Model-Based Event-Triggered Control for Systems with Quantization and Time-Varying Network Delays.

Eloy Garcia and Panos J. Antsaklis, *Fellow, IEEE*

Abstract

This paper combines two important control techniques for reducing communication traffic in control networks, namely, Model-Based Networked Control Systems (MB-NCS) and event-triggered control. The resulting framework is used for stabilization of uncertain dynamical systems and is extended to systems subject to quantization and time-varying network delays. The use of a model of the plant in the controller node not only generalizes the Zero-Order-Hold (ZOH) implementation in traditional event-triggered control schemes but it also provides stability thresholds that are robust to model uncertainties. The effects of quantized measurements are especially important in the selection of stabilizing thresholds. We are able to design error events based on the quantized variables that yield asymptotic stability compared to similar results in event-triggered control that consider non-quantized measurements which, in general, are not possible to use in digital computations. With respect to MB-NCS, the stability conditions presented here do not need explicit knowledge of the plant parameters as in previous work but are given only in terms of the parameters of the nominal model and some bounds in the model uncertainties. We consider the joint adverse effects of quantization and time delays and emphasize the expected tradeoff between the selection of quantization parameters and the admissible network induced delays.

Index Terms—Delay Systems; Event-triggering; Networked Control Systems; Quantization.

I. INTRODUCTION

In Networked Control Systems (NCS) a digital communication network is used to transfer information among the components of a control system. NCS can also help to improve efficiency, flexibility, and reliability of the network interconnected system reducing reconfiguration and maintenance costs [1].

Both authors are with the Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556 USA (e-mail: egarcia7@nd.edu, antsaklis.1@nd.edu)

In contrast, the protocols used to establish communication between nodes of the control systems introduce time delays and loss of information. In addition, a quantization technique is needed for all sensor measurements and control commands that are sent over a digital network. These situations force us to revaluate the tools that are commonly used in control design in order to account for limited feedback information in the analysis and design of NCS compared to traditional control systems.

Extensive research has been done in the area of NCS as described in the special issue [2] and references therein. Reducing the amount of communication between sensor and controller nodes without compromising the stability of the control system has been a topic of many papers [3]-[7]. In particular, Walsh, *et al.* [3] introduced a network control protocol Try-Once-Discard (TOD) to allocate network resources to the different nodes in a Networked Control System, all of which may access the network at any time assuming each access occurs before the Maximum Allowable Transfer Time (MATI). The work in [6]-[7] uses more efficiently the packet structure, that is, reduction on communication is obtained by sending packets of information using all data bits available (excluding overhead) in the structure of the packet.

A different way to address reduction of communication in a control network is by maximizing the time intervals that the nodes corresponding to a closed-loop control system need to send data to each other. A common characteristic among popular control networks is the presence of large overhead packets and small transport times. We can obtain better results, in general, by maximizing communication time intervals than by using data compression techniques or using all data bits in each packet.

Two important approaches that aim at extending the periods of time that a control system can operate without receiving feedback measurements are Model-Based Networked Control Systems (MB-NCS) and event-triggered control. The framework presented in this paper unifies the above techniques and offers multiple advantages compared to the individual approaches. Some of the results have appeared in [8]. In the present paper we offer complete proofs and present important extensions that deal with quantization of state measurements. This problem has received little attention using stabilizing event-triggered control techniques [29]. In this paper we consider quantization errors in the design process and find corresponding stabilizing thresholds. Additionally, we study control systems that are affected by both quantization and network induced delays and we provide error thresholds that consider these two problems and ensure asymptotic stability of the system. Although quantization is addressed in [9]-[10], these event-triggered techniques do not consider stabilization issues but limit themselves to studying the behavior of sensed signals in order to send them to the network at convenient time instants. The stabilizing technique in [29] considers only quantization but it does not consider time delays. The framework in this paper considers the problems of system uncertainty, time delays, data quantization, and resource utilization, all together, compared to prior approaches that consider some of these problems separately.

The paper is organized as follows: section II provides background on MB-NCS and event-triggered control techniques. Conditions for stability of MB-NCS using event-triggered control are presented in section III. Systems with quantization are considered in section IV. In section V we discuss stabilization of dynamical systems affected by both quantization and network induced delays. An illustrative example is given in section VI and conclusions are presented in section VII.

II. Preliminaries and Prior Work

Notation: Throughout this paper, $\mathbb{R}^n$ denotes the n dimensional Euclidean space. $|\bullet|$ refers to the Euclidean norm for vectors and to the induced 2-norm for matrices. The superscript $"T"$ denotes vector or matrix transposition. The notation Z^+ stands for the nonnegative integers. $\mathbb{R}^+$ denotes the positive real numbers and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. A function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and unbounded.

A. Model-Based Networked Control Systems.

One of the main problems in NCS which is studied in this paper is the design of control schemes accounting for the absence of feedback measurements for possibly long intervals of time. Model uncertainties are especially important to be considered under this situation. One of the attractive properties of a classical closed loop system with continuous feedback is that the appropriate design of closed loop controllers reduces sensitivity to model uncertainties. Naturally, this property is lost as feedback measurements are no longer received at the controller node. The MB-NCS approach represents an important framework that considers model uncertainties in the absence of continuous feedback. MB-NCS were introduced by Montestruque and Antsaklis [11]; this configuration makes use of an explicit model of the plant which is added to the actuator/controller node to compute the control input based on the state of the model rather than on the plant state. Fig. 1 shows the interconnection of several NCSs. In MB-NCS the actuator/controller node can be represented as in Fig. 2. We assume that the dynamics of each system depend only on its own state. Without loss of generality we will focus on a particular system/model pair. The dynamics of the plant and the model can be described respectively by:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \tag{2}$$

where $x, \hat{x} \in \mathbb{R}^n$, $u = K\hat{x}$, and the matrices $\hat{A}, \hat{B}$ represent the available model of the system matrices *A,B*. The plant may be unstable i.e. not all eigenvalues of A have negative real parts. The aim using this configuration is to operate in open-loop mode for as long as possible and using the estimated state provided by the model to generate the control input u. The state measurements take place every h seconds, i.e. h is constant, and are used to update the internal state of the model. Conditions for stability provide a range for h that can be used given the real plant, the model, and the controller parameters. In an extension, the same authors considered time-varying updates [12]. They also considered separately the network induced delays [13] and quantization [14]-[15] problems using periodic updates.

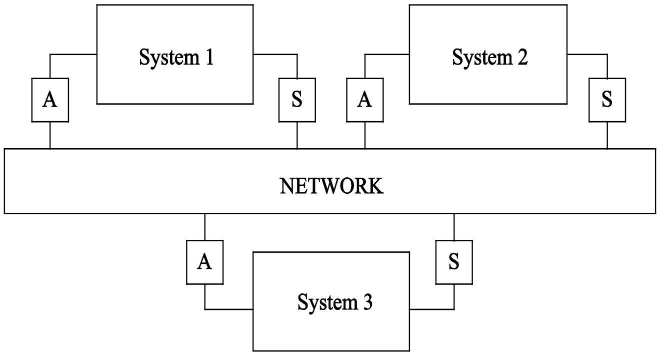

Fig. 1. Representation of Networked Control Systems. A: actuator nodes. S: sensor nodes.

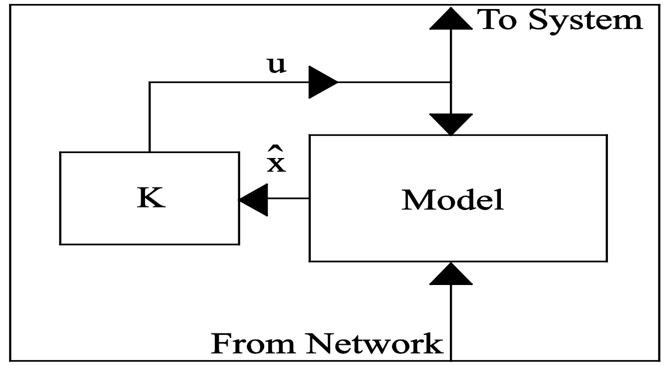

Fig. 2. Model-Based Networked Control System actuator/controller node.

In this paper we discard the periodicity assumption for updating the model. Instead we embrace a non-periodic approach that is based on events. We use the estimate of the state given by the model of the plant to compare it with the actual state. The sensor then transmits the state of the plant if the error is above some predefined tolerance. In this way the update time will be variable instead of fixed.

This approach that was introduced in [8] and that is extended in the present paper increases the time intervals that we use to update the model with respect to MB-NCS with periodic sampling by selecting the stabilizing threshold. By increasing the update interval we release the network for other uses. In case we have several control systems implemented over the network, by reducing network traffic, we are also reducing the size of time delays and reducing the probability of packet dropouts. Additionally, the conditions to select a stabilizing threshold are given only in terms of the nominal model parameters and bounds on the model uncertainties compared to previous results in MB-NCS that require explicit knowledge of the uncertainties. Finally, we use this scheme to consider the problems of network delays and quantization and how to stabilize an uncertain system in the presence of both, while the results in [14]-[15] only address the quantization problem and do not consider delays.

B. Event-triggered control.

Different authors have pursued a very intuitive framework that reduces the rate of communication among agents. In event-triggered broadcasting [16]-[33] a subsystem sends its local state to the network only when it is necessary, that is, only when a measure of the local subsystem state error is above a specified threshold. Event-triggered control schemes offer a new point of view, with respect to conventional time-driven strategies, on how information could be sampled for control purposes. One of the works that laid the foundations for this type of sampling is [16]. Tabuada [17] showed more formally the stabilizing properties of the event-triggered control strategy; he presented a triggering condition based on the norms of the state and the state error $e = x(t_i) - x(t)$, that is, the last measured state minus the current state of the system. This means that the measurement received in the controller node is held constant until a new measurement arrives. When this happens, the error is set equal to zero and starts growing until it triggers a new execution or measurement update. The use of the model to generate the control input between sampling intervals has been studied in [27]-[31]. The work in [29] deals with a similar problem, considering only quantization, and the main difference is that [29] considers external disturbances and do not consider model uncertainties. The same authors consider only delays in [30]. In contrast, we provide stability conditions considering the joint effect of both delays and quantization and in the presence of plant-model mismatch.

In all previous work on event-triggered control it is assumed that the parameters of the systems are known exactly. Our combined model-based event-triggered control framework offers an important advantage with respect to previous work in event-triggered control. The implementation of this strategy using MB-NCS accounts for the unavoidable existence of model uncertainties and its effects in control systems that operate in open-loop mode for long intervals of time. This problem has not been dealt with previously within the event-triggered approach and as it is shown it affects directly the estimated threshold values that aim to ensure stability of the system.

There exist other approaches that jointly deal with networked and quantized control systems. In particular, the work in [34] offers a general framework that applies to nonlinear systems as well. Quantization and network scheduling are considered using periodic updates (MATI). Similarly, copies of the plant dynamics are used at both ends of the communication channel. Model uncertainties are not considered, the only difference between the plant and the copies being that the copies operate using the quantized variables instead of the real ones. Our results on event-triggered strategies consider in addition robustness to model uncertainty and to network induced delays and the development is based on the MB-NCS framework.

The work in [35] presents a configuration that stabilizes a NCS with large constant delays using passivity and the scattering transformation. The works in [36] and [37] derive general models of NCSs that consider time-varying sampling intervals and delays. Although the admissible delays may be greater than the ones derived here, the authors of those papers do not consider model uncertainties which account for a conservative allowable delay in our work. In contrast, we are able to provide robustness to parameter uncertainties and quantization errors in the presence of time-varying delays.

III. STABILIZING MODEL-BASED EVENT-TRIGGERED STRATEGIES

A. A fixed threshold strategy.

We will assume in this section that the communication delay is negligible and the initial conditions of the plant are nonzero but finite. In this scheme the sensor has different functions to perform. The sensor contains a copy of the model and the controller gain so it can have access to the model state. It continuously measures the actual state and computes the model-plant state error, defined by:

$$e(t) = \hat{x}(t) - x(t). \tag{3}$$

The sensor also compares the norm of the error to a predefined threshold α, and it broadcasts the plant state to update the model state if the error is greater than the threshold.

It is clear that while $|e| \leq \alpha$ the plant is operating in open loop mode based on the model state $\hat{x}$. After substituting the input $u = K\hat{x}$ in (1) and using the definition of the error we can write:

$$\dot{x} = (A + BK)x + BKe. \tag{4}$$

In the case of the model, after substituting the input u we have a state space system of the form:

$$\dot{\hat{x}} = (\hat{A} + \hat{B}K)\hat{x}, \qquad \text{for } t_i \le t < t_{i+1}. \tag{5}$$

At the update times $t_i, i \in Z^+$, the state of the model is updated with the measurement obtained from the plant. The update intervals are non-periodic in general and are triggered by the size of the state error.

In [8] the threshold α was chosen as a constant number. By using this simple choice the state of the system can be bounded as shown in next theorem. See appendix for proof.

Theorem 1. For $|x(0)| \le \beta_1$, $0 < \beta_1 < \infty$ the system described by (4) with state feedback based on error events is bounded-input bounded-state stable with respect to the state error if the eigenvalues of $A+BK$ have negative real parts.

Note that the design of the stabilizing gain K requires a robust-type controller since we only have the nominal parameters available. Note that a specific location of the closed-loop eigenvalues is not needed in Theorem 1, they only need to be in the left-hand side of the complex plane.

B. A relative threshold strategy.

In many different applications it is desirable to asymptotically stabilize a system. It is intuitive clear that by varying the magnitude of the threshold value we can obtain longer update intervals or a smaller output size. The idea of reducing the threshold value as we approach the equilibrium point of the system is logical. The work in [17] follows this approach by comparing the norm of the state error to a function of the norm of the state of the plant; in this way the threshold can be reduced as we approach the equilibrium of the system, assuming that the zero state is the equilibrium. Previous work on event-triggered control dealt with systems controlled by static gains that generate piecewise constant inputs since the update is held constant in the controller. The main difference in this section is that we use a

Model-Based controller; the model provides an estimate of the state between updates and the model/gain controller provides an input for the plant that does not remain constant between measurement updates.

Consider again the plant and model described by (1) and (2) and by using the control input $u = K\hat{x}$ we obtain the description (4) for the plant. Assume that the control input u renders the model (2) Input-to-state stable (ISS) with respect to the measurement error e. For the definition of ISS we use the next [21]:

Definition 2. A smooth function $V:\mathbb{R}^n \to \mathbb{R}_0^+$ is said to be an ISS Lyapunov function for the dynamical system $\dot{x} = f(x,u)$, $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, t \in \mathbb{R}_0^+$ if there exist class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ and γ satisfying:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{6}$$

$$\frac{\partial V}{\partial x} f(x,u) \leq -\alpha_3(|x|) + \gamma(|u|). \tag{7}$$

The system $\dot{x} = f(x,u)$ is said to be ISS with respect to the input u if and only if there exists an ISS Lyapunov function for that system. ■

In our case, we choose a control law $u = K\hat{x}$ that renders the closed loop *model* (5) globally asymptotically stable. Any such K also renders the closed loop model Input-to-State Stable with respect to the state error. We proceed to choose a quadratic ISS-Lyapunov function, $V = x^T P x$ where P is symmetric positive definite and is the solution of the closed loop model Lyapunov function:

$$(\hat{A} + \hat{B}K)^T P + P(\hat{A} + \hat{B}K) = -Q \tag{8}$$

where Q is a symmetric positive definite matrix. Define the uncertainty matrices $\tilde{A} = A - \hat{A}$ and $\tilde{B} = B - \hat{B}$; also assume that the next bounds on the uncertainties $\left|(\tilde{A} + \tilde{B}K)^T P + P(\tilde{A} + \tilde{B}K)\right| \leq \Delta < \underline{q}$ and $\left|\tilde{B}\right| \leq \beta$ hold where $\underline{q} = \underline{\sigma}(Q)$, the smallest singular value of Q in (8). These bounds can be seen as a measure of how close the model and system parameters should be. The next theorem provides conditions on the error and its threshold value so the networked system is asymptotic stable. The error threshold is defined as a function of the norm of the state and the bounds on the uncertainty matrices. Similarly, the occurrence of an error event leads the sensor to send the current measurement of the state of the plant that is used in the controller to update the state of the model.

Theorem 3. Consider system (1) with input $u = K\hat{x}$ and the relation:

$$|e| > \frac{\sigma(\underline{q} - \Delta)}{\overline{b}}|x| \tag{9}$$

where $\overline{b} = 2\left|P\hat{B}K\right| + 2\beta|PK|$, $0 < \sigma < 1$, $\left|(\tilde{A} + \tilde{B}K)^T P + P(\tilde{A} + \tilde{B}K)\right| \le \Delta < \underline{q}$, and $\underline{q} = \underline{\sigma}(Q)$. Let the model be updated when (9) is satisfied, then the system is globally asymptotically stable.

Proof. In order to prove this theorem we will set a bound on the derivative of $V = x^T Px$ along the trajectories of the system (4) which is equal to (1) when the input $u = K\hat{x}$ has already been substituted and expressed in terms of the state error, then we can easily show that this bound can be appropriately tuned by the choice of the threshold on the error.

$$\begin{aligned} \dot{V} &= x^T[(A+BK)^T P + P(A+BK)]x + e^T K^T B^T Px + \\ &= x^T[(\hat{A} + \tilde{A} + \hat{B}K + \tilde{B}K)^T P + P(\hat{A} + \tilde{A} + \hat{B}K + \tilde{B}K)]x + 2x^T P(\hat{B} + \tilde{B})Ke \\ &= -x^T Qx + x^T[(\tilde{A} + \tilde{B}K)^T P + P(\tilde{A} + \tilde{B}K)]x + 2x^T P(\hat{B} + \tilde{B})Ke. \end{aligned} \tag{10}$$

We have just expressed $\dot{V}$ in terms of the model parameters and the uncertainty of the state matrix A. We now proceed to evaluate the contributions of each, the model, the uncertainty, and the error.

$$\dot{V} \le -\underline{q}|x|^2 + \left|(\tilde{A} + \tilde{B}K)^T P + P(\tilde{A} + \tilde{B}K)\right||x|^2 + 2\left(\left|P\hat{B}K\right| + \left|P\tilde{B}K\right|\right)|e||x| \le (-\underline{q} + \Delta)|x|^2 + \overline{b}\,|e||x|. \tag{11}$$

By updating the model according to (9) we ensure that the error becomes zero at the update instant and it satisfies $|e| \le \sigma(\underline{q} - \Delta)|x| / \overline{b}$ until a new update is generated, so we can finally write:

$$\dot{V} \le (\sigma - 1)(\underline{q} - \Delta)|x|^2. \tag{12}$$

Then V is guaranteed to decrease by updating the model state according to (9) and for $0 < \sigma < 1$. ■

Remark 1. In comparison to prior work in MB-NCS, an important advantage of this approach is that we define the controller only in terms of the model parameters. The threshold is designed using the model parameters $(\hat{A}, \hat{B})$ and some bounds on the uncertainty matrices, quantities that we specifically know.

IV. SYSTEMS WITH QUANTIZATION.

It was assumed in the last section that the sensor is able to measure the state of the system with infinite precision. Then, the sensor uses that measurement to send it through the network and to compute the event-triggered stability conditions. In reality, however, the measured variables have to be quantized in order to be represented by a finite number of bits and to be used in processor operations and carried over a digital communication network. Therefore, it becomes necessary to study the effects of quantization error on networked systems and on any computer implemented control application. In addition, we want to emphasize two important implications of quantization in event-triggered control. First, an important step in event-triggered control strategies is that the model-plant state error is set to zero at the update instants. When using quantization this step does not hold any longer since we use the quantized measurement of the plant state to update the state of the model and this measurement is not, in general, the same as the real state of the plant. Second, in previous even-triggered control techniques, the updates are triggered by comparing the norm of the state, which is not exactly available due to quantization errors, to the norm of the state error, which is not exactly available since it is a function of the real state of the plant. The problem in those approaches is that stability of the system is directly related to computations on the non-quantized measurements that are assumed to be known with certainty.

The aim is to find triggering conditions based on the available quantized variables that also ensure asymptotic stability in the presence of quantization errors. A common class of quantizers is the static one. This type of quantizer is easier to implement than their dynamic counterpart. The simple uniform quantizer is associated to fixed-point data representation. In this paper, it is also convenient to associate it to the fixed threshold event-triggered strategy (see Appendix). The type of quantizer that we use in this section is called a logarithmic quantizer. It is associated with floating-point data representations. We also associate this quantizer to the relative threshold strategy described in the previous section in order to design stabilizing thresholds using the quantized measurements of the state of the plant.

Definition 4. We define a logarithmic quantizer as a function $q:\mathbb{R}^n \to \mathbb{R}^n$ with the following property:

$$|x - q(x)| \le \delta |x|, \quad x \in \mathbb{R}^n, \quad \delta > 0. \tag{13}$$

At the update instants $t_i, i \in Z^+$ the state of the model is updated using the quantized measurement:

$$q(x(t_i)) \to \hat{x}(t_i). \tag{14}$$

Define the quantized model-plant state error:

$$e_q(t) = \hat{x}(t) - q(x(t)) \tag{15}$$

where $q(x(t))$ is the quantized value of $x(t)$ at any time $t \ge 0$ using the logarithmic quantizer (13). Note that $q(x)$ and e_q are the available variables that can be used to compute the triggering condition. Also note that $e_q(t_i) = 0$, that is, the quantized model-plant state error is set to zero at the update instants according to the update (14).

Theorem 5. Let (1) be a control system with control input based on the nominal model $u = K\hat{x}$ and assume that: there exists a symmetric positive definite solution P for the model Lyapunov equation (8) and the bounds $\left|(\tilde{A} + \tilde{B}K)^T P + P(\tilde{A} + \tilde{B}K)\right| \le \Delta < \underline{q}$ and $\left|\tilde{B}\right| \le \beta$ are satisfied. Consider the relation

$$|e_q| > \frac{\sigma\eta}{\delta + 1} |q(x)| \tag{16}$$

where $\eta = (\underline{q} - \Delta)/\overline{b}$, $0 < \sigma < \sigma' < 1$. Let the model be updated when (16) holds. Then,

$$|e| \le \sigma' \eta |x| \tag{17}$$

is always satisfied and the system is asymptotically stable when,

$$\delta \le (\sigma' - \sigma)\eta. \tag{18}$$

Proof. First observe that for the logarithmic quantizer we have:

$$|e| = |\hat{x} - x + q(x) - q(x)| \le |q(x) - x| + |e_q| \le \delta |x| + |e_q|. \tag{19}$$

Similarly,

$$|q(x)| = |q(x) + x - x| \le |q(x) - x| + |x| \le \delta |x| + |x| \quad \Rightarrow \quad \frac{|q(x)|}{\delta + 1} \le |x|. \tag{20}$$

From (20) and applying (18) we can see that:

$$\sigma\eta\frac{|q(x)|}{\delta+1}+\delta|x|\le(\sigma\eta+\delta)|x|\le\sigma'\eta|x|. \tag{21}$$

By updating the model according to (16) we ensure that e_q becomes zero at the update instant and it satisfies $|e_q|\le\sigma\eta|q(x)|/(\delta+1)$ until a new update is generated, then from (19) and (21) we obtain:

$$|e|\le\delta|x|+|e_q|\le\sigma\eta\frac{|q(x)|}{\delta+1}+\delta|x|\le\sigma'\eta|x| \tag{22}$$

then (12) is satisfied with σ', that is: $\dot{V}\le(\sigma'-1)(\underline{q}-\Delta)|x|^2$, and since $\sigma'<1$, system (1) with updates based on the error events triggered by (16) and using quantized feedback measurements to update the model is asymptotically stable. ■

V. Systems with quantization and time-varying delays

Although MB-NCS may help to reduce network induced delays we should be prepared for situations in which given peak conditions on the network produce considerable time delays between nodes. By the nature of the event-triggered strategies, it may happen that several systems attempt to access the network in order to update their corresponding controller node. In this case only one node can gain access and the rest need to wait until the network turns into an idle state. The solutions provided in previous sections assumed negligible time delays but it has been shown that the sole event-triggered control strategy is able to compensate for delays in a natural way: if some delay characteristics are known (a bound or even the exact time delay when using time-stamped messages) the next update should be scheduled before the regular one (the update when no delay is present) in such a way that stability is never compromised. In this section we take this approach along with the model dynamics in order to determine the best time to update in the presence of time delays. Two advantages are obtained by including the MB-NCS framework with respect to only using an event-triggered controller. The first one is the known property of generating an estimate of the state when operating in open loop mode to get longer update intervals.

The second advantage is that the model is able to produce almost instantaneously an estimate of the current plant state based on the delayed measurement. We can use this estimate instead of using the delayed measurement to update the model in the controller.

A. Systems with network induced delays and without quantization.

Let us first analyze the case when there exist network delays assuming that the sensor measurements can be quantized with infinite precision. When referring to the execution rule described in theorem 3 it is important to guarantee that the inter-execution update times never become too close to each other causing a model update in the controller when the previous execution has not been finished due to time delays or even resulting in a Zeno behavior. To show that this will never occur is a nontrivial task, since the execution time intervals are only implicitly defined by (9); and it is shown in the next theorem.

Theorem 6. Let (1) be a control system with control input based on the state of the nominal model $u = K\hat{x}$ and assume that: there exists a symmetric positive definite solution P for the model Lyapunov equation (8), $B = \hat{B}$, and the next bounds are satisfied: $\left|\tilde{A}\right| \le \Delta_A$ and $\left|\tilde{A}^T P + P\tilde{A}\right| \le \Delta < \underline{q}$. Then there exists $\varepsilon > 0$ such that for all network delays $\tau_N \in [0, \varepsilon]$ the system is asymptotically stable. Furthermore, there exists a time τ>0 such that for any initial condition the inter-execution times $\{t_{i+1} - t_i\}$ implicitly defined by (9) with $\sigma < 1$ are lower bounded by τ, i.e. $t_{i+1} - t_i \ge \tau \quad \forall i \in Z^+$.

Proof: In order to show asymptotic stability for the nonzero network delay case and to bound the inter-execution times let us look at the dynamics of $|e|/|x|$:

$$\frac{d}{dt}\frac{|e|}{|x|} = \frac{d}{dt}\frac{(e^T e)^{1/2}}{(x^T x)^{1/2}} = \frac{(x^T x)^{1/2}(e^T e)^{-1/2} e^T \dot{e} - (e^T e)^{1/2}(x^T x)^{-1/2} x^T \dot{x}}{x^T x}$$

$$= \frac{e^T \dot{e}}{(x^T x)^{1/2}(e^T e)^{1/2}} - \frac{(e^T e)^{1/2} x^T \dot{x}}{(x^T x)^{3/2}} = \frac{e^T(\hat{A}e - \tilde{A}x)}{|x||e|} - \frac{x^T[(\hat{A} + \tilde{A} + BK)x + BKe]}{|x||x|}\frac{|e|}{|x|}$$

$$\le \left|\tilde{A}\right| + \left|\hat{A}\right|\frac{|e|}{|x|} + \left|\hat{A} + \tilde{A} + BK\right|\frac{|e|}{|x|} + |BK|\left(\frac{|e|}{|x|}\right)^2 \le \Delta_A + (\Delta_A + \left|2\hat{A} + BK\right|)\frac{|e|}{|x|} + |BK|\left(\frac{|e|}{|x|}\right)^2. \quad (23)$$

Let us denote the term $|e|/|x|$ by θ so we have the estimate:

$$\dot{\theta} \leq \Delta_A + (\Delta_A + |2\hat{A} + BK|)\theta + |BK|\theta^2 \leq \Delta_A + |2\hat{A}| + (\Delta_A + |2\hat{A}| + |BK|)\theta + |BK|\theta^2 \tag{24}$$

and consider the differential equation:

$$\dot{\phi} = \Delta_A + |2\hat{A}| + (\Delta_A + |2\hat{A}| + |BK|)\phi + |BK|\phi^2 \tag{25}$$

then we can conclude that $\theta(t) \leq \phi(t, \phi_0)$, where $\phi(t, \phi_0)$ is the solution of (25) satisfying $\phi(0, \phi_0) = \phi_0$.

For the case when $\tau_N = 0$, the inter-execution times are bounded by the time it takes for ϕ to evolve from 0 to $\sigma(\underline{q} - \Delta)/b$, i.e. the solution $\tau \in \mathbb{R}^+$ of $\phi(\tau, 0) = \sigma(\underline{q} - \Delta)/b$. An estimate of that time can be obtained by solving (25). Such solution is given by:

$$\phi(t, 0) = \frac{-e^{dt(c-1)} + 1}{e^{dt(c-1)}/c - 1} \tag{26}$$

for $c \neq 1$. Let $y = \sigma(\underline{q} - \Delta)/b = \phi(\tau, 0)$, then

$$\tau = (\ln(y+1) - \ln(\frac{y}{c} + 1))\frac{1}{d(c-1)} \tag{27}$$

where $d = |BK|$ and $c = (\Delta_A + |2\hat{A}|)/d$. In the analysis if we have the case *c=1* we can easily avoid it by increasing the bound on the uncertainty by a very small amount. It can also be verified that $\tau > 0$ for any $y > 0$. Moreover, the range of values for τ for any positive value of the threshold y is given by $\tau \in (0, \tau_m)$, where:

$$\tau_m = \lim_{y \to \infty} \tau = \frac{\ln(c)}{d(c-1)}. \tag{28}$$

For $\tau_N > 0$, we choose some σ' such the next is satisfied $0 < \sigma < \sigma' < 1$, and let $0 < \varepsilon_1 < \tau_m$ satisfy the solution $\phi(\varepsilon_1, y) = y' = \sigma'(\underline{q} - \Delta)/b$, such ε_1 always exists since ϕ is continuous in the range $\tau \in [0, \tau_m)$ that covers all positive thresholds $0 < y, y' < \infty$, also $\dot{\phi} > 0$ and $y < y'$ since $\sigma < \sigma'$. Then, by sending the state measurement at time t_i in order to update the model in the controller, this execution is released by the condition $|e| = y|x|$, we guarantee that for $t \in [t_i, t_i + \varepsilon_1]$ we have $|e| \le y'|x|$, and since $\sigma' < 1$ asymptotic stability is still guaranteed. The inter-execution times are now bounded by $\tau_N + \tau$, where τ is the time it takes ϕ to evolve from $|e(t_i + \tau_N)| / |x(t_i + \tau_N)| = |\hat{x}(t_i + \tau_N) - x(t_i + \tau_N)| / |x(t_i + \tau_N)|$ to y, then the admissible delays τ_N need to satisfy $|e(t_i + \tau_N)| / |x(t_i + \tau_N)| < y$ since $\dot{\phi} > 0$. From continuity of $|\hat{x}(t_i + \tau_N) - x(t_i + \tau_N)| / |x(t_i + \tau_N)|$ with respect of τ_N there exists an $\varepsilon_2 > 0$ such that for any $0 \le \tau_N \le \varepsilon_2$ we have $|\hat{x}(t_i + \tau_N) - x(t_i + \tau_N)| / |x(t_i + \tau_N)| < y$. The term $|\hat{x}(t_i + \tau_N) - x(t_i + \tau_N)| / |x(t_i + \tau_N)|$ is continuous due to the fact that $|x(t_i + \tau_N)|$ is never zero since the closed loop system is asymptotically stable and never reaches zero in finite time. We complete the proof by defining $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. ■

The importance of the results in this section relate to the fact that we can find a positive lower bound on the inter-execution times in the presence of both time delays and model uncertainties. The estimation of the admissible delays is conservative at the present time and the search for better delay estimation methods will be studied in the future. We are motivated by the fact that, in general, very long sampling intervals are obtained using this framework so, it is better to search for less conservative delay estimates than trying to study delays that are longer than the inter-execution times. The extension to consider packet dropouts would be very interesting to pursue as well. Note also that the estimation of ε is an upper bound for the admissible delays $\tau_N > 0$, that is, the results apply the same way for any time-varying delay in the range $0 < \tau_N < \varepsilon$.

B. Systems with network delays and data quantization.

Based on the results of previous sections we now introduce stability thresholds that consider quantization and time delays. For the quantized scheme, we already defined the model-plant state error given in (3) and the quantized model-plant state error in (15). At the update instants t_i we update the model in the sensor node using the quantized measurement of the state. At this instant we have $e_q(t_i) = 0$, at the sensor node. When considering network delays we can reset the quantized model-state error only at the sensor node. The model-plant state error at the update instants is given by:

$$e(t_i) = \hat{x}(t_i) - x(t_i) = q(x(t_i)) - x(t_i). \tag{29}$$

It is clear that this error cannot be set to zero at the update instants as the quantized model-plant state error, due to the existence of quantization errors when measuring the state of the plant. Using the logarithmic quantizer (13) we have that:

$$|e(t_i)| = |q(x(t_i)) - x(t_i)| \le \delta |x(t_i)|. \tag{30}$$

The same conclusion can be reached by evaluating the expression in (19) at time instants t_i and by setting $|e_q(t_i)| = 0$.

The next theorem provides conditions for asymptotic stability of the control system using quantization in the presence of network induced delays. In this case, the admissible delays are also a function of the quantization parameter δ, that is, if we are able to quantize more finely the system is still stable in the presence of longer delays.

Theorem 7. Let (1) be a control system with control input based on the state of the model $u = K\hat{x}$. The event-triggering condition is computed using quantized data according to (16). The model is updated using quantized measurements of the state of the plant. Assume that: there exists a symmetric positive definite solution P for the model Lyapunov equation (8) and a small enough $\delta, 0 < \delta < 1$ such that $2\delta / (1-\delta) < \sigma\eta / (\delta + 1)$. Assume also that $B = \hat{B}$ and the next bounds are satisfied: $|\tilde{A}| \le \Delta_A$ and $|\tilde{A}^T P + P\tilde{A}| \le \Delta < \underline{q}$; then there exists an $\varepsilon(\delta) > 0$ such that for all network delays $\tau_N \in [0, \varepsilon]$ the system is asymptotically stable, furthermore, there exists a time τ>0 such that for any initial condition the inter-execution times $\{t_{i+1} - t_i\}$ implicitly defined by (16) with $\sigma < 1$ are lower bounded by τ, i.e. $t_{i+1} - t_i \ge \tau \quad \forall i \in Z^+$.

Proof: Following the approach described above for the non-zero delay case without quantization, let us look at the dynamics of the term $|e_q|/|q(x)|$, which contains the available variables for processing and broadcasting after quantization has taken place. Let us first note that:

$$|e_q| = |\hat{x} - q(x) + x - x| \le |x - q(x)| + |\hat{x} - x| \le \delta |x| + |e|. \tag{31}$$

In addition and, since $0 < \delta < 1$, we have that:

$$|x| = |x - q(x) + q(x)| \le |x - q(x)| + |q(x)| \le \delta |x| + |q(x)| \quad \Rightarrow \quad |q(x)| \ge (1-\delta)|x|. \tag{32}$$

The evolution of the term $|e_q|/|q(x)|$ can be bounded as follows:

$$\frac{|e_q|}{|q(x)|} \le \frac{\delta|x| + |e|}{|q(x)|} \le \frac{\delta|x| + |e|}{(1-\delta)|x|} = \frac{\delta}{1-\delta} + \frac{1}{1-\delta}\frac{|e|}{|x|}. \tag{33}$$

Let us denote the term $|e|/|x|$ by θ and denote the term $|e_q|/|q(x)|$ by ψ so we have the estimate:

$$\psi(t) \le \frac{\delta}{1-\delta} + \frac{1}{1-\delta}\theta(t) \le \frac{\delta}{1-\delta} + \frac{1}{1-\delta}\phi(t,\delta) \tag{34}$$

where $\phi(t,\delta)$ is the solution of (25) satisfying $\phi(0,\delta) = \delta$. The initial condition in the solution of the differential equation (25) that we are using for the quantization case is the worst case initial error in the term $\theta(t)$ given by (30). The solution $\phi(t,\delta)$ is given by:

$$\phi(t,\delta) = \frac{-c(\delta+1)e^{dt(c-1)}/(\delta+c)+1}{(\delta+1)e^{dt(c-1)}/(\delta+c)-1} \tag{35}$$

then the evolution of $|e_q|/|q(x)| = \psi$ is bounded by the following expression:

$$\psi(t) \le \xi(t) = \frac{\delta}{1-\delta} + \frac{1}{1-\delta}\frac{-c(\delta+1)e^{dt(c-1)}/(\delta+c)+1}{(\delta+1)e^{dt(c-1)}/(\delta+c)-1}. \tag{36}$$

For the case when $\tau_N = 0$, the inter-execution times for the system with quantization measurements are bounded by the time it takes for ξ to evolve from $\xi(0) = 2\delta/(1-\delta)$ to $\sigma\eta/(\delta+1)$, i.e. the solution $\tau \in \mathbb{R}^+$ of $\xi(\tau) = \sigma\eta/(\delta+1)$. An estimate of that time can be obtained in a two-step process using (35) and (36) that also provides some insight into the tradeoff between the selection of the quantization parameter δ and the admissible network delays τ_N.

First, solve for the time variable in (35) for a given $y>\delta$, that is, the solution $\phi(t,\delta)=y$, such solution is given by:

$$\tau = (\ln(y+1) - \ln(\frac{\delta+1}{\delta+c}(y+c)))\frac{1}{d(c-1)}. \tag{37}$$

For $y>\delta$, we always have $\tau>0$, since ϕ is continuous in the range $\tau \in [0,\tau_m)$ that covers all thresholds $y>\delta$ and $\dot{\phi}>0$. The last statement can be shown by analyzing directly the two factors in (37). We consider two cases and avoiding the case $c=1$. First, for $c>1$ the second factor in (37) is positive, then, in order to obtain $\tau>0$, we need the condition:

$$\frac{(y+1)(\delta+c)}{(y+c)(\delta+1)} > 1$$

which is equivalent to the condition $y>\delta$. For the case $0<c<1$ the second term is negative and we need the first factor to be negative in order to obtain a strictly positive value for τ. We can ensure that the first factor is negative by satisfying the following condition:

$$\frac{(y+1)(\delta+c)}{(y+c)(\delta+1)} < 1$$

which is equivalent to $y>\delta$.

Note that (27) is a special case of (37), when $\delta=0$. Then $\tau>0$ for any $y>0$. In this, more general case, the range of values for τ for any value of the threshold $y>\delta$ is given by $\tau \in (0,\tau_m)$, where

$$\tau_m = \lim_{y\to\infty} \tau = \frac{1}{d(c-1)}\ln(\frac{\delta+c}{\delta+1}). \tag{38}$$

Second, using the result in (37) we can obtain a solution $\tau \in \mathbb{R}^+$ of $\xi(\tau) = \sigma\eta/(\delta+1)$, i.e. the solution $\tau \in \mathbb{R}^+$ of:

$$\xi(\tau, \frac{2\delta}{1-\delta}) = \frac{\delta}{1-\delta} + \frac{1}{1-\delta}\phi(\tau,\delta) = y_q \tag{39}$$

where

$$y_q = \sigma\eta / (\delta + 1). \tag{40}$$

The solution is given by (37) with

$$y = (1-\delta)(y_q - \frac{\delta}{1-\delta}). \tag{41}$$

By assumption, we have that:

$$y_q > \frac{2\delta}{1-\delta} \tag{42}$$

which results in $y > \delta$ being satisfied, which means that $\tau > 0$. The inter-execution times are bounded by τ and strictly away from zero.

Now, for the case $\tau_N > 0$, choose $\sigma, \sigma', \sigma^\tau$ such that the next is satisfied: $0 < \sigma < \sigma' < \sigma^\tau < 1$. The last choice results in the following relation $0 < y_q < y' < y^\tau$, where y_q is defined in (40), $y' = \sigma'\eta$ and $y^\tau = \sigma^\tau \eta$.

Let an execution be triggered at time t_i by the condition (16) that in turn enforces (17) at time t_i and let ε_1, $0 < \varepsilon_1 < \tau_m$ satisfy the solution $\xi(\varepsilon_1, y') = y^\tau$ such ε_1 always exists since ξ is continuous in the range $\tau \in [0, \tau_m)$ that covers all thresholds $2\delta/(1-\delta) < y', y^\tau < \infty$, also $\dot{\xi} > 0$ and $y' < y^\tau$ by the previous choice of parameters. Then, by sending the state measurement at time t_i in order to update the model in the controller, we guarantee that for $t \in [t_i, t_i + \varepsilon_1]$ we have $|e| \leq y^\tau |x|$, and since $\sigma^\tau < 1$ asymptotic stability is still guaranteed, i.e. (12) is satisfied with $\sigma^\tau < 1$. The inter-execution times are now bounded by $\tau_N + \tau$, where τ is the time it takes ξ to evolve from $|e_q(t_i + \tau_N)| / |q(x(t_i + \tau_N))| = |\hat{x}(t_i + \tau_N) - q(x(t_i + \tau_N))| / |q(x(t_i + \tau_N))|$ to $y_q = \sigma\eta / (\delta + 1)$ then the admissible delays τ_N need to satisfy $|e_q(t_i + \tau_N)| / |q(x(t_i + \tau_N))| < y_q$ since $\dot{\xi} > 0$. From continuity of ξ with respect of τ_N there exists an $\varepsilon_2 > 0$ such that for any $0 \leq \tau_N \leq \varepsilon_2$ we have:

$$\xi(\tau_N, \frac{2\delta}{1-\delta}) < y_q$$

since $y_q > 2\delta/(1-\delta)$. We complete the proof by defining $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. ■

Remark 2. Note that by selecting a smaller parameter δ, we increase the gap between the initial value $\xi(0) = 2\delta/(1-\delta)$ and the threshold y_q, this selection allows for longer admissible delays, i.e. a larger value for the solution (37) with y defined in (41). This corresponds to our intuition in a closed-loop system in the presence of quantization and time delays. If we quantize more finely, the system can admit longer time delays and operate in a safe way, i.e. preserving asymptotic stability.

Computation of the quantization parameter δ for a fixed admissible delay ε. In developing the previous proof we first selected a quantization parameter δ (by selecting σ and σ') that ensures stability in the case of zero delay. By fixing δ and for a given choice of σ^τ we were able to estimate the longest admissible delay for which stability is still guaranteed. Now, we would like to be able to estimate the greatest quantization parameter δ for a fixed delay bound ε.

In order to compute the quantization parameter based on the admissible delays we first choose parameters σ and σ' (corresponding to σ' and σ^τ in theorem 7), and associated thresholds y and y', and find the admissible delay without quantization, i.e. find ε in theorem 6 in section V.A. Then select a new smaller delay bound than the one just found $\varepsilon_n < \varepsilon$. We select a smaller delay since we do not expect a longer or equal admissible delay when using quantization. We proceed to search for the greatest value of δ, $0 < \delta < 1$ and a $\sigma < \sigma'$, such the next two relations are satisfied:

$$\delta \le (\sigma' - \sigma)\eta \ , \qquad y_q > \frac{2\delta}{1-\delta}.$$

where,

$$y_q = \xi(\varepsilon_n, \frac{2\delta}{1-\delta}) \ , \qquad \sigma = \frac{y_q(1+\delta)}{\eta}.$$

C. Updating the model state using the delayed measurements.

The stability and performance of the networked system is affected by the use of delayed measurements for control since the difference between the delayed measurement and the current state of the system produces a large state error when the delayed data is used to update the model. A smaller state error can be obtained by estimating the current plant state based on the old measurements. This procedure represents another advantage compared to traditional ZOH event-triggered implementations. By computing a quantity that reflects more accurately the current state of the plant than the delayed measurement does, it is possible to execute a better control action over the next open-loop interval, that is, the next event will be triggered later in time than by using the old data directly.

Constant delays. Since we need to implement the model of the plant in both the controller and the sensor node, in order to compute the control input and compute the state error respectively, we have to use wisely the delayed information received by the controller so a good estimate of the current plant state is obtained to update the model in the controller and compute a better control input for the plant. In the case that the network delays are constant then we can implement the next strategy: the sensor decides to send a feedback measurement to the controller at time t_i so it updates its own state but keeps using the old input, i.e. the input generated by the same model in the case that no update has taken place, similar to the plant being fed by the model/controller that has not been updated yet. Notice that if the sensor knows the magnitude of the constant network delay τ_N then it will switch to closed loop mode at the end of the known delay. By using this strategy we need to implement a second model in the sensor node, but this is physically possible since we are considering operations performed by a single processor; that is, if we are able to implement the computations needed to measure and compute the state error and threshold comparisons then, in general, we could be able to implement a second closed loop model that only works for short intervals $[t_i, t_i + \tau_N]$. When the controller receives the measurement $x(t_i)$ at time $t_i + \tau_N$ it uses this measurement to immediately estimate the state of the model in the sensor by computing the next:

$$\hat{x}_c(t_i+\tau_N)=e^{\hat{A}\tau_N}x(t_i)+\int_0^{\tau}e^{\hat{A}(\tau-s)}Bu_c(s)ds \tag{43}$$

which can be made arbitrarily accurate by storing the sequence of inputs over the previous delay interval, i.e. $[t_i, t_i+\tau_N]$ in the controller node and since the parameters in both models are exactly the same. The subscript c indicates the quantities belonging to or available at the controller node. The result of the operation in (43) is used to update the state of the model in the controller.

Time-varying bounded delays. A more general situation in many networked systems is that the network induced delays are time-varying and bounded as discussed in the previous section. In this case the sensor does not know the maximum magnitude of the delay, but by time-stamping the measurement sent over the network the controller node does know the size of the delay for every packet containing a feedback measurement. A simple strategy in this case is to let the model in the sensor remain working in closed loop after measuring and updating its state. When the controller receives the delayed measurement it simply computes the following, which is used to update the model in the controller node:

$$\hat{x}_c(t_i+\tau_N)=e^{(\hat{A}+BK)\tau_N}x(t_i) \tag{44}$$

A slightly different strategy can be implemented in this case that, in general, results in a better performance, i.e. longer broadcast intervals, by realizing that the states of both models do not need to be the same, as long as the model in the controller produces a smaller state error than the model in the sensor. This is basically a combination of the two strategies above. The sensor updates its state and continues working in closed loop mode but the controller uses the quantity obtained by (43) in order to obtain a better estimate of the current plant state not of the current sensor model state based on the delayed measurement.

For the case of delayed and quantized measurements we follow the same approach but now we use the available quantized measurement of the state, that is, we use $q(x(t_i))$ instead of $x(t_i)$ in (43) and (44).

VI. EXAMPLE

In this section we apply the control framework described in section V to the linearized decoupled pitch and roll/yaw axes models of the Space Station described in [38]. The spacecraft is assumed to be in circular orbit with orbital angular velocity n. The variables ϕ_e, θ_e, ψ_e represent the roll, pitch, and yaw Euler angles with respect to a fixed frame; the subscript "$_e$" refers to Euler angles. ω is the absolute angular velocity vector in the body frame. When $\phi_e = \theta_e = \psi_e = 0$, the spacecraft is oriented in the desired location. The goal is to keep the spacecraft oriented in the desired attitude in the presence of quantized measurements, time delays, and an expected range change in the moments of inertia.

Consider first the stabilization of the decoupled pitch axis dynamics by an appropriate reaction control jet. The simplified nominal model for the pitch axis is given by:

$$\begin{bmatrix} \dot{\hat{\theta}}_e \\ \ddot{\hat{\theta}}_e \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -3n^2\hat{D}_2 & 0 \end{pmatrix} \begin{bmatrix} \hat{\theta}_e \\ \dot{\hat{\theta}}_e \end{bmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2$$

where $\hat{D}_2 = (\hat{I}_3 - \hat{I}_1) / \hat{I}_2$, and $\hat{I}_i$, i=1,2,3, are the nominal values of the main diagonal elements of the inertia matrix. We assume the cross-products of inertia are negligible.

In this example, we consider variations on the main diagonal elements of the inertia matrix due to different operations, maneuvers, and some disturbances affecting the dynamics of the spacecraft. The plant can be described by:

$$\begin{bmatrix} \dot{\theta}_e \\ \ddot{\theta}_e \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -3n^2 D_2 & 0 \end{pmatrix} \begin{bmatrix} \theta_e \\ \dot{\theta}_e \end{bmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2$$

where $D_2 = (I_3 - I_1) / I_2$ and I_i are the real values of the principle moments of inertia.

The control gain is obtained based on the nominal model parameters using a linear quadratic regulator (LQR) that considers the consumption of fuel on the reaction control jets. In particular, the next quadratic performance index is minimized:

$$J = \int_0^{\infty} (x^T Q_l x + u^T R_l u) dt .$$

The nominal moments of inertia are as given in [38]. The orbital angular velocity is *n*=.0011 rad/sec. We assume the variations on the moments of inertia to be in the range: $I_i = \hat{I}_i \pm 0.9 \hat{I}_i$.

Example 1. We select $\sigma = 0.5, \sigma' = 0.62, \sigma^{\tau} = 0.99$ and $\delta = 0.015$. The longest admissible delay using these parameters is found to be 0.0295 seconds.

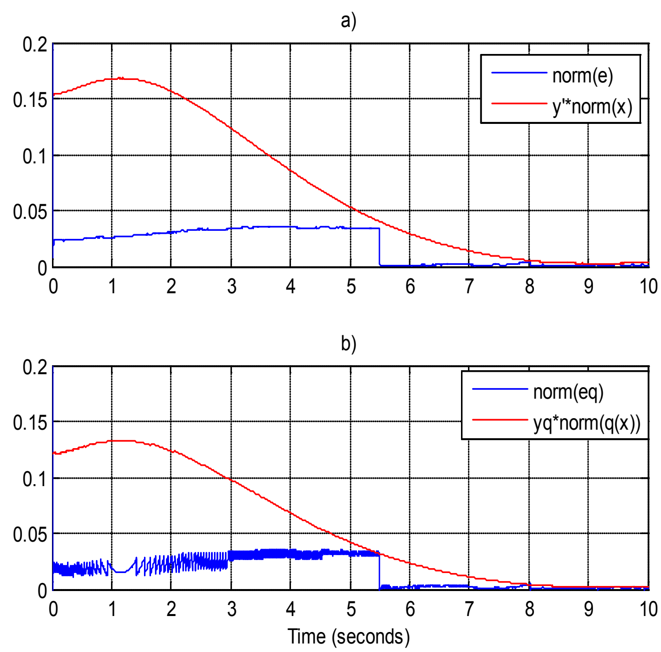

Fig. 3. Response of: a) $|e(t)|$ and $y'|x(t)|$. b) $|e_q(t)|$ and $y_q|q(x(t))|$, for fixed quantization parameter δ.

Results of simulations are shown in Fig. 3. Part a) shows the response of the norm of the state of the plant and the norm of the error for time-varying delays bounded by 0.03 seconds. It can be seen that $|e(t)|$ is never greater than $y'|x(t)|$. Part b) of Fig. 3 shows the norm of e_q and the norm of *q(x)*. The events are triggered when the relation $|e_q| \le y_q |q(x)|$ is not satisfied. The discrete variations on the error correspond to the events generated at the sensor node i.e. when the sensor decides to transmit the current measurement and updates its internal model, resetting the quantized model-plant error as measured by the sensor.

Example 2. Now we would like to find the greatest quantization parameter for a selected admissible delay. We select $\sigma' = 0.55, \sigma^{\tau} = 0.99$. The admissible delay without quantization is 0.0352 seconds. Then we select $\varepsilon_n = 0.034$ seconds and we obtain $\sigma = 0.4061, \delta = 0.0027$.

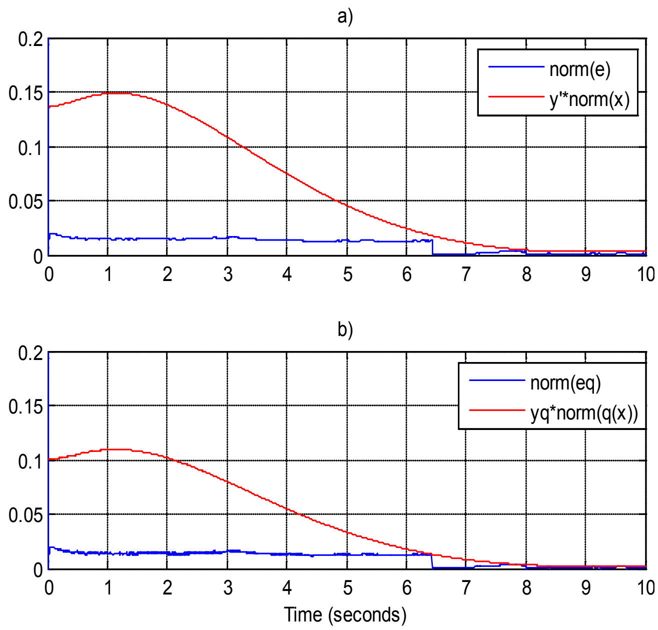

Fig. 4. Response of: a) $|e(t)|$ and $y'|x(t)|$. b) $|e_q(t)|$ and $y_q|q(x(t))|$, for fixed admissible time delay.

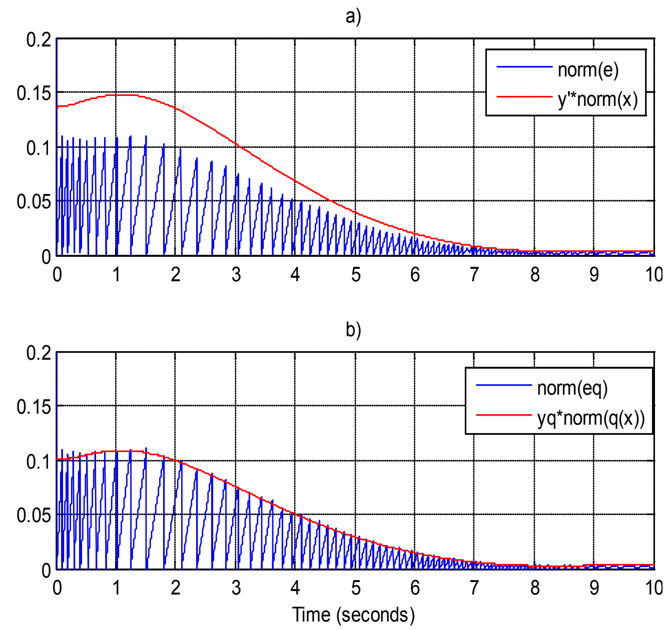

Fig. 5. Response of: a) $|e(t)|$ and $y'|x(t)|$. b) $|e_q(t)|$ and $y_q|q(x(t))|$, for fixed admissible time delay and ZOH model using the same parameters as in example 2.

Results of simulations are shown in Fig. 4. As in example 1 above, part a) shows the response of the norm of the state of the plant and the norm of the error for time-varying delays now bounded by of 0.034 seconds. It can also be seen that $|e(t)|$ is never greater than $y'|x(t)|$. Similarly, part b) of Fig. 4 shows the norm of e_q and the norm of *q(x)*. In order to draw a comparison to the case when a ZOH model is used in the controller node, that is, the received measurement is held constant until a new measurement arrives we execute similar simulations using the same parameters, controller gains, and time delays that we found in example 2, i.e. the model parameters are used to compute the controller and the error thresholds but the model is not used as part of the controller. Fig. 5 shows the simulation results. It can be seen that error events are triggered more frequently in this case that no model of the plant is used to generate an estimate of the state between sampling intervals. A significant saving in communication bandwidth is obtained by using the model to control the system when no feedback measurements are sent from the sensor to the controller.

Example 3. The simplified nominal model for the roll/yaw axes is given by:

$$\begin{bmatrix} \dot{\hat{\phi}}_e \\ \dot{\hat{\psi}}_e \\ \dot{\hat{\omega}}_1 \\ \dot{\hat{\omega}}_3 \end{bmatrix} = \begin{pmatrix} 0 & n & 1 & 0 \\ -n & 0 & 0 & 1 \\ -3n^2\hat{D}_1 & 0 & 0 & -n\hat{D}_1 \\ 0 & 0 & -n\hat{D}_3 & 0 \end{pmatrix} \begin{bmatrix} \hat{\phi}_e \\ \hat{\psi}_e \\ \hat{\omega}_1 \\ \hat{\omega}_3 \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} u_1 \\ u_3 \end{bmatrix}$$

where $\hat{D}_1 = (\hat{I}_2 - \hat{I}_3)/\hat{I}_1$ and $\hat{D}_2 = (\hat{I}_1 - \hat{I}_2)/\hat{I}_3$. The plant can be described by:

$$\begin{bmatrix} \dot{\phi}_e \\ \dot{\psi}_e \\ \dot{\omega}_1 \\ \dot{\omega}_3 \end{bmatrix} = \begin{pmatrix} 0 & n & 1 & 0 \\ -n & 0 & 0 & 1 \\ -3n^2 D_1 & 0 & 0 & -nD_1 \\ 0 & 0 & -nD_3 & 0 \end{pmatrix} \begin{bmatrix} \phi_e \\ \psi_e \\ \omega_1 \\ \omega_3 \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} u_1 \\ u_3 \end{bmatrix}$$

where $D_1 = (I_2 - I_3)/I_1$ and $D_2 = (I_1 - I_2)/I_3$ are given by the real moments of inertia.

We select $\sigma = 0.58, \sigma' = 0.76, \sigma^\tau = 0.99$ and $\delta = 0.03$. The longest admissible delay using these parameters is found to be 0.0175 seconds.

Fig. 6 shows the simulation results for this example. Part a) shows the response of the norm of the state of the plant and the norm of the error and part b) shows the norm of e_q and the norm of *q(x)* for time-varying delays bounded by 0.0175 seconds. Similar to example 2, we plot in Fig.7 the response of the system using the same parameters but without using the model to control the system between event intervals. Instead, the measurements are held constant until a new measurement arrives at the controller. The error events are triggered more frequently using the ZOH model than using the nominal model of the system as in the simulation results shown in Fig. 6.

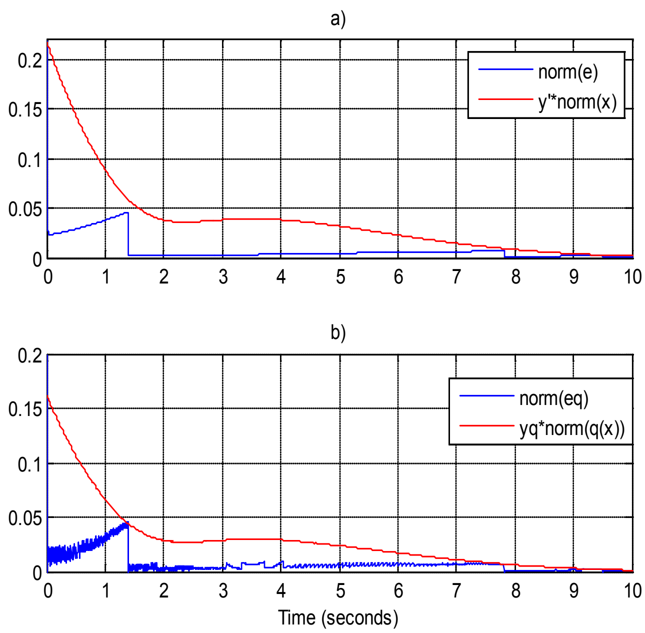

Fig. 6. Response of: a) $|e(t)|$ and $y'|x(t)|$. b) $|e_q(t)|$ and $y_q|q(x(t))|$, for the roll/yaw axes and fixed quantization parameter δ.

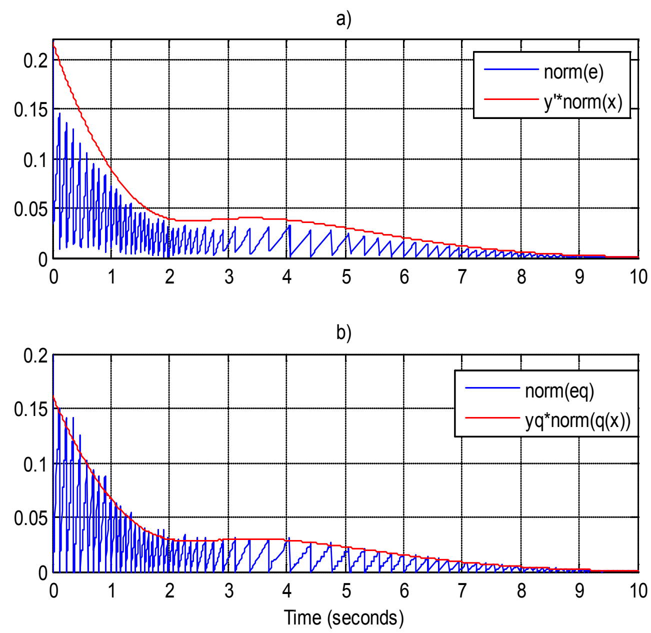

Fig. 7. Response of: a) $|e(t)|$ and $y'|x(t)|$. b) $|e_q(t)|$ and $y_q|q(x(t))|$, for the roll/yaw axes, fixed quantization parameter δ and ZOH model using same parameters as in example 3.

VII. Conclusion

Stabilization of systems subject to both quantization and time-varying network induced delays has been addressed in this paper. The framework presented here combines two approaches that also reduce the rate at which sensor nodes need to send feedback measurements to the controller nodes. This new control strategy generalizes the standard event-triggered control scheme. It implements a nominal model of the system that is part of the actuator/controller node in order to generate an estimate of the state of the system between update intervals, which is an improvement compared to the ZOH that generates a constant input during the same interval. Additionally, we are able to design stabilizing thresholds considering robustness to parameter uncertainties and quantization errors. The event-triggered strategy provides a different way to update the model of the system in MB-NCS without compromising stability. The resulting stability conditions can be easily checked. Future work will lift the restriction that the controller should be adjacent to the plant, considering network channels in both sides of the control loop.

Since the results in the paper pertain to the case of full state feedback, an important extension will consider the case of output feedback using the MB-ET framework.

APPENDIX

Proof of theorem 1. The response of the plant (4) with *t(0)=0* and Hurwitz matrix *A+BK* at any given time $t \geq 0$ is given by:

$$x(t) = e^{(A+BK)t}x(0) + \int_0^t e^{(A+BK)(t-\tau)} BKe(\tau)d\tau \tag{45}$$

where *e(t)* is a piecewise continuous input bounded by: $|e| \leq \alpha$. We can show that the state of the plant is bounded by evaluating its norm which is done next:

$$|x(t)| = \left| e^{(A+BK)t}x(0) + \int_0^t e^{(A+BK)(t-\tau)} BKe(\tau)d\tau \right| \leq \left| e^{(A+BK)t} \right| |x(0)| + \int_0^t \left| e^{(A+BK)(t-\tau)} \right| |BK| |e(\tau)| d\tau.$$

By the assumption on the initial condition and the triggering condition, and using the bound $\left| e^{(A+BK)t} \right| \leq k_1 e^{-\lambda t} \quad k_1, \lambda > 0$, we can write:

$$|x(t)| \leq \beta_1 k_1 e^{-\lambda t} + \alpha k_1 |BK| \int_0^t e^{-\lambda(t-\tau)} d\tau = (\beta_1 k_1 - \frac{\alpha k_1 |BK|}{\lambda}) e^{-\lambda t} + \frac{\alpha k_1 |BK|}{\lambda}. \tag{46}$$

Also note that:

$$\lim_{t \to \infty} |x(t)| \leq \frac{\alpha k_1 |BK|}{\lambda}.$$

A choice of a stabilizing controller *K*, or in other words, the fact that the closed loop plant poles are in the left hand side of the complex plane ensure that the first term in the right hand side of (46) decreases exponentially with time and the second term is bounded for all time *t>0*.

Note also that by considering *y=x*, then (*y,e*) is BIBO stable when *A+BK* is asymptotically stable. If *A,B* is controllable then the relation is if and only if. Then we need to ensure that the error is bounded by updating the model when $|e| \leq \alpha$ is not satisfied.■

We define a uniform quantizer as a function $q : \mathbb{R}^n \to \mathbb{R}^n$ with the following property:

$$\left|x - q(x)\right| \le \delta, \quad x \in \mathbb{R}^n, \quad \delta > 0. \tag{47}$$

Here, it is convenient to associate this type of quantizer to the event-triggered fixed threshold strategy. The main reason is that we can preserve the above results under uniform quantization. In contrast, the asymptotic stability property obtained by using a relative threshold strategy cannot be achieved using a uniform quantizer but we have to rely to another type of static quantizer, the logarithmic one, as it was shown in section IV.

Corollary 3. For $\left|x(0)\right| \le \beta_1, \quad 0 < \beta_1 < \infty$ system (4) with state feedback based on error events and using a uniform quantizer with parameter $\delta < \alpha$ is bounded-input bounded- state stable with respect to the measurement error if the eigenvalues of $A+BK$ have negative real parts.

Proof. We only have available the quantized measurement *q(x)*. Using the quantized model-plant state error e_q defined in (15) we can enforce (triggering an update otherwise) the following

$$\left|e_q\right| = \left|\hat{x} - q(x)\right| \le \alpha - \delta. \tag{48}$$

The term $\alpha - \delta > 0$ is the new threshold that is applied to the available error e_q. Equation (48) in turn enforces the following:

$$\left|e\right| = \left|\hat{x} - x\right| \le \left|q(x) - x\right| + \left|e_q\right| \le \delta + \alpha - \delta \le \alpha \tag{49}$$

then the norm of the state of the plant is still bounded by (46). ■

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