ACC '12

Stability of Interconnected Switched Systems using QSR Dissipativity with Multiple Supply Rates

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***Abstract*— This paper considers a notion of QSR dissipativity for discrete-time switched systems that uses multiple supply rates. The focus on switched systems is motivated by cyber physical systems (CPS) where physical dynamics interact with discrete-event or logical dynamics. The notion of QSR dissipativity used in this paper is based on previous work on QSR dissipativity for non-switched systems and decomposable dissipativity for switched systems. The notion of QSR dissipativity is well established for non-switched systems and generalizes many system results including both the passivity theorem and the small gain theorem. In decomposable dissipativity, separate supply rates are developed for each subsystem depending on whether it is currently active or inactive. This paper presents this definition of QSR dissipativity for switched systems and then uses it to prove stability for single systems and interconnected systems. Beyond stability, the dissipative property for interconnected systems is shown. This allows for large scale systems to be studied using successive dissipative assessments. When considering passive systems, these results are studied in more detail.**

I. INTRODUCTION

Cyber-physical systems (CPS) arise from the tight interconnection between physical and cyber processes. Physical processes include naturally occurring systems as well as man-made systems that follow physical laws. These are modeled using differential or difference equations with a strong dependence on time. The cyber processes evolve based on the occurrence of events, both physically and in software, and often have little or no dependence on time. These include computational systems, communication systems, or any discrete-state based system. The combination of these different components results in system models that are hybrid. In CPS, these more complex models are used because performance standards are high so that the system can't be adequately modeled using only a continuous model or a discrete-event model.

An important class of hybrid systems is switched systems. These systems are modeled as a finite set of dynamics with a rule that determines switching between them. Each set of dynamics is modeled as a single subsystem of the switched system. In this paper, we consider subsystems that are nonlinear and time-invariant. We also assume the switching rule is not specified. In this case, the switching is allowed to be arbitrary. Although results that assume arbitrary switching have the most restrictive stability conditions, this assumption allows for stability that is robust to variations in switching. This is an important consideration when systems are interconnected; while the switching signal may be fixed when a system operates in isolation, switching is often unknown when systems are interconnected. For more on switched systems, refer to the following surveys and the references therein [1], [2], [3].

Another facet of CPS is that these systems are often complex systems that are made up of several interacting heterogeneous components. Traditional approaches for analyzing large scale systems have focused on energy [4]. While the main approach for stability of a single system has been Lyapunov theory, the approaches for large scale system have been passivity and dissipativity theory [5], [6]. These system properties are useful in guaranteeing stability results for single as well interconnected systems. While stability of the feedback interconnection of two passive systems is guaranteed, stability for dissipative systems is more challenging. This is both because the energy supply rate is typically different for different systems and because the two supply rates have to be complementary to maintain stability. When considering a quadratic supply rate, as in QSR-dissipativity, a general test may be applied to determine whether a feedback interconnection is stable [7], [8]. This result is a generalization of both the passivity theorem and the small gain theorem and may be applied to a large class of interconnections including interconnections that don't fit those other frameworks such as unstable systems.

Much of the research in dissipativity has focused on continuous time systems. In this paper we chose to focus instead on the discrete time case. This choice highlights the application of this theory to CPS which include discrete-time systems. These dynamics arise in cyber processes, in discrete-time communication between components of a system, and in sampling physical processes to interact with computational systems. Dissipativity theory has been considered in continuous time switched systems [9], [10], [11], [12], [13] and discrete time switched systems [14], [15]. A few of these works have considered a notion of decomposable dissipativity that breaks the energy supply rate for each subsystem into an active rate and an inactive rate. However, none of these papers have considered the QSR dissipativity framework. As a result, the stability problem for dissipative switched systems in feedback has been largely unexplored. There are some exceptions in continuous-time when using the special case of passivity [9] and the special case of conic systems

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[13]. By using the QSR decomposable dissipativity approach, we present stability results for the feedback interconnection of two dissipative switched systems.

The remainder of this paper is organized as follows. Section 2 covers the switched system model and discusses related notation that will be used throughout this paper. Section 3 briefly reviews QSR dissipativity for non-switched systems before moving on to present a notion of QSR dissipativity for switched systems using multiple supply rates. The stability of a single switched system is shown at this point. Stability of interconnections of dissipative switched systems is shown in Section 4. Additionally, the dissipative rate of a feedback interconnection is shown. The special case of passivity for switched systems is considered in detail in Section 5. This case is important both because passive systems are prevalent in application and because passive systems form stable feedback loops without additional conditions. A definition of passivity for switched systems is given that implies stability and is preserved in feedback and in parallel. Finally, concluding remarks are given in Section 6.

II. Switched System Model

A nonlinear switched system consists of a finite set of subsystems with nonlinear dynamics. We consider discrete-time systems with time index $t \in \mathbb{Z}$. At any point in time, a single subsystem is active and the dynamics are nonlinear and time-invariant according to the model

$$\begin{aligned} x(t+1) &= f_i(x(t), u(t)) \\ y(t) &= h_i(x(t), u(t)) \end{aligned} \tag{1}$$

where $i \in \{1, 2, ..., M\}$ for M subsystems. The signals are all functions of the discrete-time t and are vectors, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$.

The time-varying nature of these systems comes from the switching behavior. The switching signal $\sigma(t)$ is a function that maps the current time to the index of the active subsystem, $\sigma : \mathbb{Z}^+ \to \{1, ..., M\}$. This function is piecewise constant and only changes at switching instants. The model with the switching signal is given by

$$\begin{aligned} x(t+1) &= f_{\sigma(t)}(x(t), u(t)) \\ y(t) &= h_{\sigma(t)}(x(t), u(t)). \end{aligned} \tag{2}$$

The switching instants can be listed in order t_1, t_2, etc. Alternatively, the notation t_{i_k} will be used to denote the k^{th} time that subsystem i becomes active. For example, the first subsystem ($i = 1$) becomes active for the first time ($k = 1$) at time t_0 ($t_0 = t_{1_1}$). The second subsystem $i = 2$ becomes active at time t_1 ($t_1 = t_{2_1}$) and so forth. By using these two notations in conjunction, we can list completely the times that a system becomes active as well as the times it becomes inactive. Subsystem i becomes active the k^{th} time at time t_{i_k} and then inactive at time $t_{(i_k+1)}$. That subsystem becomes active again at time $t_{i_{(k+1)}}$. When dealing with discrete-time switched systems, a system can only switch a finite number of times on any finite time interval, thus Zeno phenomena is not an issue. When considering the switching behavior of a system from initial time t_0 to an arbitrary time T, we denote the number of times the system switches as K. When needed, we will also denote the number of times each subsystem i switches on that same time interval as K_i.

We define an indicator set to signify regions where a particular subsystem is active. Consider subsystem i that is active from t_{i_1} to $t_{(i_1+1)}$, t_{i_2} to $t_{(i_2+1)}$, etc. We define a set of times I_i to indicate those time intervals where subsystem i is active,

$$I_i = \bigcup_{k=1}^{K_i} \{t_{i_k}, ..., t_{(i_k+1)}\}. \tag{3}$$

This notation will be used to draw a distinction between the active and inactive time intervals of a system.

III. QSR Dissipativity

A. Dissipativity for Non-Switched Systems

Let a nonlinear non-switched discrete-time system be described by

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)). \end{aligned} \tag{4}$$

Since the dynamics are unchanging, the conditions of dissipativity involve a single energy storage function and single energy supply rate. The energy storage function $V(x)$ represents a generalized notion of energy. Since it is a notion of energy, it must be non-negative for all x, ($V(x) \geq 0$). The energy supplied to the system is captured by an energy supply rate $\omega(u, y)$. The supply rate is allowed to be general as long as it is finite for all finite u and y. A system is dissipative [5] if it satisfies the following inequality for all times $t \geq t_0$

$$V(x(t+1)) \leq V(x(t)) + \omega(u, y). \tag{5}$$

In QSR dissipativity [7], the supply rate has a quadratic form,

$$\omega(u, y) = \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}. \tag{6}$$

More on dissipativity for discrete-time systems including stability results can be found in [16], [17].

Important special cases of QSR dissipativity are found in passivity and finite-gain $\mathcal{L}_2$ stability. A passive system is dissipative with supply rate given by $Q = 0$, $R = 0$, and $S = \frac{1}{2}I$. An $\mathcal{L}_2$ stable system is dissipative with supply rate $S = 0$, $Q = -\frac{1}{\gamma}I$, and $R = \gamma I$ where γ is the $\mathcal{L}_2$ gain of the system.

B. QSR Dissipativity for Switched Systems

The notion of decomposable dissipativity for switched systems has been studied in continuous-time [10], [12] and in discrete-time [15]. The concept of decomposable dissipativity is based on the fact that systems typically store energy differently when they are active or inactive. The solution is to decompose the supply rate into an active portion and an inactive portion. When a subsystem is inactive, it may have a different supply rate depending on which other subsystem is active. The definition presented here is a special case of [15]. While that work presented a very general definition, the authors didn't consider the problem of stability for interconnected systems. Traditionally, stability of feedback interconnections is one of the main benefits of dissipativity. In this paper, we present a definition of QSR dissipativity that is used to prove stability of feedback interconnections as well as dissipativity properties of those connections.

The notion of dissipativity presented in this paper is modeled after the traditional notion of QSR dissipativity for non-switched systems. In the simplest case, consider applying the traditional notion of QSR dissipativity to a switched system with a fixed supply rate. If there exists a common energy storage function for all subsystems of the switched system for that supply rate, then the system can be considered QSR dissipative. At this point, the traditional results related to stability and stability of interconnections can be used directly.

However, this notion is somewhat restrictive. For one, it is rare for a common storage function to exist for a set of dynamics that may vary significantly. Additionally, when each subsystem is inactive it may store energy differently depending on which other subsystem is currently active. This happens because the storage function is determined by the dynamics of the active subsystem while energy supplied when inactive is based on the dynamics of the active subsystem which may be quite different. This discrepancy may result in large jumps in the energy storage function at each switch even if the size of the state changes little or not at all.

To relax these restrictions for switched systems, the multiple storage function approach is taken. The energy stored in a system may be different for each subsystem i by defining different functions $V_i(x)$. Additionally, the notion of supplied energy may be different for each subsystem, captured by a different ω_i. When each subsystem is active (i.e. $t \in I_i$) the following inequality holds ($\forall i$)

$$V_i(x(t+1)) \leq V_i(x(t)) + \omega_i(u, y). \tag{7}$$

For the notion of QSR dissipativity used in this paper, the supply rate will take the following form,

$$\omega_i(u, y) = \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}. \tag{8}$$

The notion of supplied energy for a subsystem i while it is inactive may be unique for each active subsystem j. This results in several inactive energy supply rates for each i and j, $\omega_i^j(u, y, x, t)$. When each subsystem is inactive, the following inequality holds for each active subsystem j at an appropriate time $t \in I_j$ ($\forall i$)

$$V_i(x(t+1)) \leq V_i(x(t)) + \omega_i^j(u, y, x, t). \tag{9}$$

When we combine these two conditions, we have the following definition. Recall that a function $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is class K_∞ if $\alpha(0) = 0$, α is non-decreasing, and α is radially unbounded.

Definition 1. *Consider a discrete-time switched system (2). This system is QSR dissipative if there exists a positive storage function $V_i(x)$, for each subsystem i, with the property that for some K_∞ functions $\underline{\alpha_i}$ and $\overline{\alpha_i}$,*

$$\underline{\alpha_i}(||x||) \leq V_i(x) \leq \overline{\alpha_i}(||x||), \tag{10}$$

such that the following two conditions hold.

1) *During the active time period $t \in I_i$ of each subsystem i, the system is dissipative with respect to a QSR-supply rate (7-8).*
2) *When each subsystem i is inactive, it is dissipative with respect to a supply rate $\omega_i^j(u, y, x, t)$ that may be specific to each active subsystem j (9).*

This definition is a natural extension of the QSR dissipativity notion for non-switched systems. Consider a case when $Q = Q_i$, $S = S_i$, and $R = R_i$ for all i. If there exists a common storage function for the switched system such that equation (7) holds, the system is QSR dissipative.

Many of the functions of interest (V, ω_i, ω_i^j, etc.) are functions of signals that are time dependent. In the remainder of the paper, these will be denoted simply as functions of time, i.e. $V(t)$, $\omega_i(t)$, $\omega_i^j(t)$, etc.

The following stability theorem concerns stability of a single dissipative switched system. It is based on traditional dissipativity results with a consideration for how the switching signal effects stability.

Theorem 1. *Consider an unforced ($u(t) = 0$) QSR dissipative switched system with storage functions $V_i(x)$. This system is stable if $Q_i \leq 0$ for all i and the cross supply rates are absolutely summable for all switching sequences $\forall i$ and $\forall j \neq i$,*

$$\sum_{t=t_0}^{\infty} |\omega_i^j(t)| < L. \tag{11}$$

Proof. We prove stability by showing that $||x(t)|| < \epsilon, \forall t$ when $||x(t_0)|| < \delta$. Recall the functions $\underline{\alpha_i}$ such that $\underline{\alpha_i}(x) \leq V_i(x), \forall x$. Define a function ρ such that

$$\rho(c) = \min_{i=1,\ldots,M} \{\underline{\alpha_i}(c)\}.$$

By the assumption that the energy added over the infinite time horizon is bounded, there exists a time T such that, $\forall t \geq T$ and $\forall i$,

$$V_i(t) - V_i(T) \leq \frac{1}{2M}\rho(\epsilon).$$

On the interval $[t_0, T]$, the system switches K times where $K \leq T - t_0$. Using the definition of dissipativity, we can bound the value of $V_i(T)$ by the following expression

$$V_i(T) \leq V_i(t_K) + \sum_{t=t_K}^{T-1} \omega_i(t).$$

Since all Q_i are negative semi-definite, $V_i(T) \leq V_i(t_K)$. At the switching instant t_K, the system switches from an arbitrary subsystem j to the system of interest i with some amount of added energy that is bounded. Define a $\delta_{K-1} > 0$ such that $V_j(t_{K-1}) \leq r(\delta_{K-1}) \rightarrow V_i(t_K) \leq \frac{1}{2}\rho(\epsilon)$. This implies that, when $||x(t_{K-1})|| < \delta_{K-1}$, then $V_i(t_K) \leq \frac{1}{2}\rho(\epsilon)$. This process can be iterated backwards from switching instant t_K back to t_1 with $\frac{1}{2}\epsilon > \delta_{K-1} > ... > \delta_1$. The final step is defining $\delta_0 > 0$ such that when $||x(t_0)|| \leq \delta_0 = \delta$ then $V_i(t_0) \leq \delta$. With δ_i defined this way, the bound $V_i(t_K) \leq \frac{1}{2}\rho(\epsilon)$ holds. Overall, the bound on $||x(t_0)||$ implies the following inequality,

$$V_i(T) \leq V_i(t_0) + \frac{1}{2M}\rho(\epsilon).$$

Now we sum over all systems and define $V(x) = \sum_{i=1}^{M} V_i(x)$ to arrive at the following

$$\sum_{i=1}^{M} V_i(T) \leq \sum_{i=1}^{M} V_i(t_0) + \sum_{i=1}^{M} \frac{1}{2M}\rho(\epsilon)$$
$$V(T) \leq V(t_0) + \frac{1}{2}\rho(\epsilon).$$

This expression with the previous bound on $V_i(t)$ for $t > T$ implies that $V(t) \leq \epsilon, \forall t$. This in turn implies that that, for all $\epsilon > 0$, there exists a δ such that $||x(t)|| \leq \epsilon, \forall t$ whenever $||x(t_0)|| \leq \delta$. Since this proof is valid for arbitrary ϵ, the system is stable. □

This extension of QSR dissipativity for switched systems allows for stability to be shown for systems with different energy storage functions and several energy supply rates. The energy is even allowed to increase at switching instants as long as the increase is bounded on the infinite time horizon. This extension allows for dissipativity theory to apply to switched systems with a range of dynamics.

IV. Stability of Systems in Feedback

Dissipativity theory can be used to show stability of systems in feedback. Consider the feedback interconnection of two switched systems G_1 and G_2 (Fig. 1). This interconnection forms a new switched system G which is a mapping from $r \rightarrow y$ where

$$r(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} \quad \text{and} \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

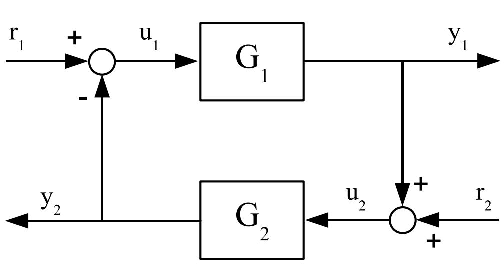

Fig. 1. The negative feedback interconnection of two systems.

The subsystems of the new system depend on the subsystems of both G_1 and G_2. This causes the number of subsystems of the combined system to grow to as many as $M = M_1 M_2$ where G_1 and G_2 have M_1 and M_2 subsystems, respectively. Whenever either system G_1 or G_2 switches, the overall system G switches. This means that the set of switching instants of the new system G is the union of the sets of switching instants of the two individual systems.

The following theorem considers stability of the feedback interconnection of two dissipative switched systems. System G_1 has supply rates $\omega_i^{(1)}$ parametrized by $\{Q_i, S_i, R_i\}$ and G_2 has supply rates $\omega_{\hat{i}}^{(2)}$ with $\{Q_{\hat{i}}, S_{\hat{i}}, R_{\hat{i}}\}$. The result considers the active supply rates and inactive supply rates to establish a bound on the storage functions. Finally, a bound on the system state is inferred from the bound on the storage functions.

Theorem 2. *Consider the feedback interconnection of two QSR dissipative switched systems G_1 and G_2. If the supply rates for the two systems satisfy*

$$\hat{Q}_{i\hat{i}} = \begin{bmatrix} Q_i + R_{\hat{i}} & S_{\hat{i}}^T + S_i \\ S_{\hat{i}} + S_i^T & Q_{\hat{i}} + R_i \end{bmatrix} \leq 0,$$
$$\forall i = 1, ..., M_1, \forall j = 1, ..., M_2,$$

and the energy accumulated while inactive is bounded for each subsystem, the unforced ($r(t) = 0$) feedback interconnection G is stable.

Proof. Since both systems are QSR dissipative, there exists $V_i^{(1)}$ for G_1 and $V_{\hat{i}}^{(2)}$ for G_2. The following summed storage functions can be defined

$$V^{(1)} = \sum_{i=1}^{M_1} V_i^{(1)} \quad \text{and} \quad V^{(2)} = \sum_{\hat{i}=1}^{M_2} V_{\hat{i}}^{(1)}.$$

Define $V(t) = V^{(1)}(t) + V^{(2)}(t)$ and $\rho(\epsilon) = \rho^{(1)}(\epsilon) + \rho^{(2)}(\epsilon)$. By the assumption that energy accumulated while inactive is bounded for each subsystem, $\forall\epsilon$ there exists a time T such that, $\forall t \geq T$,

$$V(t) - V(T) \leq \frac{1}{2}\rho(\epsilon). \tag{12}$$

Using the dissipative relationships we can find a bound on $V(T)$ based on previous switching instants,

$$V(T) \leq V(t_K) + \sum_{t=t_K}^{T-1} \left[\omega_i^{(1)}(t) + \omega_{\hat{i}}^{(2)}(t)\right] + \frac{1}{2}(r(\epsilon)).$$

The next step is to inspect the quantity inside the sum, $\omega_i^{(1)}(t) + \omega_j^{(2)}(t)$. This term can be written out,

$$\begin{bmatrix} y_1 \\ u_1 \end{bmatrix}^T \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} y_2 \\ u_2 \end{bmatrix}^T \begin{bmatrix} Q_{\hat{i}} & S_{\hat{i}} \\ S_{\hat{i}}^T & R_{\hat{i}} \end{bmatrix} \begin{bmatrix} y_2 \\ u_2 \end{bmatrix}.$$

At this point, the loop relationships between r, u, and y can be substituted and the resulting expression simplified to arrive at

$$y^T \hat{Q}_{i\hat{i}} y + 2y^T \hat{S}_{i\hat{i}} r + r^T \hat{R}_{i\hat{i}} r$$

where

$$\hat{Q}_{i\hat{i}} = \begin{bmatrix} Q_i + R_{\hat{i}} & S_{\hat{i}}^T + S_i \\ S_{\hat{i}} + S_i^T & Q_{\hat{i}} + R_i \end{bmatrix},$$

$$\hat{S}_{i\hat{i}} = \begin{bmatrix} S_i & R_{\hat{i}} \\ -R_i & S_{\hat{i}} \end{bmatrix} \quad \text{and} \quad \hat{R}_{i\hat{i}} = \begin{bmatrix} R_i & 0 \\ 0 & R_{\hat{i}} \end{bmatrix}.$$

By assumption, $\hat{Q}_{i\hat{i}} \leq 0, \forall i = 1, ..., M_1$ and $\forall j = 1, ..., M_2$. This implies that the unforced system ($r(t) = 0$) satisfies $V_i(T) \leq V_i(t_K)$ for each active subsystem i.

As in Theorem 1, a series of δ_0, δ_1, ... , δ_K can be defined to bound the sum of the storage functions at each switch. This argument can be repeated from the last switch t_K to the first one t_1 to show that, for any $\epsilon > 0$, there exists a $\delta > 0$ such that when $||x(t_0)|| \leq \delta$ then $V(t_0) \leq \rho(\delta)$ and eventually $V(t_K) \leq \frac{1}{2}\rho(\epsilon)$. Overall, the bound on $||x(t_0)||$ implies the following inequality,

$$V(T) \leq \frac{1}{2}\rho(\epsilon).$$

This statement along with (12) confirms that $V(t) \leq \rho(\epsilon), \forall t$ which implies that $||x(t)|| \leq \epsilon, \forall t$ whenever $||x(t_0)|| \leq \delta$. Since this is valid for all ϵ, the feedback interconnection is stable. $\square$

This result gives stability conditions for the feedback of two dissipative systems with no input. However, we are often interested in studying systems that have further systems interconnected as is the case in the study of large scale systems. Alternatively, we might simply be interested in the input-output properties of a single feedback interconnection of two systems. This sort of analysis can be done if we consider the dissipative properties of a feedback system. The following theorem assesses dissipative rate of a feedback connection based on the dissipative rates of the two systems in feedback.

Corollary 1. *The feedback interconnection of two dissipative switched systems is dissipative with respect to the supply rate*

$$\omega_i(t) = y^T \hat{Q}_{i\hat{i}} y + 2y^T \hat{S}_{i\hat{i}} r + r^T \hat{R}_{i\hat{i}} r$$

where

$$\hat{Q}_{i\hat{i}} = \begin{bmatrix} Q_i + R_{\hat{i}} & S_{\hat{i}}^T + S_i \\ S_{\hat{i}} + S_i^T & Q_{\hat{i}} + R_i \end{bmatrix},$$

$$\hat{S}_{i\hat{i}} = \begin{bmatrix} S_i & R_{\hat{i}} \\ -R_i & S_{\hat{i}} \end{bmatrix} \quad \textit{and} \quad \hat{R}_{i\hat{i}} = \begin{bmatrix} R_i & 0 \\ 0 & R_{\hat{i}} \end{bmatrix}.$$

This result was derived as part of the proof of the previous theorem so it will not be derived again. Beyond simply assessing stability, this result allows us to assess the level of dissipativity of the interconnection. This assessment can be used as additional systems are added in feedback. As long as the dissipative rate of each loop is considered, stability of the overall interconnection can be given if the final connection satisfies the conditions of Theorem 2.

V. Passive Discrete-time Switched Systems

For non-switched systems, an important class of dissipative systems is passive systems. Passivity is a property that implies stability and the property is preserved when systems are combined in feedback. Combining these two results gives open loop conditions for closed loop stability. Additionally, large scale systems can be shown to be stable if each component is passive and the components are sequentially combined in feedback or in parallel. The concept of passivity for switched systems has been explored for discrete-time [14], but the previous work has not considered the feedback interconnection. To the best of our knowledge, the only paper that investigated the feedback of passive switched systems was presented in continuous-time [10]. This section presents a notion of passivity for switched systems. This will include results showing that passive switched systems are stable and that passivity is preserved in feedback and in parallel.

Definition 2. *A discrete-time switched system is passive if it is dissipative with respect to a QSR supply rate where $Q_i = 0$, $R_i = 0$, and $S_i = \frac{1}{2}I$ ($\forall i$), and the cross supply rates are bounded for all switching signals.*

It is important to note that S_i being a square matrix implies that the dimension of the input u and output y must be the same. With this definition, a corollary can be stated. It is a special case of Theorem 2 so it will not be proven.

Corollary 2. *A passive switched system is stable for zero input ($u(t) = 0$).*

The passivity property can be used when considering interconnections of systems. The following result shows stability of the feedback interconnection of two passive systems.

Theorem 3. *The feedback interconnection (Fig. 1) of two passive switched systems G_1 and G_2 forms a passive switched system.*

Proof. By each system being passive, there exists $V_i^{(1)}$ and $V_{\hat{i}}^{(2)}$ such that the following inequalities hold, for $t \in I_i$ for G_1 and $t \in I_{\hat{i}}$ for G_2

$$V_i^{(1)}(t+1) \leq V_i^{(1)}(t) + u_1^T y_1$$
$$V_{\hat{i}}^{(2)}(t+1) \leq V_{\hat{i}}^{(2)}(t) + u_2^T y_2.$$

Additionally, the following hold $\forall i, \hat{i}$ when $t \in I_j$ for G_1 and $t \in I_{\hat{j}}$ for G_2

$$V_i^{(1)}(t+1) \leq V_i^{(1)}(t) + \phi_i^j(t)^{(1)}$$
$$V_{\hat{i}}^{(2)}(t+1) \leq V_{\hat{i}}^{(2)}(t) + \phi_{\hat{i}}^{\hat{j}}(t)^{(2)}.$$

Define $V_{i\hat{i}}(t) = V_i^{(1)}(t) + V_{\hat{i}}^{(2)}(t)$ and $\phi_{i\hat{i}}^{j\hat{j}}(t) = \phi_i^j(t)^{(1)} + \phi_{\hat{i}}^{\hat{j}}(t)^{(2)}$. Since each of $\phi_i^j(t)^{(1)}$ and $\phi_{\hat{i}}^{\hat{j}}(t)^{(2)}$ are absolutely summable then the sum of the two is absolutely summable. Note that $u_1^T y_1 + u_2^T y2 = r_1^T y_1 + r_2^T y_2 = r^T y$. At this point, these definitions can be used to demonstrate that when subsystems i and $\hat{i}$ are active

$$V_{i\hat{i}}(t+1) \leq V_{i\hat{i}}(t) + r^T y$$

and when subsystems j and $\hat{j}$ are active then $\forall i, \hat{i}$

$$V_{i\hat{i}}(t+1) \leq V_{i\hat{i}}(t) + \phi_{i\hat{i}}^{j\hat{j}}(t).$$

Since the $\phi_{i\hat{i}}^{j\hat{j}}$ are absolutely summable for all switching sequences, the feedback interconnection G is a passive switched system. □

As in the non-switched case, these results can be used to verify closed loop stability by showing that the two systems in feedback are passive. This result can also be used from a design perspective. When controlling a passive switched system, any passive controller is stabilizing without additional conditions. This allows for a large class of controllers to be applied directly including traditional PI controllers.

A similar argument can be used to show that the parallel interconnection of two passive switched systems is a passive switched system.

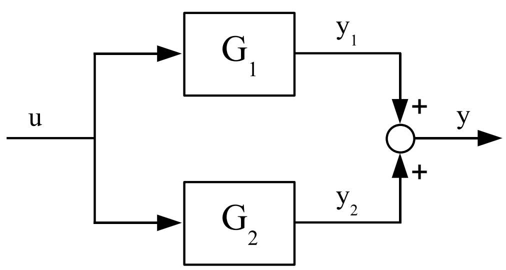

Fig. 2. The parallel interconnection of two systems.

Theorem 4. *The parallel interconnection (Fig. 2)of two passive switched systems G_1 and G_2 forms a passive switched system, G.*

Proof. The proof is similar to the feedback case with the main difference being the signal relationships. The new output y is defined $y = y_1 + y_2$. Since both G_1 and G_2 are passive then the mapping from u to y (system G) is also passive. □

These connection results can be used to demonstrate stability for large scale systems. As long as each component is a passive (switched or non-switched) system and each component is sequentially combined in feedback or parallel, then the resulting large scale system is passive and stable. These results greatly simplify stability analysis for large scale systems with switching. Demonstrating stability without these results is typically an involved process where any individual subsystem or connection can cause the entire interconnection to lose stability.

VI. Conclusions

This paper presented a notion of dissipativity for nonlinear discrete-time switched systems. This draws upon the well-established notion of QSR dissipativity for non-switched systems. The definition is a natural extension as it simplifies to the traditional notion of QSR dissipativity when applied to a system without switching dynamics. This paper included analysis tools for assessing stability and stability of feedback interconnections. The special case of passive switched systems was considered. Passivity was shown to imply stability and the feedback interconnection of two passive systems remains passive.

VII. Acknowledgement

The support of the National Science Foundation under Grant No. CNS-0819865 and Grant No. CNS-1035655 is gratefully acknowledged.

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