \forall t \geq 0 \tag{55}$$

for some $\delta_6 \in (0,1]$, where $0 < \gamma < 1$ and $0 < \beta < 2$. Then the output synchronization error of the studied multi-agent system is ultimately bounded by the quantization errors of agents' latest transmitted outputs information in the networks.

Proof: The proof is provided in the Appendix D. ■

Example 5. We consider again the "asynchronous consensus" problem as studied in section V, the underlying information exchange graph is given by (46), thus the topology of the underlying information exchange graph satisfies **A2**. Let the communication delay between each coupled agents be constant and we randomly choose the delays from the interval $[1,4]s$. We use the set-up shown in Fig.16 for each coupled agents, and we choose $M_{11} = M_{21} = 1$, $M_{22} = 2$, $\delta = 1$, $\gamma = 0.9$ and $\beta = 0.1$. The quantizer of each agent is a uniform mid-tread quantizer with quantization level 0.5. The simulation results with initial condition $x_1(0) = -20, x_2(0) = 15, x_3(0) = 32, x_4(0) = 68, x_5(0) = 0$ are shown in Fig.17-Fig.19, the states of agents finally converge to a value around 3.4. However, if we randomly choose the delays from the interval $[0.5,1]s$, the states of agents finally converge to a different value which is around 9.5 as shown in Fig.19.

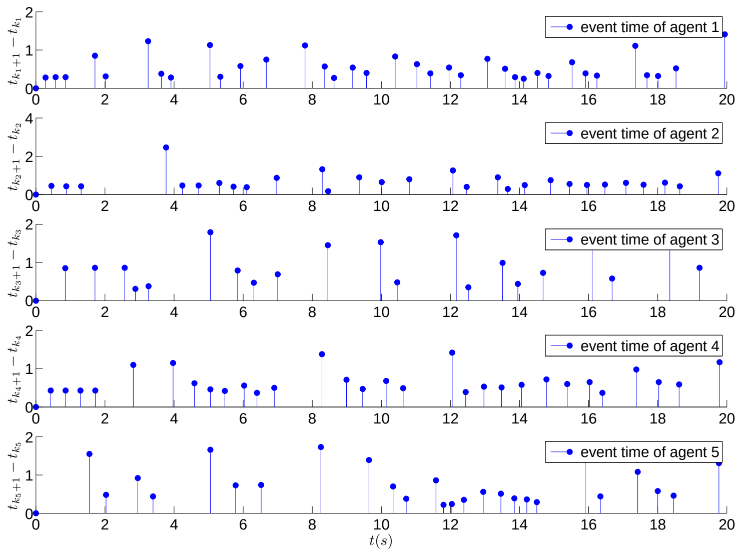

Fig. 17: example 5: event time

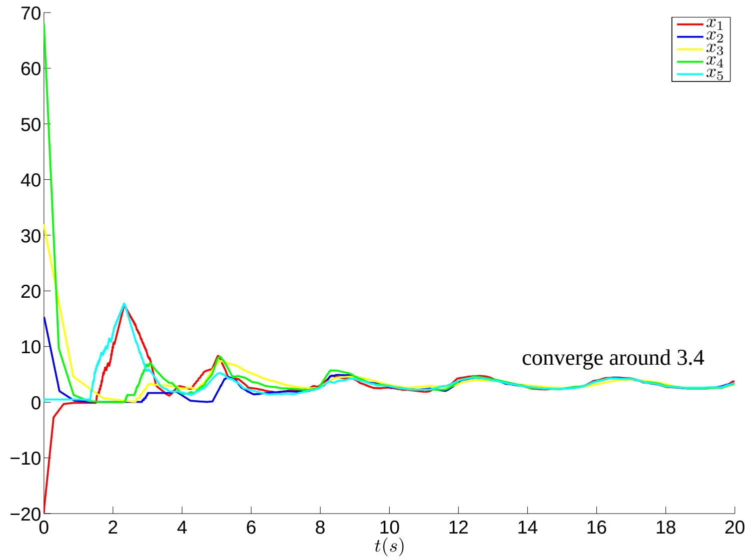

Fig. 18: example 5: evolution of the agent's state for delays chosen from the interval [1, 4]s

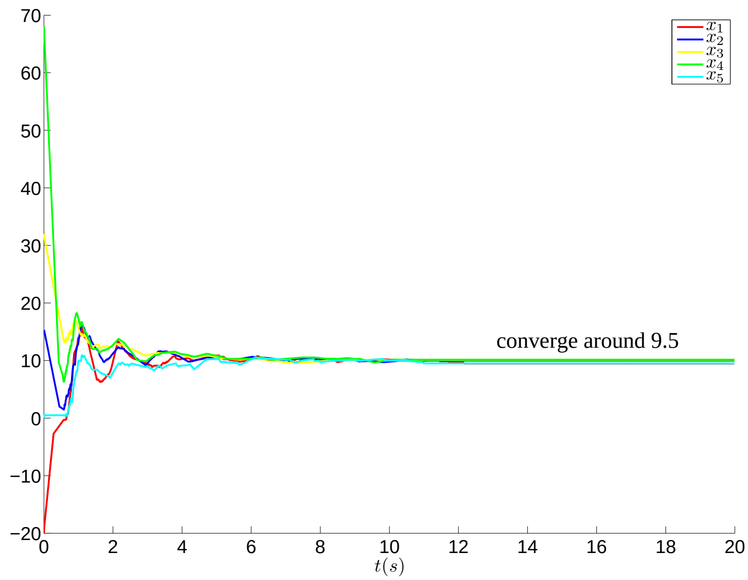

Fig. 19: example 5: evolution of the agent's state for delays chosen from the interval [0.5, 1]s

Remark 3. It could be seen from Example 3 and Example 5 that when arbitrary constant network induced delays are considered in our proposed set-up, the event-driven consensus problem may not be able to achieve agreement at the agents' initial average, and with different constant delays, the final agreement value could be different. However, as seen from Example 2 and Example 4, it is still possible to achieve average consensus with event-driven communication (and with signal quantization) in the presence of network induced delays as long as the delays are upper bounded by the inter-event time implicitly determined by the triggering condition. It is interesting to further study distributed algorithm which could achieve average consensus with event-driven communication in the presence of arbitrary constant network induced delays. However, it is not the focus of our interests in the current paper. Note that in many control applications of multi-agent systems, in order for all the agents to achieve output synchronization at some specific value or within some pre-determined set (i.e., leader following problem or rendezvous problem), we could designate certain agents as leaders in the group and send important leading information to the leaders from time to time, while the information exchange between leaders and their followers are still event-based as the event-driven communication set-up shown in the current paper.

XI. Conclusion

In this paper, we study the output synchronization problem of multi-agent systems with event-driven communication. We assume all the agents in the network are lossless and we use scattering transformation to deal with network induced delays between coupled agents. Whenever the agent satisfies its triggering condition, a scattering variable which contains the sampled output information of the agent will be sent to its coupled neighbors, and the neighbors will extract reference information from its received scattering variables for its control action update. The proposed set-up allows us to find a composite storage function (which is derived from the scattering transformation) to analyze the stability of the entire system. The result presented in this paper is an important extension of applying event-driven communication to control of multi-agent systems, especially when it is difficult to derive a common upper bound on the admissible network induced delays or when the network induced delay between coupled agents is larger than the inter-event time implicitly determined by the event triggering condition. Quantization effects on output synchronization with event-driven communication have also been investigated in this paper. We have shown that output synchronization error of coupled agents is ultimately bounded by the quantization errors with the event-driven communication set-up proposed in this paper.

Appendix A
Proof of Theorem 3

Proof: With T_{ij} and T_{ji} being constant and finite, we can verify that

$$\int_0^t \|\upsilon_{ji}^+(\tau)\|_2^2 d\tau \le \int_0^t \|\upsilon_{ij}^+(\tau)\|_2^2 d\tau \text{ and } \int_0^t \|\upsilon_{ij}^-(\tau)\|_2^2 d\tau \le \int_0^t \|\upsilon_{ji}^-(\tau)\|_2^2 d\tau. \tag{56}$$

Since

$$\int_0^t \|\upsilon_{ij}^+(\tau)\|_2^2 d\tau = \sum_{k_i=0}^{n_i} \delta(t - t_{k_i}) M_{11}^2 \|\widehat{y}_{k_i}\|_2^2 \le \sum_{k_i=0}^{n_i} \int_{t_{k_i}}^{t_{k_i+1}} M_{11}^2 \|\widehat{y}_{k_i}\|_2^2 d\tau \tag{57}$$

where $\delta(\cdot)$ is the Dirac delta function, n_i is the number of scattering variables sent from agent i to agent j during the time interval $[0, t]$. Similarly, one can obtain

$$\int_0^t \|\upsilon_{ji}^-(\tau)\|_2^2 d\tau = \sum_{k_j=0}^{n_j} \delta(t - t_{k_j}) M_{11}^2 \|\widehat{y}_{k_j}\|_2^2 \le \sum_{k_j=0}^{n_j} \int_{t_{k_j}}^{t_{k_j+1}} M_{11}^2 \|\widehat{y}_{k_j}\|_2^2 d\tau \tag{58}$$

where n_j is the number of scattering variables sent from agent j to agent i during the time interval $[0, t]$. Denote

$$\begin{aligned} \int_0^t \|\widetilde{\upsilon}_{ij}^+(\tau)\|_2^2 d\tau &= \sum_{k_i=0}^{n_i} \int_{t_{k_i}}^{t_{k_i+1}} M_{11}^2 \|\widehat{y}_{k_i}\|_2^2 d\tau \ , \\ \int_0^t \|\widetilde{\upsilon}_{ji}^-(\tau)\|_2^2 d\tau &= \sum_{k_j=0}^{n_j} \int_{t_{k_j}}^{t_{k_j+1}} M_{11}^2 \|\widehat{y}_{k_j}\|_2^2 d\tau \ . \end{aligned} \tag{59}$$

If we let $\widehat{n}_i$ denote the number of scattering variables received by agent j during the time interval $[0, t]$, then we can obtain

$$\int_0^t \|\upsilon_{ji}^+(\tau)\|_2^2 d\tau = \sum_{k_i=0}^{\widehat{n}_i} \delta(t - t_{k_i} - T_{ij}) M_{11}^2 \|\widehat{y}_{k_i}\|_2^2. \tag{60}$$

Note that due to delay T_{ij} from agent i to agent j, we have $\widehat{n}_i < n_i$. Since $\widetilde{\upsilon}_{ji}^+(t)$ holds the last sample of $\upsilon_{ji}^+(t)$, we have

$$\widetilde{\upsilon}_{ji}^+(t) = M_{11}\widehat{y}_{k_i}, \text{ for } t \in [t_{k_i} + T_{ij}, t_{k_i+1} + T_{ij}], \tag{61}$$

therefore

$$\begin{aligned} \int_0^t \|\widetilde{\upsilon}_{ji}^+(\tau)\|_2^2 d\tau &= \sum_{k_i=0}^{\widehat{n}_i} \int_{t_{k_i}+T_{ij}}^{t_{k_i+1}+T_{ij}} \|M_{11}\widehat{y}_{k_i}\|_2^2 d\tau \\ &= \sum_{k_i=0}^{\widehat{n}_i} \int_{t_{k_i}}^{t_{k_i+1}} M_{11}^2 \|\widehat{y}_{k_i}\|_2^2 d\tau. \end{aligned} \tag{62}$$

Similarly, since $\widetilde{\upsilon}_{ij}^-(t)$ holds the last sample of $\upsilon_{ij}^-(t)$, we can get

$$\widetilde{\upsilon}_{ij}^-(t) = M_{11}\widehat{y}_{k_j}, \text{ for } t \in [t_{k_j} + T_{ji}, t_{k_j+1} + T_{ji}], \tag{63}$$

therefore

$$\begin{aligned} \int_0^t \|\widetilde{\upsilon}_{ij}^-(\tau)\|_2^2 d\tau &= \sum_{k_j=0}^{\widehat{n}_j} \int_{t_{k_j}+T_{ji}}^{t_{k_j+1}+T_{ji}} \|M_{11}\widehat{y}_{k_j}\|_2^2 d\tau \\ &= \sum_{k_j=0}^{\widehat{n}_j} \int_{t_{k_j}}^{t_{k_j+1}} M_{11}^2 \|\widehat{y}_{k_j}\|_2^2 d\tau. \end{aligned} \tag{64}$$

Since $n_i \ge \widehat{n}_i$ and $n_j \ge \widehat{n}_j$, thus we have

$$\int_0^t \|\widetilde{\upsilon}_{ij}^+(\tau)\|_2^2 d\tau - \int_0^t \|\widetilde{\upsilon}_{ji}^+(\tau)\|_2^2 d\tau + \int_0^t \|\widetilde{\upsilon}_{ji}^-(\tau)\|_2^2 d\tau - \int_0^t \|\widetilde{\upsilon}_{ij}^-(\tau)\|_2^2 d\tau \ge 0. \tag{65}$$

Since for $t \in [t_{k_j}, t_{k_j+1}]$, we have

$$\begin{aligned} &\frac{1}{M_{22}}\widetilde{\upsilon}_{ji}^{+}(t) - \frac{M_{21}}{M_{22}}\widehat{y}_{k_j} = a\big[y_{is}(t) - \widehat{y}_{k_j}\big] \\ &\Rightarrow \widetilde{\upsilon}_{ji}^{+}(t) = \big[M_{21}\widehat{y}_{k_j} + M_{22}a\big(y_{is}(t) - \widehat{y}_{k_j}\big)\big], \end{aligned} \tag{66}$$

and for $t \in [t_{k_i}, t_{k_i+1}]$, we have

$$\begin{aligned} &\frac{1}{M_{22}}\widetilde{\upsilon}_{ij}^{-}(t) - \frac{M_{21}}{M_{22}}\widehat{y}_{k_i} = a\big[y_{js}(t) - \widehat{y}_{k_i}\big] \\ &\Rightarrow \widetilde{\upsilon}_{ij}^{-}(t) = \big[M_{21}\widehat{y}_{k_i} + M_{22}a\big(y_{js}(t) - \widehat{y}_{k_i}\big)\big], \end{aligned} \tag{67}$$

therefore

$$\begin{aligned} \int_0^t \big\|\widetilde{\upsilon}_{ji}^{+}(\tau)\big\|_2^2 d\tau &= \sum_{k_j=0}^{n_j}\int_{t_{k_j}}^{t_{k_j+1}} \big\|M_{21}\widehat{y}_{k_j} + M_{22}a\big(y_{is}(t) - \widehat{y}_{k_j}\big)\big\|_2^2 d\tau \\ \int_0^t \big\|\widetilde{\upsilon}_{ij}^{-}(\tau)\big\|_2^2 d\tau &= \sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}} \big\|M_{21}\widehat{y}_{k_i} + M_{22}a\big(y_{js}(t) - \widehat{y}_{k_i}\big)\big\|_2^2 d\tau, \end{aligned} \tag{68}$$

with $M_{11} = M_{21} = \frac{\sqrt{a}}{2}$ and $M_{22} = \frac{1}{\sqrt{a}}$, we can get

$$\begin{aligned} \int_0^t \big\|\widetilde{\upsilon}_{ji}^{+}(\tau)\big\|_2^2 d\tau &= \sum_{k_j=0}^{n_j}\int_{t_{k_j}}^{t_{k_j+1}} \Big[\frac{a}{4}\big\|\widehat{y}_{k_j}\big\|_2^2 - a y_{is}^T\widehat{y}_{k_j} + a\big\|y_{is}\big\|_2^2\Big] d\tau \\ \int_0^t \big\|\widetilde{\upsilon}_{ij}^{-}(\tau)\big\|_2^2 d\tau &= \sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}} \Big[\frac{a}{4}\big\|\widehat{y}_{k_i}\big\|_2^2 - a y_{js}^T\widehat{y}_{k_i} + a\big\|y_{js}\big\|_2^2\Big] d\tau \\ \int_0^t \|\widetilde{\upsilon}_{ij}^{+}(\tau)\|_2^2 d\tau &= \sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}} \frac{a}{4}\big\|\widehat{y}_{k_i}\big\|_2^2 d\tau \\ \int_0^t \|\widetilde{\upsilon}_{ji}^{-}(\tau)\|_2^2 d\tau &= \sum_{k_j=0}^{n_j}\int_{t_{k_j}}^{t_{k_j+1}} \frac{a}{4}\big\|\widehat{y}_{k_j}\big\|_2^2 d\tau, \end{aligned} \tag{69}$$

thus if we define

$$V^{ij} = \int_0^t \big\|\widetilde{\upsilon}_{ij}^{+}(\tau)\big\|_2^2 d\tau - \int_0^t \big\|\widetilde{\upsilon}_{ji}^{+}\big\|_2^2 d\tau + \int_0^t \big\|\widetilde{\upsilon}_{ji}^{-}(\tau)\big\|_2^2 d\tau - \int_0^t \big\|\widetilde{\upsilon}_{ij}^{-}(\tau)\big\|_2^2 d\tau, \tag{70}$$

then we can get

$$\begin{aligned} V^{ij} &= \sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}} \Big[a y_{js}^T(\tau)\widehat{y}_{k_i} - a\big\|y_{js}(\tau)\big\|_2^2\Big] d\tau \\ &+ \sum_{k_j=0}^{n_j}\int_{t_{k_j}}^{t_{k_j+1}} \Big[a y_{is}^T(\tau)\widehat{y}_{k_j} - a\big\|y_{is}(\tau)\big\|_2^2\Big] d\tau. \end{aligned} \tag{71}$$

Consider a storage function for the multi-agent system given by

$$V = \sum_{i=1}^{N} V_i + \frac{1}{2}\sum_{(i,j)\in E(G)} V^{ij}, \tag{72}$$

where V_i is the storage function of agent i, such that $\dot{V}_i = u_i^T(t)y_i(t)$, $\forall t \geq 0$. Since

$$\begin{aligned} \sum_{i=1}^{N} V_i &= \sum_{i=1}^{N}\sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}} u_i^T(\tau)y_i(\tau)d\tau \\ &= \sum_{i=1}^{N}\sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}} a\sum_{j\in\mathcal{N}_i}\Big[y_{js}(t) - \widehat{y}_{k_i}\Big]^T\Big[e_i(\tau) + \widehat{y}_{k_i}\Big] d\tau \end{aligned} \tag{73}$$

and

$$\frac{1}{2}\sum_{(i,j)\in E(G)} V^{ij} = \sum_{i=1}^{N}\sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}} a\sum_{j\in\mathcal{N}_i}\Big[y_{js}^T(\tau)\widehat{y}_{k_i} - \big\|y_{js}(\tau)\big\|_2^2\Big]d\tau, \tag{74}$$

therefore, we further get

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N} a\sum_{j\in\mathcal{N}_i}\big[y_{js}^T(t)-\widehat{y}_{k_i}\big]^T e_i(t) + \sum_{i=1}^{N} a\sum_{j\in\mathcal{N}_i}\big[y_{js}^T(t)-\widehat{y}_{k_i}\big]^T\widehat{y}_{k_i}\\ &\quad+\sum_{i=1}^{N} a\sum_{j\in\mathcal{N}_i}\big[y_{js}^T(t)\widehat{y}_{k_i} - \|y_{js}(t)\|_2^2\big]\\ &= \sum_{i=1}^{N} a\sum_{j\in\mathcal{N}_i}\big[y_{js}^T(t)-\widehat{y}_{k_i}\big]^T e_i(t) - \sum_{i=1}^{N} a\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t)-\widehat{y}_{k_i}\big\|_2^2\\ &\le \sum_{i=1}^{N} a\big\|e_i(t)\big\|_2\Big\|\sum_{j\in\mathcal{N}_i}\big[y_{js}(t)-\widehat{y}_{k_i}\big]\Big\|_2 - \sum_{i=1}^{N} a\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t)-\widehat{y}_{k_i}\big\|_2^2,\end{aligned} \tag{75}$$

so if

$$\|e_i(t)\|_2 \le \frac{\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t)-\widehat{y}_{k_i}\big\|_2^2}{\big\|\sum_{j\in\mathcal{N}_i}\big[y_{js}(t)-\widehat{y}_{k_i}\big]\big\|_2},\ \forall t\ge 0, \tag{76}$$

then $\dot{V}\le 0$. Notice that the triggering condition (41) guarantees that (76) holds. Invoking LaSalle's Invariance principle[1], we can conclude that $\lim_{t\to\infty}\dot{V}=0$, thus we can further conclude that

$$\lim_{t\to\infty}\big[y_{js}(t)-\widehat{y}_{k_i}\big] = \lim_{t\to\infty}\big[y_{is}(t)-\widehat{y}_{k_j}\big] = 0,\quad \forall (i,j)\in E(G). \tag{77}$$

Under the triggering condition (41), (76) and (77) also implies that

$$\lim_{t\to\infty} e_i(t) = \lim_{t\to\infty}\big[y_i(t)-\widehat{y}_{k_i}\big] = 0,\ \forall i, \tag{78}$$

which yields

$$\begin{aligned}&\lim_{t\to\infty} y_{js}(t) = \lim_{t\to\infty}\widehat{y}_{k_i} = \lim_{t\to\infty} y_i(t) \text{ and}\\ &\lim_{t\to\infty} y_{is}(t) = \lim_{t\to\infty}\widehat{y}_{k_j} = \lim_{t\to\infty} y_j(t)\end{aligned} \tag{79}$$

$\forall (i,j)\in E(G)$. Since

$$\lim_{t\to\infty} M_{11}\widehat{y}_{k_i} = \lim_{t\to\infty}\widetilde{v}_{ji}^{+}(t) = \lim_{t\to\infty}\big[M_{21}\widehat{y}_{k_j} + M_{22}a\big(y_{is}(t)-\widehat{y}_{k_j}\big)\big], \tag{80}$$

thus $\lim_{t\to\infty}\widehat{y}_{k_i} = \lim_{t\to\infty}\widehat{y}_{k_j}$. Similarly, we have

$$\lim_{t\to\infty} M_{11}\widehat{y}_{k_j} = \lim_{t\to\infty}\widetilde{v}_{ij}^{-}(t) = \lim_{t\to\infty}\big[M_{21}\widehat{y}_{k_i} + M_{22}a\big(y_{js}(t)-\widehat{y}_{k_i}\big)\big], \tag{81}$$

thus $\lim_{t\to\infty}\widehat{y}_{k_j} = \lim_{t\to\infty}\widehat{y}_{k_i}$. Under **A2** and in view of (79), we can conclude that $\lim_{t\to\infty} y_j(t) = \lim_{t\to\infty} y_i(t)$, $\forall (i,j)\in E(G)$, which completes the proof. ■

Appendix B
Proof of Theorem 4

Proof: Consider a storage function for the multi-agent system given by $V = \sum_{i=1}^{N} V_i$, where $V_i \geq 0$ is the storage function for agent H_i such that $\dot{V}_i = u_i^T(t) y_i(t), \forall t \geq 0$. Since all the agents are lossless, then we have

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]^T y_i(t) \\
&= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]^T \big[\widetilde{e}_i(t) + q(\widehat{y}_{k_i})\big] \\
&= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]^T \widetilde{e}_i(t) + \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]^T q(\widehat{y}_{k_i}) \\
&= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]^T \widetilde{e}_i(t) + \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a q(\widehat{y}_{k_j})^T q(\widehat{y}_{k_i}) - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a q(\widehat{y}_{k_i})^T q(\widehat{y}_{k_i}),
\end{aligned} \tag{82}$$

as the underlying information exchange graph is balanced, we have

$$\sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a q(\widehat{y}_{k_i})^T q(\widehat{y}_{k_i}) = 0.5 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a q(\widehat{y}_{k_i})^T q(\widehat{y}_{k_i}) + 0.5 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a q(\widehat{y}_{k_j})^T q(\widehat{y}_{k_j}), \tag{83}$$

and therefore, it follows that

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]^T \widetilde{e}_i(t) - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \frac{a}{2} \big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2 \\
&= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]^T \big[e_i(t) + \varepsilon_{k_i}\big] - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \frac{a}{2} \big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2 \\
&\leq \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2 \big\|e_i(t)\big\|_2 + \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2 \big\|\varepsilon_{k_i}\big\|_2 \\
&\quad - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \frac{a}{2} \big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2 \\
&\leq \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2 \big\|e_i(t)\big\|_2 + \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \frac{a}{2\beta} \big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2 \\
&\quad + \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \frac{a\beta}{2} \big\|\varepsilon_{k_i}\big\|_2^2 - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \frac{a}{2} \big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2, \text{ where } \beta > 0,
\end{aligned} \tag{84}$$

choose $0 < \kappa < 1$ such that $1 < \frac{1}{1-\kappa} < \beta$, then we can further get

$$\begin{aligned}
\dot{V} \leq& \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2 \big\|e_i(t)\big\|_2 - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \left[\frac{a(1-\kappa)}{2} - \frac{a}{2\beta}\right] \big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2 \\
&+ \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \frac{a\beta}{2} \big\|\varepsilon_{k_i}\big\|_2^2 - \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \frac{a\kappa}{2} \big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2,
\end{aligned} \tag{85}$$

so if we can guarantee that

$$\big\|e_i(t)\big\|_2 \leq \frac{\sum_{j \in \mathcal{N}_i} \left(\frac{1-\kappa}{2} - \frac{1}{2\beta}\right) \|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\|_2^2}{\sum_{j \in \mathcal{N}_i} \big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2}, \ \forall t \geq 0, \tag{86}$$

then we will have

$$\dot{V} \le \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\beta}{2}\big\|\varepsilon_{k_i}\big\|_2^2 - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\kappa}{2}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2,\ \forall t\ge 0. \tag{87}$$

Note that

$$\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2 \ge \frac{1}{|\mathcal{N}_i|}\Big(\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2\Big)^2,$$

thus

$$\begin{aligned}\frac{\sum_{j\in\mathcal{N}_i}\big[\frac{(1-\kappa)}{2}-\frac{1}{2\beta}\big]\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\|_2^2}{\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2} &\ge \frac{\big(\frac{1-\kappa}{2}-\frac{1}{2\beta}\big)\frac{1}{|\mathcal{N}_i|}\Big(\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2\Big)^2}{\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2} \\ &= \big(\frac{1-\kappa}{2}-\frac{1}{2\beta}\big)\frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2,\end{aligned} \tag{88}$$

so if

$$\|e_i(t)\|_2 \le \big(\frac{1-\kappa}{2}-\frac{1}{2\beta}\big)\frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2,\ \forall t\ge 0, \tag{89}$$

then (86) holds and so does (87). Note that the triggering condition (48) actually guarantees that (89) is satisfied. In view of (89), we can further get

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|e_i(t)\|_2 = \sum_{i=1}^{N}|\mathcal{N}_i|\|e_i(t)\|_2 \le \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big(\frac{1-\kappa}{2}-\frac{1}{2\beta}\big)\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2 \tag{90}$$

since $\widetilde{e}_i(t) = e_i(t) + \varepsilon_{k_i}$, we can get

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|\widetilde{e}_i(t)\|_2 &\le \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|e_i(t)\|_2 + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|\varepsilon_{k_i}\|_2 = \sum_{i=1}^{N}|\mathcal{N}_i|\|e_i(t)\|_2 + \sum_{i=1}^{N}|\mathcal{N}_i|\|\varepsilon_{k_i}\|_2 \\ &\le \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big(\frac{1-\kappa}{2}-\frac{1}{2\beta}\big)\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2 + \sum_{i=1}^{N}|\mathcal{N}_i|\|\varepsilon_{k_i}\|_2.\end{aligned} \tag{91}$$

Since the underlying information graph is balanced, we have

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|\widetilde{e}_i(t)\|_2 = \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|\widetilde{e}_j(t)\|_2. \tag{92}$$

Now, let's integrate both sides of (87) from t_0 to t, $\forall t\ge t_0\ge 0$, then we will get

$$V(x_t) - V(x_{t_0}) \le \int_{t_0}^{t}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\beta}{2}\|\varepsilon_{k_i}\|_2^2 d\tau - \int_{t_0}^{t}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\kappa}{2}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2 d\tau, \tag{93}$$

thus

$$0 \le V(x_t) \le \int_{t_0}^{t}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\beta}{2}\|\varepsilon_{k_i}\|_2^2 d\tau - \int_{t_0}^{t}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\kappa}{2}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2 d\tau + V(x_{t_0}). \tag{94}$$

Since we can arbitrarily choose $t\ge t_0$, we can conclude that

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\beta}{2}\|\varepsilon_{k_i}\|_2^2 + V(x_{t_0}) \ge \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\kappa}{2}\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2,\ \forall t\ge t_0. \tag{95}$$

Moreover, since

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\kappa}{2}\big\|q(\widehat{y}_{k_j})-q(\widehat{y}_{k_i})\big\|_2^2 &\geq \sum_{i=1}^{N}\frac{a\kappa}{2|\mathcal{N}_i|}\Big(\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j})-q(\widehat{y}_{k_i})\big\|_2\Big)^2\\ &\geq \frac{1}{N}\Big(\sum_{i=1}^{N}\sqrt{\frac{a\kappa}{2|\mathcal{N}_i|}}\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j})-q(\widehat{y}_{k_i})\big\|_2\Big)^2,\end{aligned}\tag{96}$$

we have

$$\begin{aligned}\frac{1}{N}\Big(\sum_{i=1}^{N}\sqrt{\frac{a\kappa}{2|\mathcal{N}_i|}}\sum_{j\in\mathcal{N}_i}\big\|q(\widehat{y}_{k_j})-q(\widehat{y}_{k_i})\big\|_2\Big)^2 &\leq \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\beta}{2}\|\varepsilon_{k_i}\|_2^2+V(x_{t_0})\\ &=\sum_{i=1}^{N}\frac{a\beta}{2}|\mathcal{N}_i|\|\varepsilon_{k_i}\|_2^2+V(x_{t_0}).\end{aligned}\tag{97}$$

Taking the square root of both sides of (97) yields

$$\sqrt{\frac{a\kappa}{2N}}\sum_{i=1}^{N}\frac{1}{\sqrt{|\mathcal{N}_i|}}\sum_{j\in\mathcal{N}_i}\|q(\widehat{y}_{k_j})-q(\widehat{y}_{k_i})\|_2\leq\sum_{i=1}^{N}\sqrt{|\mathcal{N}_i|}\sqrt{\frac{a\beta}{2}}\|\varepsilon_{k_i}\|_2+\sqrt{V(x_{t_0})}.\tag{98}$$

Let $N_m=\max_i\{|\mathcal{N}_i|\}$, then (98) also implies

$$\frac{1}{\sqrt{N_m}}\sqrt{\frac{a\kappa}{2N}}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|q(\widehat{y}_{k_j})-q(\widehat{y}_{k_i})\|_2\leq\sum_{i=1}^{N}\sqrt{|\mathcal{N}_i|}\sqrt{\frac{a\beta}{2}}\|\varepsilon_{k_i}\|_2+\sqrt{V(x_{t_0})},\ \text{or}$$

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|q(\widehat{y}_{k_j})-q(\widehat{y}_{k_i})\|_2\leq\sqrt{\frac{\beta NN_m}{\kappa}}\sum_{i=1}^{N}\sqrt{|\mathcal{N}_i|}\|\varepsilon_{k_i}\|_2+\sqrt{\frac{2NN_m}{a\kappa}}\sqrt{V(x_{t_0})}.\tag{99}$$

Since $\big\|q(\widehat{y}_{k_j})-q(\widehat{y}_{k_i})\big\|_2=\big\|y_j(t)-\widetilde{e}_j(t)-y_i(t)+\widetilde{e}_i(t)\big\|_2\geq\big\|y_j(t)-y_i(t)\big\|_2-\|\widetilde{e}_i(t)\|_2-\|\widetilde{e}_j(t)\|_2$, replace it into (99), we can get

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|y_j(t)-y_i(t)\|_2\leq{}&\sqrt{\frac{\beta NN_m}{\kappa}}\sum_{i=1}^{N}\sqrt{|\mathcal{N}_i|}\|\varepsilon_{k_i}\|_2+\sqrt{\frac{2NN_m}{a\kappa}}\sqrt{V(x_{t_0})}\\ &+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|\widetilde{e}_i(t)\|_2+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|\widetilde{e}_j(t)\|_2,\end{aligned}\tag{100}$$

in view of (91) and (92), we get

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|y_j(t)-y_i(t)\|_2\leq{}&\sqrt{\frac{\beta NN_m}{\kappa}}\sum_{i=1}^{N}\sqrt{|\mathcal{N}_i|}\|\varepsilon_{k_i}\|_2+\sqrt{\frac{2NN_m}{a\kappa}}\sqrt{V(x_{t_0})}\\ &+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big(1-\kappa-\frac{1}{\beta}\big)\big\|q(\widehat{y}_{k_j})-q(\widehat{y}_{k_i})\big\|_2+\sum_{i=1}^{N}2|\mathcal{N}_i|\|\varepsilon_{k_i}\|_2.\end{aligned}\tag{101}$$

In view of (99), (101) further yields

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\|y_j(t)-y_i(t)\|_2\leq{}&\sum_{i=1}^{N}\Big[(2-\kappa-\frac{1}{\beta})\sqrt{\frac{\beta NN_m}{\kappa}}\sqrt{|\mathcal{N}_i|}+2|\mathcal{N}_i|\Big]\|\varepsilon_{k_i}\|_2\\ &+\big(2-\kappa-\frac{1}{\beta}\big)\sqrt{\frac{2NN_m}{a\kappa}}\sqrt{V(x_{t_0})},\ \forall t\geq t_0.\end{aligned}\tag{102}$$

(102) implies that the output synchronization error in the multi-agent system is bounded by the quantization errors of agents' latest transmitted output information by the time t. The proof is completed. ■

APPENDIX C
PROOF OF THEOREM 5

Proof: Based on Lemma 1, choose $V_i(x_i) = \int_0^{x_i} q(\sigma)d\sigma$ as the storage function for each agent, then we have $\dot{V}_i = u_i(t)q(y_i(t)),\ \forall t \geq 0$. Consider a storage function for the multi-agent system given by $V = \sum_{i=1}^{N} V_i$, then we have

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N} u_i(t)y_i(t) = \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]\left[\varepsilon_i(t) + q(\widehat{y}_{k_i})\right] \\ &= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]\varepsilon_i(t) + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]q(\widehat{y}_{k_i}) \\ &= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]\varepsilon_i(t) + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} aq(\widehat{y}_{k_i})q(\widehat{y}_{k_j}) - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} aq(\widehat{y}_{k_i})^2,\end{aligned} \tag{103}$$

as the underlying information exchange graph is balanced, we have

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} aq(\widehat{y}_{k_i})^2 = 0.5\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} aq(\widehat{y}_{k_i})^2 + 0.5\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} aq(\widehat{y}_{k_j})^2,$$

and therefore it follows that

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]\varepsilon_i(t) - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} 0.5a\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2 \\ &\leq \sum_{i=1}^{N}\big\|\varepsilon_i(t)\big\|_2\Big\|\sum_{j\in\mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]\Big\|_2 - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} 0.5a\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2^2,\end{aligned} \tag{104}$$

so if we can guarantee that

$$\|\varepsilon_i(t)\|_2 \leq \frac{\sum_{j\in\mathcal{N}_i} 0.5a\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2}{\big\|\sum_{j\in\mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]\big\|_2} = \frac{\sum_{j\in\mathcal{N}_i} 0.5\big\|q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big\|_2}{\big\|\sum_{j\in\mathcal{N}_i}\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]\big\|_2}, \quad \forall t \geq 0, \tag{105}$$

then we will have $\dot{V} \leq 0,\ \ \forall t \geq 0$. Note that the triggering condition (49) actually guarantees that (105) is satisfied. Moreover, we can rewrite (103) as

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big[q(\widehat{y}_{k_j}) - q(\widehat{y}_{k_i})\big]q(y_i(t)) \\ &= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big[q(y_j(t)) - \varepsilon_j(t) - q(y_i(t)) + \varepsilon_i(t)\big]q(y_i(t)) \\ &= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big[q(y_j(t)) - q(y_i(t))\big] - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big[\varepsilon_j(t) - \varepsilon_i(t)\big]q(y_i(t)) \\ &= -q(Y)^T Lq(Y) + \widetilde{E}^T Lq(Y),\end{aligned} \tag{106}$$

where $Y = [y_1, y_2, \ldots, y_N]^T$, $\widetilde{E} = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N]^T$, $q(\cdot)$ acts component wise on the vector Y, and L is the graph Laplacian of the underlying information exchange graph. Since under the triggering condition, we have $\dot{V} \leq 0$, in view of (106), based on LaSalle's Invariance Principle and assumption **A1**, $\dot{V} \leq 0$ also implies that

$$\lim_{t\to\infty}\big[q(y_i(t)) - q(y_j(t))\big] = 0, \quad \forall (i,j) \in E(G). \tag{107}$$

Further more, since

$$\frac{1}{N}\sum_{i=1}^{N}\dot{x}_i(t)=\frac{1}{N}\sum_{i=1}^{N}u_i(t)=\frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}a\big[q(\widehat{y}_{k_j})-q(\widehat{y}_{k_i})\big]$$
$$=\frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}a\big[q(x_j(t))-q(x_i(t))\big]-\frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}a\big[\varepsilon_j(t)-\varepsilon_i(t)\big], \tag{108}$$

under assumption **A1**, we have

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}a\big[q(x_j(t))-q(x_i(t))\big]=0, \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}a\big[\varepsilon_j(t)-\varepsilon_i(t)\big]=0,$$

thus $\frac{1}{N}\sum_{i=1}^{N}\dot{x}_i(t)=0$ and $\frac{1}{N}\sum_{i=1}^{N}x_i(t)=\sum_{i=1}^{N}x_i(0)=0$, $\forall t\geq 0$. In view of (107), we can further conclude that the state of each agent will converge to a value around their initial average asymptotically. The proof is completed. ■

Appendix D
Proof of Theorem 6

Proof: With T_{ij} and T_{ji} being constant and finite, one can verify that

$$\int_0^t \|\upsilon_{ji}^{+}(\tau)\|_2^2 d\tau \leq \int_0^t \|\upsilon_{ij}^{+}(\tau)\|_2^2 d\tau \quad \text{and} \quad \int_0^t \|\upsilon_{ij}^{-}(\tau)\|_2^2 d\tau \leq \int_0^t \|\upsilon_{ji}^{-}(\tau)\|_2^2 d\tau.$$

Since

$$\int_0^t \|\upsilon_{ij}^{+}(\tau)\|_2^2 d\tau=\sum_{k_i=0}^{n_i}\delta(t-t_{k_i})\|q(M_{11}\widehat{y}_{k_i})\|_2^2\leq\sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}}\|q(M_{11}\widehat{y}_{k_i})\|_2^2 d\tau,$$

where $\delta(\cdot)$ is the Dirac delta function, n_i is the number of scattering variables sent from agent i to agent j during the time interval $[0,t]$. Similarly, one can get

$$\int_0^t \|\upsilon_{ji}^{-}(\tau)\|_2^2 d\tau=\sum_{k_j=0}^{n_j}\delta(t-t_{k_j})\|q(M_{11}\widehat{y}_{k_j})\|_2^2\leq\sum_{k_j=0}^{n_j}\int_{t_{k_j}}^{t_{k_j+1}}\|q(M_{11}\widehat{y}_{k_j})\|_2^2 d\tau,$$

where n_j is the number of scattering variables sent from agent j to agent i during the time interval $[0,t]$. Let's denote

$$\int_0^t \big\|\widetilde{\upsilon}_{ij}^{+}(\tau)\big\|_2^2 d\tau=\sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}}\big\|q(M_{11}\widehat{y}_{k_i})\big\|_2^2 d\tau,$$
$$\int_0^t \big\|\widetilde{\upsilon}_{ji}^{-}(\tau)\big\|_2^2 d\tau=\sum_{k_j=0}^{n_j}\int_{t_{k_j}}^{t_{k_j+1}}\big\|q(M_{11}\widehat{y}_{k_j})\big\|_2^2 d\tau,$$

let $\widehat{n}_i$ denote the number of scattering variables received by agent j during the time interval $[0,t]$, then we have

$$\int_0^t \big\|\upsilon_{ji}^{+}(\tau)\big\|_2^2 d\tau=\sum_{k_i=0}^{\widehat{n}_i}\delta(t-t_{k_i}-T_{ij})\big\|q(M_{11}\widehat{y}_{k_i})\big\|_2^2. \tag{109}$$

Note that due to the delay T_{ij} from agent i to agent j, we have $\widehat{n}_i<n_i$. Since $\widetilde{\upsilon}_{ji}^{+}(t)$ holds the last sample of $\upsilon_{ji}^{+}(t)$, we have

$$\widetilde{\upsilon}_{ji}^{+}(t)=q(M_{11}\widehat{y}_{k_i}), \quad \text{for } t\in[t_{k_i}+T_{ij},t_{k_i+1}+T_{ij}],$$

therefore

$$\int_0^t \big\|\widetilde{\upsilon}_{ji}^{+}(\tau)\big\|_2^2 d\tau=\sum_{k_i=0}^{\widehat{n}_i}\int_{t_{k_i}+T_{ij}}^{t_{k_i+1}+T_{ij}}\big\|q(M_{11}\widehat{y}_{k_i})\big\|_2^2 d\tau=\sum_{k_i=0}^{\widehat{n}_i}\int_{t_{k_i}}^{t_{k_i+1}}\big\|q(M_{11}\widehat{y}_{k_i})\big\|_2^2 d\tau.$$

Similarly, since $\widetilde{v}_{ij}^{-}(t)$ holds the last sample of $v_{ij}^{-}(t)$, we have

$$\widetilde{v}_{ij}^{-}(t) = q(M_{11}\widehat{y}_{k_j}), \quad \text{for } t \in [t_{k_j} + T_{ji}, t_{k_j+1} + T_{ji}],$$

therefore

$$\int_0^t \left\|\widetilde{v}_{ij}^{-}(\tau)\right\|_2^2 d\tau = \sum_{k_j=0}^{\widehat{n}_j} \int_{t_{k_j}+T_{ji}}^{t_{k_j+1}+T_{ji}} \left\|q(M_{11}\widehat{y}_{k_j})\right\|_2^2 d\tau = \sum_{k_j=0}^{\widehat{n}_j} \int_{t_{k_j}}^{t_{k_j+1}} \left\|q(M_{11}\widehat{y}_{k_j})\right\|_2^2 d\tau.$$

Since $n_i \geq \widehat{n}_i$ and $n_j \geq \widehat{n}_j$, we have

$$\int_0^t \left\|\widetilde{v}_{ij}^{+}(\tau)\right\|_2^2 d\tau - \int_0^t \left\|\widetilde{v}_{ji}^{+}(\tau)\right\|_2^2 d\tau + \int_0^t \left\|\widetilde{v}_{ji}^{-}(\tau)\right\|_2^2 d\tau - \int_0^t \left\|\widetilde{v}_{ij}^{-}(\tau)\right\|_2^2 d\tau \geq 0.$$

Moreover, based on the scattering transformation (52) and (53), we have

$$\begin{aligned} \widetilde{v}_{ij}^{-}(t) &= q(M_{21}\widehat{y}_{k_i}) + aM_{22}\big[y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big], \text{ for } t \in [t_{k_i}, t_{k_i+1}) \\ \widetilde{v}_{ji}^{+}(t) &= q(M_{21}\widehat{y}_{k_j}) + aM_{22}\big[y_{is}(t) - q(M_{21}\widehat{y}_{k_j})\big], \text{ for } t \in [t_{k_j}, t_{k_j+1}), \end{aligned} \tag{110}$$

therefore

$$\begin{aligned} \int_0^t \left\|\widetilde{v}_{ij}^{-}(\tau)\right\|_2^2 d\tau &= \sum_{k_i=0}^{n_i} \int_{t_{k_i}}^{t_{k_i+1}} \left\|q(M_{21}\widehat{y}_{k_i}) + aM_{22}\big[y_{js}(\tau) - q(M_{21}\widehat{y}_{k_i})\big]\right\|_2^2 d\tau \\ &= \sum_{k_i=0}^{n_i} \int_{t_{k_i}}^{t_{k_i+1}} \Big[(1 - 2aM_{22} + a^2M_{22}^2)\left\|q(M_{21}\widehat{y}_{k_i})\right\|_2^2 \\ &\quad - (2a^2M_{22}^2 - 2aM_{22})q^T(M_{21}\widehat{y}_{k_i})y_{js}(\tau) + a^2M_{22}^2\left\|y_{js}(\tau)\right\|_2^2\Big] d\tau. \end{aligned} \tag{111}$$

Similarly, we can get

$$\begin{aligned} \int_0^t \left\|\widetilde{v}_{ji}^{+}(\tau)\right\|_2^2 d\tau &= \sum_{k_j=0}^{n_j} \int_{t_{k_j}}^{t_{k_j+1}} \Big[(1 - 2aM_{22} + a^2M_{22}^2)\left\|q(M_{21}\widehat{y}_{k_j})\right\|_2^2 \\ &\quad - (2a^2M_{22}^2 - 2aM_{22})q^T(M_{21}\widehat{y}_{k_j})y_{is}(\tau) + a^2M_{22}^2\left\|y_{is}(\tau)\right\|_2^2\Big] d\tau. \end{aligned} \tag{112}$$

With $M_{11} = M_{21}$, we have

$$\begin{aligned} &\int_0^t \left\|\widetilde{v}_{ij}^{+}(\tau)\right\|_2^2 d\tau - \int_0^t \left\|\widetilde{v}_{ji}^{+}(\tau)\right\|_2^2 d\tau + \int_0^t \left\|\widetilde{v}_{ji}^{-}(\tau)\right\|_2^2 d\tau - \int_0^t \left\|\widetilde{v}_{ij}^{-}(\tau)\right\|_2^2 d\tau \\ &= \sum_{k_i=0}^{n_i} \int_{t_{k_i}}^{t_{k_i+1}} \Big[(2aM_{22} - a^2M_{22}^2)\left\|q(M_{21}\widehat{y}_{k_i})\right\|_2^2 + (2a^2M_{22}^2 - 2aM_{22})q^T(M_{21}\widehat{y}_{k_i})y_{js}(\tau) \\ &- a^2M_{22}^2\left\|y_{js}(\tau)\right\|_2^2\Big] d\tau + \sum_{k_j=0}^{n_j} \int_{t_{k_j}}^{t_{k_j+1}} \Big[(2aM_{22} - a^2M_{22}^2)\left\|q(M_{21}\widehat{y}_{k_j})\right\|_2^2 \\ &+ (2a^2M_{22}^2 - 2aM_{22})q^T(M_{21}\widehat{y}_{k_j})y_{is}(\tau) - a^2M_{22}^2\left\|y_{is}(\tau)\right\|_2^2\Big] d\tau, \end{aligned} \tag{113}$$

with $aM_{22} = 2$, we can further obtain

$$\begin{aligned} V_{ij} &= \int_0^t \left\|\widetilde{v}_{ij}^{+}(\tau)\right\|_2^2 d\tau - \int_0^t \left\|\widetilde{v}_{ji}^{+}(\tau)\right\|_2^2 d\tau + \int_0^t \left\|\widetilde{v}_{ji}^{-}(\tau)\right\|_2^2 d\tau - \int_0^t \left\|\widetilde{v}_{ij}^{-}(\tau)\right\|_2^2 d\tau \\ &= \sum_{k_i=0}^{n_i} \int_{t_{k_i}}^{t_{k_i+1}} \Big[4q^T(M_{21}\widehat{y}_{k_i})y_{js}(\tau) - 4\left\|y_{js}(\tau)\right\|_2^2\Big] d\tau \\ &+ \sum_{k_j=0}^{n_j} \int_{t_{k_j}}^{t_{k_j+1}} \Big[4q^T(M_{21}\widehat{y}_{k_j})y_{is}(\tau) - 4\left\|y_{is}(\tau)\right\|_2^2\Big] d\tau, \end{aligned} \tag{114}$$

consider a storage function for the multi-agent system given by

$$\begin{aligned} V &= M_{11}\sum_{i=1}^{N} V_i + \frac{a}{8}\sum_{(i,j)\in E(G)} V_{ij} \\ &= M_{11}\sum_{i=1}^{N} V_i + \sum_{i=1}^{N}\sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}} a\sum_{j\in\mathcal{N}_i}\Big[q^T(M_{21}\widehat{y}_{k_i})y_{js}(\tau) - \big\|y_{js}(\tau)\big\|_2^2\Big]d\tau, \end{aligned} \tag{115}$$

where V_i is the storage function for agent i such that $\dot{V}_i = u_i^T(t)y_i(t),\ \forall t \geq 0$. For $t \in [t_{k_i}, t_{k_i+1}]$, let

$$\begin{aligned} \overline{e}_i(t) &= M_{11}y_i(t) - q(M_{11}\widehat{y}_{k_i}), \\ \widehat{e}_i(t) &= M_{11}\big[y_i(t) - \widehat{y}_{k_i}\big], \\ \overline{\varepsilon}_{k_i} &= M_{11}y_{k_i} - q(M_{11}\widehat{y}_{k_i}), \end{aligned} \tag{116}$$

and one can see that $\overline{e}_i(t) = \widehat{e}_i(t) + M_{11}\widehat{y}_{k_i} - q(M_{11}\widehat{y}_{k_i}) = \widehat{e}_i(t) + \overline{\varepsilon}_{k_i}$, for $t \in [t_{k_i}, t_{k_i+1}]$. So

$$\begin{aligned} M_{11}\sum_{i=1}^{N} V_i &= \sum_{i=1}^{N}\sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}} M_{11}a\sum_{j\in\mathcal{N}_i}\Big[y_{js}(\tau) - q(M_{21}\widehat{y}_{k_i})\Big]^T\frac{1}{M_{11}}\Big[\overline{e}_i(\tau) + q(M_{11}\widehat{y}_{k_i})\Big]d\tau \\ &= \sum_{i=1}^{N}\sum_{k_i=0}^{n_i}\int_{t_{k_i}}^{t_{k_i+1}} a\sum_{j\in\mathcal{N}_i}\Big[y_{js}(\tau) - q(M_{21}\widehat{y}_{k_i})\Big]^T\Big[\overline{e}_i(\tau) + q(M_{11}\widehat{y}_{k_i})\Big]d\tau \end{aligned} \tag{117}$$

with $0 < \beta < 1$, we can get

$$\begin{aligned} \dot{V} &= M_{11}\sum_{i=1}^{N}\dot{V}_i + \frac{a}{8}\sum_{(i,j)\in E(G)}\dot{V}_{ij} \\ &= \sum_{i=1}^{N} a\sum_{j\in\mathcal{N}_i}\Big[y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\Big]^T\Big[\overline{e}_i(t) + q(M_{11}\widehat{y}_{k_i})\Big] \\ &+ \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\Big[q^T(M_{21}\widehat{y}_{k_i})y_{js}(t) - \big\|y_{js}(t)\big\|_2^2\Big] \\ &= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\Big[y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\Big]^T\overline{e}_i(t) - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big\|q(M_{21}\widehat{y}_{k_i}) - y_{js}(t)\big\|_2^2 \\ &= \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\Big[y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\Big]^T[\widehat{e}_i(t) + \overline{\varepsilon}_{k_i}] - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big\|q(M_{21}\widehat{y}_{k_i}) - y_{js}(t)\big\|_2^2 \end{aligned} \tag{118}$$

thus

$$\begin{aligned} \dot{V} &\leq \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2\big\|\widehat{e}_i(t)\big\|_2 + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2\big\|\overline{\varepsilon}_{k_i}\big\|_2 \\ &- \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big\|q(M_{21}\widehat{y}_{k_i}) - y_{js}(t)\big\|_2^2 \\ &\leq \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2\|\widehat{e}_i(t)\|_2 + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a}{2\beta}\big\|\overline{\varepsilon}_{k_i}\big\|_2^2 \\ &+ \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a\beta}{2}\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2 - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big\|q(M_{21}\widehat{y}_{k_i}) - y_{js}(t)\big\|_2^2, \end{aligned} \tag{119}$$

choose $0 < \gamma < 1$, then we have

$$\begin{aligned}\dot{V} \leq& \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2\|\widehat{e}_i(t)\|_2 - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(1-\frac{\beta}{2})\gamma\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2^2 \\ &+ \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a}{2\beta}\big\|\overline{\varepsilon}_{k_i}\big\|_2 - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(1-\frac{\beta}{2})(1-\gamma)\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2^2,\end{aligned} \tag{120}$$

so of we can guarantee that

$$\big\|\widehat{e}_i(t)\big\|_2 \leq \frac{\sum_{j\in\mathcal{N}_i}(1-\frac{\beta}{2})\gamma\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2^2}{\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2},\ \forall t\in[t_{k_i}, t_{k_i+1}),\quad i = 1,2,\ldots,N, \tag{121}$$

then we will have

$$\dot{V} \leq \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a}{2\beta}\big\|\overline{\varepsilon}_{k_i}\big\|_2^2 - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(1-\frac{\beta}{2})(1-\gamma)\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2^2,\ \forall t\geq 0. \tag{122}$$

Note that

$$\begin{aligned}\frac{\sum_{j\in\mathcal{N}_i}(1-\frac{\beta}{2})\gamma\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2^2}{\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2} &\geq \frac{\frac{(1-\frac{\beta}{2})\gamma}{|\mathcal{N}_i|}\Big(\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2\Big)^2}{\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2} \\ &= \frac{(1-\frac{\beta}{2})\gamma}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2\end{aligned} \tag{123}$$

thus a sufficient condition for (121) to be hold is given by

$$\big\|\widehat{e}_i(t)\big\|_2 \leq \frac{(1-\frac{\beta}{2})\gamma}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2,\ \forall\ t\in[t_{k_i}, t_{k_i+1}),\quad i = 1,2,\ldots,N. \tag{124}$$

Note that the triggering condition (55) actually assures that (124) is satisfied. Now integrating both sides of (122) from t_0 to t, $\forall t \geq t_0$, then we have

$$V(x_t) - V(x_{t_0}) \leq \int_{t_0}^{t}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a}{2\beta}\big\|\overline{\varepsilon}_{k_i}\big\|_2^2 d\tau - \int_{t_0}^{t}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(1-\frac{\beta}{2})(1-\gamma)\big\|y_{js}(\tau) - q(M_{21}\widehat{y}_{k_j})\big\|_2^2 d\tau,$$

and

$$0 \leq V(x_t) \leq V(x_{t_0}) + \int_{t_0}^{t}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a}{2\beta}\big\|\overline{\varepsilon}_{k_i}\big\|_2^2 d\tau - \int_{t_0}^{t}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(1-\frac{\beta}{2})(1-\gamma)\big\|y_{js}(\tau) - q(M_{21}\widehat{y}_{k_j})\big\|_2^2 d\tau,$$

thus

$$\int_{t_0}^{t}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(1-\frac{\beta}{2})(1-\gamma)\big\|y_{js}(\tau) - q(M_{21}\widehat{y}_{k_j})\big\|_2^2 d\tau \leq V(x_{t_0}) + \int_{t_0}^{t}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a}{2\beta}\big\|\overline{\varepsilon}_{k_i}\big\|_2^2 d\tau, \tag{125}$$

since we can arbitrarily choose $t \geq t_0$, (125) also indicates that

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(1-\frac{\beta}{2})(1-\gamma)\big\|y_{js}(t) - q(M_{21}\widehat{y}_{k_i})\big\|_2^2 \leq V(x_{t_0}) + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a}{2\beta}\big\|\overline{\varepsilon}_{k_i}\big\|_2^2,\ \forall t\geq t_0. \tag{126}$$

Moreover, because

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a(1-\frac{\beta}{2})(1-\gamma)\big\|y_{js}(t)-q(M_{21}\widehat{y}_{k_i})\big\|_2^2 \\
&\qquad\geq \sum_{i=1}^{N}\frac{a(1-\frac{\beta}{2})(1-\gamma)}{|\mathcal{N}_i|}\Big(\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t)-q(M_{21}\widehat{y}_{k_i})\big\|_2\Big)^2 \\
&\qquad\geq \frac{a(1-\frac{\beta}{2})(1-\gamma)}{N}\Big(\sum_{i=1}^{N}\frac{1}{\sqrt{|\mathcal{N}_i|}}\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t)-q(M_{21}\widehat{y}_{k_i})\big\|_2\Big)^2
\end{aligned}\tag{127}$$

so (126) also implies

$$\frac{a(1-\frac{\beta}{2})(1-\gamma)}{N}\Big(\sum_{i=1}^{N}\frac{1}{\sqrt{|\mathcal{N}_i|}}\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t)-q(M_{21}\widehat{y}_{k_i})\big\|_2\Big)^2 \leq V(x_{t_0})+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\frac{a}{2\beta}\big\|\overline{\varepsilon}_{k_i}\big\|_2^2,\ \forall\ t\geq t_0$$

which further indicates

$$\begin{aligned}
&\sqrt{\frac{a(1-\frac{\beta}{2})(1-\gamma)}{N}}\sum_{i=1}^{N}\frac{1}{\sqrt{N_m}}\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t)-q(M_{21}\widehat{y}_{k_i})\big\|_2 \\
&\leq \sqrt{\frac{a(1-\frac{\beta}{2})(1-\gamma)}{N}}\sum_{i=1}^{N}\frac{1}{\sqrt{|\mathcal{N}_i|}}\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t)-q(M_{21}\widehat{y}_{k_i})\big\|_2 \\
&\leq \sqrt{V(x_{t_0})}+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sqrt{\frac{a}{2\beta}}\big\|\overline{\varepsilon}_{k_i}\big\|_2,\ \forall\ t\geq t_0,
\end{aligned}\tag{128}$$

where $N_m=\max_i\{|\mathcal{N}_i|\}$, or we can rewrite (128) as

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t)-q(M_{21}\widehat{y}_{k_i})\big\|_2 &\leq \sqrt{\frac{NN_m}{a(1-\frac{\beta}{2})(1-\gamma)}}\sqrt{V(x_{t_0})} \\
&\quad+\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sqrt{\frac{NN_m}{\beta(2-\beta)(1-\gamma)}}\big\|\overline{\varepsilon}_{k_i}\big\|_2,\ \forall\ t\geq t_0.
\end{aligned}\tag{129}$$

Since $aM_{22}=2$, in view of (52), we can conclude that

$$\lim_{t\to\infty}\big[y_{js}(t)-q(M_{21}\widehat{y}_{k_i})\big]=\lim_{t\to\infty}\frac{1}{2}\big[\widetilde{v}_{ij}^{-}(t)-q(M_{21}\widehat{y}_{k_i})\big]$$

where $\lim_{t\to\infty}q(M_{21}\widehat{y}_{k_i})$ could be considered as the latest transmitted scattering variable of agent i by the time $t\to\infty$. Moreover, since $\lim_{t\to\infty}\widetilde{v}_{ij}^{-}(t)=\lim_{t\to\infty}q(M_{11}\widehat{y}_{k_j})$, where $\lim_{t\to\infty}q(M_{11}\widehat{y}_{k_j})$ could be considered as the latest transmitted scattering variable of agent j by the time $t\to\infty$, we have

$$\lim_{t\to\infty}\big[y_{js}(t)-q(M_{21}\widehat{y}_{k_i})\big]=\lim_{t\to\infty}=\frac{1}{2}\big[q(M_{11}\widehat{y}_{k_j})-q(M_{21}\widehat{y}_{k_i})\big],$$

replace it into (129), we can further get

$$\begin{aligned}
&\lim_{t\to\infty}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|q(M_{11}\widehat{y}_{k_j})-q(M_{21}\widehat{y}_{k_i})\big\|_2\leq 2\sqrt{\frac{NN_m}{a(1-\frac{\beta}{2})(1-\gamma)}}\sqrt{V(x_{t_0})} \\
&+\lim_{t\to\infty}2\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sqrt{\frac{NN_m}{\beta(2-\beta)(1-\gamma)}}\big\|\overline{\varepsilon}_{k_i}\big\|_2,\ \forall\ t\geq t_0.
\end{aligned}\tag{130}$$

Note that

$$\begin{aligned}
\big\|q(M_{11}\widehat{y}_{k_j})-q(M_{21}\widehat{y}_{k_i})\big\|_2 &= \big\|M_{11}y_j(t)-\overline{e}_j(t)-M_{11}y_i(t)+\overline{e}_i(t)\big\|_2 \\
&\geq M_{11}\big\|y_j(t)-y_i(t)\big\|_2-\big\|\overline{e}_j(t)\big\|_2-\big\|\overline{e}_i(t)\big\|_2,\ \forall t\geq 0,
\end{aligned}\tag{131}$$

so we can conclude that

$$\begin{aligned}
&\lim_{t\to\infty}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|y_j(t)-y_i(t)\big\|_2 \le \lim_{t\to\infty}\frac{1}{M_{11}}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|q(M_{11}\widehat{y}_{k_j})-q(M_{21}\widehat{y}_{k_i})\big\|_2\\
&+\lim_{t\to\infty}\frac{1}{M_{11}}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|\overline{e}_j(t)\big\|_2+\lim_{t\to\infty}\frac{1}{M_{11}}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|\overline{e}_i(t)\big\|_2\\
&\le \lim_{t\to\infty}\frac{2}{M_{11}}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sqrt{\frac{NN_m}{\beta(2-\beta)(1-\gamma)}}\big\|\overline{\varepsilon}_{k_i}\big\|_2+\lim_{t\to\infty}\frac{1}{M_{11}}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|\overline{e}_j(t)\big\|_2\\
&+\lim_{t\to\infty}\frac{1}{M_{11}}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|\overline{e}_i(t)\big\|_2+\frac{2}{M_{11}}\sqrt{\frac{NN_m}{a(1-\frac{\beta}{2})(1-\gamma)}}\sqrt{V(x_{t_0})}
\end{aligned}\tag{132}$$

because the underlying information exchange graph is balanced, we have $\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|\overline{e}_j(t)\big\|_2=\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|\overline{e}_i(t)\big\|_2$, where

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|\overline{e}_i(t)\big\|_2=\sum_{i=1}^{N}|\mathcal{N}_i|\big\|\overline{e}_i(t)\big\|_2\le\sum_{i=1}^{N}|\mathcal{N}_i|\big\|\widehat{e}_i(t)\big\|_2+\sum_{i=1}^{N}|\mathcal{N}_i|\big\|\overline{\varepsilon}_{k_i}\big\|_2,$$

in view of (124), we have

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|\overline{e}_i(t)\big\|_2\le\sum_{i=1}^{N}(1-\frac{\beta}{2})\gamma\sum_{j\in\mathcal{N}_i}\big\|y_{js}(t)-q(M_{21}\widehat{y}_{k_i})\big\|_2+\sum_{i=1}^{N}|\mathcal{N}_i|\big\|\overline{\varepsilon}_{k_i}\big\|_2,$$

thus

$$\begin{aligned}
&\lim_{t\to\infty}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|\overline{e}_i(t)\big\|_2\le\frac{(1-\frac{\beta}{2})\gamma}{2}\lim_{t\to\infty}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|q(M_{11}\widehat{y}_{k_j})-q(M_{21}\widehat{y}_{k_i})\big\|_2+\sum_{i=1}^{N}|\mathcal{N}_i|\big\|\overline{\varepsilon}_{k_i}\big\|_2\\
&\le(1-\frac{\beta}{2})\gamma\sqrt{\frac{NN_m}{a(1-\frac{\beta}{2})(1-\gamma)}}\sqrt{V(x_{t_0})}+(1-\frac{\beta}{2})\gamma\sqrt{\frac{NN_m}{\beta(2-\beta)(1-\gamma)}}\lim_{t\to\infty}\sum_{i=1}^{N}|\mathcal{N}_i|\big\|\overline{\varepsilon}_{k_i}\big\|_2\\
&+\lim_{t\to\infty}\sum_{i=1}^{N}|\mathcal{N}_i|\big\|\overline{\varepsilon}_{k_i}\big\|_2,
\end{aligned}\tag{133}$$

and we can obtain

$$\begin{aligned}
\lim_{t\to\infty}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|y_j(t)-y_i(t)\big\|_2 &\le \lim_{t\to\infty}\frac{2}{M_{11}}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\big\|\overline{e}_i(t)\big\|_2+\frac{2}{M_{11}}\sqrt{\frac{NN_m}{a(1-\frac{\beta}{2})(1-\gamma)}}\sqrt{V(x_{t_0})}\\
&+\lim_{t\to\infty}\frac{2}{M_{11}}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\sqrt{\frac{NN_m}{\beta(2-\beta)(1-\gamma)}}\big\|\overline{\varepsilon}_{k_i}\big\|_2\\
&\le\frac{1}{M_{11}}\Big\{\big[2+(2-\beta)\gamma\big]\sqrt{\frac{NN_m}{\beta(2-\beta)(1-\gamma)}}+2\Big\}\lim_{t\to\infty}\sum_{i=1}^{N}|\mathcal{N}_i|\big\|\overline{\varepsilon}_{k_i}\big\|_2\\
&+\frac{(1-\beta)\gamma+2}{M_{11}}\sqrt{\frac{NN_m}{a(1-\frac{\beta}{2})(1-\gamma)}}\sqrt{V(x_{t_0})},
\end{aligned}\tag{134}$$

which shows that the output synchronization error of the studied multi-agent system is ultimately bounded by the quantization error of agents' latest transmitted sampled output information. The proof is completed. ■

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