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Characterization of Robust Feedback Nash Equilibrium for Multi-Channel Systems

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Abstract— **In this paper, we consider the problem of robust state-feedback stabilization for multi-channel systems from a game-theoretic framework. In such a framework, we characterize the feedback Nash equilibria via a set of stabilizing state-feedback solutions corresponding to a family of perturbed multi-channel systems with dissipativity properties. Specifically, we show that the existence of a weak-optimal solution to a set of constrained dissipativity problems is a sufficient condition for the existence of a feedback Nash equilibrium, whereas the set of robust stabilizing state-feedback solutions is completely described in terms of a set of extended linear matrix inequalities.**

I. INTRODUCTION

In this paper, we consider a multi-channel system governed by several players (or *decision makers*) where the stability of the overall closed-loop system is a common objective while each player aims to maximize different types of objective functions. In such a scenario, Nash strategy offers a suitable framework to study an inherent robustness or non-fragile property of the strategies under a family of information structures, since no player can improve his payoff by deviating unilaterally from the Nash strategy once the equilibrium is attained (e.g., see references [1]-[5]).

In the past, several theoretical results have been established to characterize control related problems in the context of Nash equilibria via a game theoretic interpretation [5]–[9]. For example, the existence of open-loop Nash strategies for linear-quadratic games over a finite time-horizon, assuming that all strategies lie in compact subsets of an admissible strategy space, has been addressed in [10], [11] and [1]; the existence of Nash Strategies for linear-quadratic differential games over an infinite-horizon has been studied in detail in [7], [8], [5] and [12]. We also note that some of these works have discussed the uniqueness of the optimal strategies for linear-quadratic games with structured uncertainties, where the bound for the objective function is based on the existence of a set of solutions for appropriately parameterized Riccati equations. Moreover, in the area of multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ control theory, the concept of differential games has been applied by interpreting uncertainty (or neglected dynamics) as a fictitious player while the model of the system is supposed to be well known; where the fictitious player is usually introduced in the criteria through a weighting matrix (e.g., see references [13]-[16]).

On the other hand, the use of different simplified models of the same system has been employed for capturing certain information structures, models or objective functions that individual players may hold about the overall system. Thus, the resulting problem can be best described by nonzero-sum differential games where the individual players are allowed to minimize different types of objective functions (e.g., see references [17]-[19]). An extensive survey on the area of noncooperative dynamic games is provided in the book by Başar and Olsder [5].

Our main focus in this paper is to take this line of approach, where individual players have different objective functions that are associated with certain information structures, i.e., the dissipativity property of the multi-channel system, where the optimality concept is that of Nash equilibrium. We characterize the feedback Nash equilibria via a set of stabilizing state-feedback solutions corresponding to a family of perturbed multi-channel systems with dissipativity properties (see [20], [21] and references therein for a review of systems with dissipative properties). We specifically consider two fundamental problems: (i) Firstly, we isolate a condition guaranteeing that the control/strategy space is sufficiently *decentralized* to make the game-theoretic interpretation sensible, and (ii) Secondly, we provide a sufficient condition for the existence of *robust* feedback Nash equilibrium, where the individual players have different objective functions that are associated with certain information structures, i.e., the dissipativity inequalities, of the system. Moreover, we show that the existence of a weak-optimal solution to a set of constrained dissipativity problems is a sufficient condition for the existence of a feedback Nash equilibrium.

The rest of the paper is organized as follows. In Section II, we present a stability condition for a multi-channel system in terms of a set of extended linear matrix inequalities (LMIs), with a certain dissipativity property being used to extend the stability condition when there is a model perturbation in the system. Section III presents the main results, where we provide a sufficient condition for the existence of a feedback Nash equilibrium via a weak-optimal solution corresponding to a set of constrained dissipativity problems. Finally, Section IV provides some concluding remarks.

Notation. For a matrix $A \in \mathbb{R}^{n\times n}$, $\mathrm{He}\,(A)$ denotes a hermitian matrix defined by $\mathrm{He}\,(A) \triangleq (A + A^T)$, where A^T is the transpose of A. For a matrix $B \in \mathbb{R}^{n\times p}$ with $r = \mathrm{rank}\, B$,

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$B^{\perp} \in \mathbb{R}^{(n-r)\times n}$ denotes an orthogonal complement of B, which is a matrix that satisfies $B^{\perp}B = 0$ and $B^{\perp}B^{\perp T} \succ 0$. $\mathbb{S}_{+}^{n}$ denotes the set of strictly positive definite $n \times n$ real matrices and $\mathbb{C}^{-}$ denotes the set of complex numbers with negative real parts, that is $\mathbb{C}^{-} \triangleq \{s \in \mathbb{C} \,|\, \mathrm{Re}\{s\} < 0\}$. $\mathrm{Sp}(A)$ denotes the spectrum of a matrix $A \in \mathbb{R}^{n\times n}$, i.e., $\mathrm{Sp}(A) \triangleq \{\lambda \in \mathbb{C} \,|\, \mathrm{rank}(A - \lambda I) < n\}$ and $\mathrm{GL}_n(\mathbb{R})$ denotes the general linear group consisting of all $n \times n$ real nonsingular matrices.

II. Preliminaries

Consider a continuous-time N-channel system

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{N} B_j u_j(t), \quad x(0) = x_0, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times r_j}$, $x(t) \in \mathbb{R}^n$ is the state of the system, and $u_j(t) \in \mathbb{R}^{r_j}$ is a control input to the jth-channel of the system.

For this system, consider the set of all stabilizing state-feedback gains

$$\mathcal{K}_N \subseteq \left\{ \left(K_1, K_2, \ldots, K_N \right) \in \prod_{j=1}^{N} \mathcal{K}_j \subseteq \prod_{j=1}^{N} \mathbb{R}^{r_j\times n} \;\middle|\; \mathrm{Sp}\left(A + \sum_{j=1}^{N} B_j K_j \right) \in \mathbb{C}^{-} \right\}, \tag{2}$$

where $K_j \in \mathbb{R}^{r_j\times n}$ for $j = 1, 2, \ldots, N$.

Let us introduce the following matrices that will be used in Theorem 1 (and also in the following section).

$$E = [\underbrace{I_{n\times n} \quad I_{n\times n} \quad \cdots \quad I_{n\times n}}_{(N+1)\ times}],$$

$$[A, B]_{U,\widetilde{L}} = [\underbrace{AU \quad B_1L_1 \quad B_2L_2 \quad \cdots \quad B_NL_N}_{(N+1)\ times}],$$

$$\langle U, \widetilde{W} \rangle = \mathrm{block\,diag}\{\underbrace{U, W_1, W_2, \ldots, W_N}_{(N+1)\ times}\}.$$

Then, we characterize the set $\mathcal{K}_N$ in terms of a set of extended LMIs as follow.

Theorem 1: For any stabilizable pair $\left(A, [\, B_1 \quad B_2 \quad \cdots \quad B_N \,]\right)$, there exist $X \in \mathcal{S}_{+}^{n}$, $U \in \mathcal{GL}_n(\mathbb{R})$, $W_j \in \mathcal{GL}_n(\mathbb{R})$, $\epsilon > 0$ and $L_j \in \mathbb{R}^{r_j\times n}$ for $j = 1, 2, \ldots, N$ such that

$$\begin{bmatrix} 0_{n\times n} & XE \\ E^TX & 0_{(N+1)n\times(N+1)n} \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} [A, B]_{U,\widetilde{L}} \\ -\langle U, \widetilde{W} \rangle \end{bmatrix} \times \begin{bmatrix} E^T & \epsilon I_{(N+1)n\times(N+1)n} \end{bmatrix} \right) \prec 0, \tag{3}$$

Moreover, for any N-tuple family $(\, L_1, \; L_2, \; \ldots, \; L_N \,)$ and $(\, W_1, \; W_2, \; \ldots, \; W_N \,)$ as above, and setting $K_j = L_jW_j^{-1}$ for each $j = 1, 2, \ldots, N$, the matrix $\left(A + \sum_{j=1}^{N} B_jK_j\right)$ is a Hurwitz matrix.[1]

Proof: Note that

$$\begin{bmatrix} [A, B]_{U,\widetilde{L}} \\ -\langle U, \widetilde{W} \rangle \end{bmatrix}^{\perp} = \begin{bmatrix} I & [A, B]_{U,\widetilde{L}} \langle U, \widetilde{W} \rangle^{-1} \end{bmatrix},$$

$$\begin{bmatrix} E \\ \epsilon I \end{bmatrix}^{\perp} = \begin{bmatrix} \epsilon I & -E \end{bmatrix}.$$

Then, eliminating $\langle U, \widetilde{W} \rangle$ from (3) by using these matrices, we have two inequalities

$$\begin{aligned} &\begin{bmatrix} I & [A, B]_{U,\widetilde{L}} \langle U, \widetilde{W} \rangle^{-1} \end{bmatrix} \begin{bmatrix} 0 & XE \\ E^TX & 0 \end{bmatrix} \times \begin{bmatrix} I \\ (\langle U, \widetilde{W} \rangle^{-1})^T [A, B]_{U,\widetilde{L}}^T \end{bmatrix} \\ &\quad = \mathrm{He}\left((A + \sum_{i=1}^{N} B_iK_i)X \right) \prec 0, \end{aligned} \tag{4}$$

$$\begin{bmatrix} \epsilon I & -E \end{bmatrix} \begin{bmatrix} 0 & XE \\ E^TX & 0 \end{bmatrix} \begin{bmatrix} \epsilon I \\ -E^T \end{bmatrix} = -2\epsilon(N+1)X \prec 0. \tag{5}$$

Hence, we see that (4) and (5) state exactly the Lyapunov stability condition with $X \in \mathcal{S}_{+}^{n}$ and state-feedback gains $K_j = L_jW_j^{-1}$ for $j = 1, 2, \ldots, N$.

Suppose the system in (1) is stable with state-feedback gains $K_j = L_jW_j^{-1}$ for $W_j \in \mathcal{GL}_n(\mathbb{R})$, $j = 1, 2, \ldots, N$. Then, there exists a sufficiently small $\epsilon > 0$ that satisfies

$$\mathrm{He}\left((A + \sum_{j=1}^{N} B_jK_j)X \right) + \frac{1}{2}\epsilon \left[A, B \right]_{X,\widetilde{L}} \times \langle X, \widetilde{X} \rangle \left[A, B \right]_{X,\widetilde{L}}^T \prec 0, \tag{6}$$

where $\langle X, \widetilde{X} \rangle = \mathrm{block\,diag}\{ \underbrace{X, X, \ldots, X}_{(N+1)\ times} \}$ and $[A, B]_{X,\widetilde{L}} = [\underbrace{AX \quad B_1L_1 \quad B_2L_2 \quad \cdots \quad B_NL_N}_{(N+1)\ times}]$.

Note that $\langle X, \widetilde{X} \rangle \succ 0$ and $\langle X, \widetilde{X} \rangle E^T = E^TX$, employing the Schur complement for (6), then we have

$$\begin{aligned} &\begin{bmatrix} \mathrm{He}\left((A + \sum_{j=1}^{N} B_jK_j)X \right) & \epsilon\,[A, B]_{X,\widetilde{L}} \langle X, \widetilde{X} \rangle \\ \epsilon \langle X, \widetilde{X} \rangle ([A, B]_{X,\widetilde{L}})^T & -2\epsilon \langle X, \widetilde{X} \rangle \end{bmatrix} \\ &= \begin{bmatrix} 0 & XE \\ E^TX & 0 \end{bmatrix} \\ &+ \mathrm{He}\left(\begin{bmatrix} [A, B]_{X,\widetilde{L}} \langle U, \widetilde{W} \rangle^{-1} \\ -I \end{bmatrix} \langle X, \widetilde{X} \rangle \begin{bmatrix} E^T & \epsilon I \end{bmatrix} \right) \\ &\prec 0. \end{aligned}$$

[1]Recently, a similar LMIs condition has been investigated by Fujisaki and Befekadu [23] in the context of reliable decentralized stabilization for multi-channel systems.

This means that (3) holds with $\langle U, \widetilde{W}\rangle = \langle X, \widetilde{X}\rangle$ for $U \in \mathcal{GL}_n(\mathbb{R})$. ■

Consider next a multi-channel system with a perturbation term, i.e.,

$$\dot{x}(t) = \left(A + u_\rho A^\delta\right) x(t) + \sum_{j=1}^{N} B_j u_j(t), \tag{7}$$

where $u_\rho \in [-\rho, \rho]$, $\rho \in \mathbb{R}_+$ is the uncertainty level and $A^\delta \in \mathbb{R}^{n\times n}$ is a fixed perturbation term in the system. Here we assume that the perturbed matrix $(A + u_\rho A^\delta)$ lies in a compact uncertainty set $\mathcal{U}_\rho$.[2]

In what follows, we assume there exits a set of stabilizing state-feedback gains $\mathcal{K}_N$ that maintains the stability of the system in (1) and this set is completely characterized via a solution of (3). Then, we will estimate an upper bound $\hat{\rho} \in \mathbb{R}_+$ on the uncertainty level for which the state-feedback gains preserve robust stability property of the perturbed multi-channel system.

Lemma 1: Let $X \in \mathcal{S}_+^n$, $U \in \mathcal{GL}_n(\mathbb{R})$, $W_i \in \mathcal{GL}_n(\mathbb{R})$, $L_j \in \mathbb{R}^{r_j\times n}$, $j = 1, 2, \ldots, N$ and $\epsilon > 0$ satisfy Theorem 1. Suppose $\alpha > 0$, $\beta \geq 1$ and $Z \in \mathcal{S}_+^n$, then there exist an upper bound $\hat{\rho} \in \mathbb{R}_+$ and $Y \in \mathcal{S}_+^n$ such that

$$\beta^{-1} Z \preceq Y \preceq Z, \tag{8}$$

$$\begin{bmatrix} I \\ \langle U, \widetilde{W}\rangle^{-1} E^T \end{bmatrix}^T \begin{bmatrix} u_{\hat{\rho}}\,\mathrm{He}((A^\delta)^T Y) & Y\,[A, B]_{U,\widetilde{L}} \\ ([A, B]_{U,\widetilde{L}})^T Y & 0 \end{bmatrix} \times \begin{bmatrix} I \\ \langle U, \widetilde{W}\rangle^{-1} E^T \end{bmatrix} \preceq -\alpha Z. \tag{9}$$

Moreover, the perturbed multi-channel system in (7) is stable for all instances of perturbation $u_{\hat{\rho}} \in [-\hat{\rho}, \hat{\rho}]$ in the system.

Proof: To prove the above theorem, we require the following system

$$\begin{aligned} \dot{x}(t) &= \left(A + u_\rho A^\delta + \sum_{j=1}^{N} B_j K_j\right) x(t) + 0_{n\times 1}\tilde{u}(t), \\ \tilde{y}(t) &= x(t) + 0_{n\times 1}\tilde{u}(t), \end{aligned} \tag{10}$$

to satisfy certain dissipativity property for all instances of perturbation in the system.

Define the following supply rate

$$w_{(\alpha,Z)}(\tilde{y}(t), \tilde{u}(t)) = \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}^T \begin{bmatrix} -\alpha Z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}, \tag{11}$$

with $Z \in \mathcal{S}_+^n$ and $\alpha > 0$. We clearly see that if the system in (10) is stable for all instances of perturbation. Then, the following *dissipation inequality* will hold

$$V(x(0)) + \int_0^t w_{(\alpha,Z)}(\tilde{y}(t), \tilde{u}(t))dt \geq V(x(t)), \tag{12}$$

for all $t \geq 0$ with non-negative quadratic storage function $V(x(t)) = x(t)^T Y x(t)$, $Y \in \mathcal{S}_+^n$ that satisfies $V(0) = 0$.

Condition (12) with (11) further implies the following

$$\mathrm{He}\left(\, (A + u_\rho A^\delta + \sum_{j=1}^{N} B_j K_j)^T Y \,\right) \preceq -\alpha Z. \tag{13}$$

Therefore, there exists an upper bound $\hat{\rho} \in \mathbb{R}_+$ for which the dissipativity condition in (13) will hold true for all instances of perturbation in the system.

Then, we have the following result

$$\begin{aligned} &\mathrm{He}\left(\left([A, B]_{U,\widetilde{L}}\langle U, \widetilde{W}\rangle^{-1} E^T + u_{\hat{\rho}} A^\delta\right)^T Y\right) = \\ &\begin{bmatrix} I \\ \langle U, \widetilde{W}\rangle^{-1} E^T \end{bmatrix}^T \begin{bmatrix} u_{\hat{\rho}}\,\mathrm{He}\left((A^\delta)^T Y\right) & Y[A, B]_{U,\widetilde{L}} \\ \left([A, B]_{U,\widetilde{L}}\right)^T Y & 0 \end{bmatrix} \\ &\times \begin{bmatrix} I \\ \langle U, \widetilde{W}\rangle^{-1} E^T \end{bmatrix} \preceq -\alpha Z, \end{aligned} \tag{14}$$

with $u_{\hat{\rho}} \in [-\hat{\rho},\ \hat{\rho}]$.[3]

On the other hand, let us define the following matrix interval

$$\mathcal{I}_{(\beta,Z)} = \left\{ Y \;\middle|\; \beta^{-1} Z \preceq Y \preceq Z \right\}, \tag{15}$$

where $Z \in \mathcal{S}_+^n$ and $\beta \geq 1$ are assumed to be known $a\ priori$. Suppose that Y satisfies the conditions in (8) and (9), then the trajectories of the perturbed closed-loop system

$$\dot{x}(t) = \left(A + u_\rho \delta A + \sum_{j=1}^{N} B_j K_j\right) x(t),$$

satisfy

$$\begin{aligned} &\frac{d}{dt}(x^T(t) Y x(t)) \\ &\quad = x^T(t)\,\mathrm{He}\left(\left(\, A + u_\rho \delta A + \sum_{j=1}^{N} B_j K_j \,\right)^T Y\right) x(t), \\ &\quad \leq -\alpha x^T(t) Z x(t), \\ &\quad \leq -\alpha x^T(t) Y x(t). \end{aligned} \tag{16}$$

Hence, condition (16) stating, equivalently, that $Y \in \mathcal{I}_{(\beta,Z)}$ is a dissipativity certificate with supply rate (11) for all instances of perturbation in (10) (e.g., see references [24] and [25]). ■

Remark 1: We remark that if there exists a solution set X for Lemma 1 that gives a minimum distance between X and the set $\mathcal{I}_{(\beta,Z)}$, i.e., $\varrho(X, Y) \triangleq \inf_{Y\in\mathcal{I}_{(\beta,Z)}} \|X - Y\|$, then we essentially have a weak-optimal solution. This solution is unique since $\mathcal{I}_{(\beta,Z)}$ is a convex and compact set [26]. Moreover, finding an upper bound $\hat{\rho} \in \mathbb{R}_+$ and Y from a closed and convex set $\mathcal{I}_{(\beta,Z)}$ is equivalent to solving the verification problem, i.e., the constrained dissipativity control problem (e.g., see reference [27]).

In the next section, we will see that such additional information structure, i.e., the dissipativity property, about the system is indeed useful in the context of game-theoretic framework.

[2]Note that the existence of a solution for state trajectories is well-defined and it is always upper semicontinuous in x_0 (e.g., see reference [22]).

[3]Note that the upper bound $\hat{\rho}$ continuously depends (*in the weak sense*) on x_0 and K_j, $j = 1, 2, \ldots, N$.

III. Main Results

In this section, we establish an equivalence result between the set of robust state-feedback gains corresponding to constrained dissipativity problem and the feedback Nash equilibria. Specifically, we consider two fundamental problems in this framework: (i) we first isolate a condition guaranteeing that the control/strategy space is sufficiently *decentralized* to make the game-theoretic scenario/interpretation meaningful, and (ii) then we provide a sufficient condition for the existence of *robust* feedback Nash equilibrium, where the individual players have different objective functions that are associated with certain information structures, i.e., the dissipativity inequalities, of the following system.

$$\begin{aligned}\dot{x}(t) &= \left(A + u_{\rho_j} A_j^{\delta} + \sum_{i=1}^{N} B_i K_i\right) x(t) + 0_{n\times 1}\tilde{u}(t),\\ \tilde{y}(t) &= x(t) + 0_{n\times 1}\tilde{u}(t),\end{aligned} \tag{17}$$

where $u_{\rho_j} \in [-\rho_j, \rho_j]$, $\rho_j \in \mathbb{R}_+$ and $A_j^{\delta} \in \mathbb{R}^{n\times n}$ are the uncertainty levels and the perturbation terms associated with the jth-player, respectively. We further assume that each perturbed system matrix $(A + u_{\rho_j} A_j^{\delta})$ lies in a compact uncertainty set $\mathcal{U}_{\rho_j}$ for $j = 1, 2, \ldots, N$ and $(K_1, K_2, \ldots, K_N) \in \mathcal{K}_N$.

Next it will be convenient to identify each objective function $J_j\colon \mathbb{R}^n \times \mathcal{U}_{\rho_j} \times \mathcal{K}_N \to \mathbb{R}_+$ with related function

$$\mathbb{R}^n \times \mathcal{U}_{\rho_j} \times \mathcal{K}_{N_{\neg j}} \to \mathbb{R}_+ \colon (x_0, u_{\rho_j}, K_{\neg j}) \mapsto J_j(x_0, u_{\rho_j}, K_{\neg j}), \tag{18}$$

for $j = 1, 2, \ldots, N$.

Then, we can specify a game Γ in in strategic form, i.e., the feedback Nash game, by the following data:

$$\Gamma\left(\mathcal{N}, \mathcal{K}_N, (J_j)_{j\in\mathcal{N}}, \left(A + u_{\rho_j} A_j^{\delta}, [B_j]_{j\in\mathcal{N}}\right)_{j\in\mathcal{N}}\right),$$

where $\mathcal{N} \triangleq \{1, 2, \ldots, N\}$ - is the players set.

Therefore, for such a game in strategic form, an N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$, (i.e., $K^* \triangleq (K_1^*, K_2^*, \ldots, K_N^*)$) is called a feedback Nash equilibrium if for all $j \in \mathcal{N}$, $K_j \in \mathbb{R}^{r_j\times n}$, all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ and each $x_0 \in \mathbb{R}^n$

$$J_j(x_0, u_{\hat{\rho}_j}, K_{\neg j}^*) \le J_j(x_0, u_{\hat{\rho}_j}, K^*), \tag{19}$$

where $K_{\neg j}^* \triangleq (K_1^*, \ldots, K_{j-1}^*, K_j, K_{j+1}^*, \ldots, K_N^*) \in \mathcal{K}_N$.[4]

In the following, we assume that the strategy space for each player is restricted to linear time-invariant state-feedback gains, and the resulting multi-channel closed-loop system is also assumed to be stable for all (or some) initial conditions $x_0 \in \mathbb{R}^n$.

[4]In this paper, the game is essentially defined in the framework of an incomplete information, since the jth-player's objective function involves different uncertainty information, i.e., u_{ρ_j}, about the system. However, we remark that the jth-player decides his own strategy by solving the optimization problem with the opponents' strategies $(K_1^*, \ldots, K_{j-1}^*, K_{j+1}^*, \ldots, K_N^*)$ fixed.

Introduce the following set of supply rate functions

$$\mathcal{W} = \left\{ w_{(\alpha_j, Z_j)}(\tilde{y}(t), \tilde{u}(t)) \;\middle|\; w_{(\alpha_j, Z_j)}(\tilde{y}(t), \tilde{u}(t)) = \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix}^T \begin{bmatrix} -\alpha_j Z_j & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{y}(t) \\ \tilde{u}(t) \end{bmatrix} \right\}, \tag{20}$$

for $j = 1, 2, \ldots, N$, and a matrix interval set $\mathcal{I}_{(\beta_j, Z_j)}$

$$\mathcal{I}_{(\beta_j, Z_j)} = \left\{ Y_j \;\middle|\; \beta_j^{-1} Z_j \preceq Y_j \preceq Z_j \right\}, \tag{21}$$

where $\alpha_j > 0$, $\beta_j \ge 1$ and $Z_j \in \mathcal{S}_+^n$ for $j = 1, 2, \ldots, N$.

In light of Lemma 1 and above discussion, we have the following theorem which provides a sufficient condition for the existence of feedback Nash equilibria.

Theorem 2: Let $W_j \in \mathcal{GL}_n(\mathbb{R})$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$. Assume that $\alpha_j > 0$, $\beta_j \ge 1$ and $Z_j \in \mathcal{S}_+^n$ for $j = 1, 2, \ldots, N$. Then, there exit $X_j \in \mathcal{S}_+^n$, $U_j \in \mathcal{GL}_n(\mathbb{R})$, $j = 1, 2, \ldots, N$ and an N-tuple $(L_1^*, L_2^*, \ldots, L_N^*) \in \prod_{j=1}^{N} \mathbb{R}^{r_j\times n}$ such that

$$\begin{bmatrix} 0 & X_j E \\ E^T X_j & 0 \end{bmatrix} + \operatorname{He}\left(\begin{bmatrix} [A, B]_{U_j, L_{\neg j}^*} \\ -\langle U_j, W\rangle \end{bmatrix} \begin{bmatrix} E^T & \epsilon_j I \end{bmatrix}\right) \prec 0, \tag{22}$$

where, for some $L_j \in \mathbb{R}^{r_j\times n}$,

$$\begin{aligned}&[A, B]_{U_j, L_{\neg j}^*}\\ &= [AU_j\; B_1 L_1^* \cdots B_{j-1} L_{j-1}^*\; B_j L_j\; B_{j+1} L_{j+1}^* \cdots B_N L_N^*]\end{aligned}$$

and

$$\langle U_j, W\rangle = \operatorname{block\,diag}\{U_j, W_1, W_2, \ldots, W_N\}.$$

Moreover, there exist $Y_j \in \mathcal{I}_{(\beta_j, Z_j)}$, $j = 1, 2, \ldots, N$ that satisfies the following related mapping

$$\sup_{(x_0, u_{\rho_j}, K_j) \in \mathbb{R}^n \times \mathcal{U}_{\rho_j} \times \mathbb{R}^{r_j\times n}} J_j(x_0, u_{\rho_j}, K_{\neg j}^*) \ni \hat{\rho}_j. \tag{23}$$

for which the closed-loop system in (17) is robustly stable for all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ with $K_j^* \in \arg\sup_{K_j \in \mathbb{R}^{r_j\times n}} J_j(x_0, u_{\hat{\rho}_j}, K_{\neg j}^*) \{\triangleq\; \hat{\rho}_j(x_0, u_{\hat{\rho}_j}, K^*)\}$ for all $j = 1, 2, \ldots, N$.[5]

Proof: Suppose all the perturbed systems in (17) satisfy the following dissipativity inequalities

$$V_j(x(0)) + \int_0^t w_{(\alpha_j, Z_j)}(\tilde{y}(t), \tilde{u}(t))dt \ge V_j(x(t)), \tag{24}$$

for all $t \ge 0$ with non-negative quadratic storage functions $V_j(x(t)) = x(t)^T Y_j x(t)$ and $Y_j \in \mathcal{I}_{Y_j}$ that satisfy $V_j(0) = 0$ for $j = 1, 2, \ldots, N$.

[5]In general, simultaneously solving a set of optimization problems, i.e., solving (23) together with (22), is not easy since it is a non-convex optimization problem which involves finding a solution satisfying at the intersection of a set of constrained quadratic functionals [28] (c.f. Remark 2, Section II above).

Thus, the trajectories of each perturbed closed-loop system (i.e., for $j = 1, 2, \ldots, N$)

$$\dot{x}(t) = \left(A + u_{\rho_j} A_j^{\delta} + \sum_{i=1}^{N} B_i K_i^*\right) x(t),$$

satisfy

$$\begin{aligned} &\frac{d}{dt}\left(x^T(t) Y_j x(t)\right) \\ &\quad = x^T(t)\,\mathrm{He}\left(\left(A + u_{\rho_j} A_j^{\delta} + \sum_{i=1}^{N} B_i K_i^*\right)^T Y_j\right) x(t), \\ &\quad \leq -\alpha_j x^T(t) Z_j x(t), \\ &\quad \leq -\alpha_j x^T(t) Y_j x(t). \end{aligned} \tag{25}$$

for all instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$ in the system.

Then, the rest of the proof follows the same lines as that of Theorem 1. In fact, replacing the following

$$[A, B]_{U, \widetilde{L}} \to [A, B]_{U_j, L^*_{\neg j}}, \quad \langle U, \widetilde{W} \rangle \to \langle U_j, \widetilde{W} \rangle \quad \text{and} \quad X \to X_j,$$

in Theorem 1 immediately gives the condition in (22) of Theorem 2. Note that K_j^* and K_j are given by

$$K_j^* = L_j^* W_j^{-1} \qquad \text{and} \qquad K_j = L_j W_j^{-1},$$

for $j = 1, 2, \ldots, N$.

Moreover, the N-tuple $(Y_1, Y_2, \cdots, Y_N) \in \prod_{j=1}^{N} \mathcal{I}_{(\beta_j, Z_j)}$ is a collection of dissipativity certificates corresponding to a set of supply rates (20) for all instances of perturbation in (17). ■

We next present a more realistic game-theoretic interpretation in terms of the upper uncertainty bounds $\hat{\rho}_j \in \mathbb{R}_+$ for all $j \in \mathcal{N}$ that describe the N-tuple uncertainty set $(u_{\hat{\rho}_1}, u_{\hat{\rho}_2}, \cdots, u_{\hat{\rho}_N}) \in \prod_{j=1}^{N} [-\hat{\rho}_j, \hat{\rho}_j]$ together with the existence of stabilizing state-feedback gains that provide a sufficient condition for obtaining a set of feedback Nash equilibria.

Hence, we have the following equivalent statements:

(i). $\exists K^* \in \mathcal{K}_N,\ \forall x_0,\ \forall u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j],\ \forall K^*_{\neg j} \in \mathcal{K}_N,\ \forall j \in \{1, 2, \ldots, N\}$ such that

$$J_j(x_0, u_{\hat{\rho}_j}, K^*_{\neg j}) \leq J_j(x_0, u_{\hat{\rho}_j}, K^*). \tag{26}$$

(ii). The extended LMIs condition in (22) and the dissipativity inequalities of (24) with a set of supply rates $\mathcal{W}$ in (20) completely describes the set of robust stabilizing state-feedback gains.

The equivalence between (i) and (ii) leads to characterization of feedback Nash equilibria over an infinite-time horizon in terms of stabilizing solutions of a set of extended LMIs.

Furthermore, the exact characterization of the feedback Nash equilibria is given by the following two theorems.

Theorem 3: Let $W_j \in \mathcal{GL}_n(\mathbb{R})$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$. Suppose $X_j \in \mathcal{S}_+^n$, $U_j \in \mathcal{GL}_n(\mathbb{R})$, $L_j^* \in \mathbb{R}^{r_j \times n}$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$ satisfy the extended LMIs condition in (22). Then, there exists an N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$ feedback Nash equilibrium with respect to the upper uncertainty bounds $\hat{\rho}_j \in \mathbb{R}_+$ for $j = 1, 2 \ldots, N$ of (23).

Proof: The first part of this theorem is already provided in Theorem 2, i.e., from the standard argument of the stabilizability of the pair $(A, [\, B_1 \quad B_2 \quad \cdots \quad B_N \,])$, we can always find an N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$ and for all $K_j = L_j W_j^{-1} \in \mathbb{R}^{r_j \times n}$ and $j \in \{1, 2 \ldots, N\}$ such that (22) holds. Applying (23) of Theorem 2 together with the dissipativity certificates $Y_j \in \mathcal{I}_{Y_j}$ and a set of supply rates $\mathcal{W}$ (20). Then, for a fixed $(x_0, K^*) \in \mathbb{R}^n \times \mathcal{K}_N$, we will obtain an upper bound $\hat{\rho}_j \in \mathbb{R}_+$ for all instances of perturbation in (17) and so that

$$J_j(x_0, u_{\hat{\rho}_j}, K^*_{\neg j}) \leq J_j(x_0, u_{\hat{\rho}_j}, K^*),$$

for all $j \in \{1, 2, \ldots, N\}$.

Hence, we immediately see that the N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$ satisfies the feedback Nash equilibrium. ■

Remark 2: The class of admissible strategies for all players are generated through a set of individual objective functions that are induced from dissipativity inequalities of (24) with a set of supply rates (20).

Theorem 4: Suppose the N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$ is a feedback Nash equilibrium with respect to the objective function values of (23). Assume that $W_j \in \mathcal{GL}_n(\mathbb{R})$ and $\epsilon_j > 0$ for $j = 1, 2, \ldots, N$. Then, there exists a solution set $X_j \in \mathcal{S}_+^n$, $U_j \in \mathcal{GL}_n(\mathbb{R})$ and $L_j^* \in \mathbb{R}^{r_j \times n}$ for $j = 1, 2, \ldots, N$ that satisfies the extended LMIs condition of (22).

Proof: Suppose the N-tuple $(K_1^*, K_2^*, \ldots, K_N^*) \in \mathcal{K}_N$ is a feedback Nash equilibrium such that

$$J_j(x_0, u_{\hat{\rho}_j}, K^*_{\neg j}) \leq J_j(x_0, u_{\hat{\rho}_j}, K^*),$$

where the value for the continuous objective function $J_j \colon \mathbb{R}^n \times \mathcal{U}_{\rho_j} \times \mathcal{K}_N \to \mathbb{R}_+$ is claimed as

$$\sup_{(x_0, u_{\rho_j}, K_j) \in \mathbb{R}^n \times \mathcal{U}_{\rho_j} \times \mathbb{R}^{r_j \times n}} J_j(x_0, u_{\rho_j}, K_{\neg j}) \mapsto \hat{\rho}_j,$$

with $K_j^* \in \arg \sup_{K_j \in \mathbb{R}^{r_j \times n}} J_j(x_0, u_{\hat{\rho}_j}, K^*_{\neg j}) \{\triangleq \hat{\rho}_j(x_0, u_{\hat{\rho}_j}, K^*)\}$ for all $j \in \mathcal{N}$.

Then, we can always find a solution set that satisfies the extended LMIs condition in (22) for which the closed-loop systems in (17) are robustly stable for all instances of perturbations $(u_{\hat{\rho}_1}, u_{\hat{\rho}_2}, \cdots, u_{\hat{\rho}_N}) \in \prod_{j=1}^{N} [-\hat{\rho}_j, \hat{\rho}_j]$. ■

Remark 3: Note that all closed-loop systems in (17) satisfy the dissipative inequalities of (24) with a set of supply rates (20) for all $j \in \mathcal{N}$ and instances of perturbation $u_{\hat{\rho}_j} \in [-\hat{\rho}_j, \hat{\rho}_j]$.

Note that the equivalence between (i) and (ii) (i.e., Theorem 3: (ii) $\Rightarrow$ (i) and Theorem 4: (i) $\Rightarrow$ (ii)) leads exactly to characterization of the feedback Nash equilibrium via a set of robust stabilizing state-feedback solutions of the extended LMIs.

IV. Concluding remarks

In this paper, we have looked the problem of state-feedback stabilization for a multi-channel system from a game-theoretic framework, where the class of admissible strategies for the players is induced from a solution set of the objective functionals that are realized through certain dissipativity inequalities. In such a scenario, we characterized the feedback Nash equilibria via a set of robust stabilizing state-feedback gains corresponding to constrained dissipativity problems. Moreover, we showed that the existence of a weak-optimal solution to the constrained dissipativity problem is a sufficient condition for the existence of a feedback Nash equilibrium.

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