Passivity Analysis for Discrete-Time Periodically Controlled Nonlinear Systems

Technical Report of the ISIS Group

at the University of Notre Dame

ISIS-2012-003

March, 2012

Yue Wang, Vijay Gupta, and Panos J. Antsaklis

Department of Electrical Engineering

University of Notre Dame

Notre Dame, IN 46556

Interdisciplinary Studies in Intelligent Systems

Passivity Analysis for Discrete-Time Periodically Controlled Nonlinear Systems

Yue Wang, Vijay Gupta, and Panos J. Antsaklis

Abstract

In this paper, we generalize the classical definition of passivity to discrete-time periodically controlled nonlinear systems that are open loop nonpassive and closed-loop feedback passive. In the classical passivity theory, a system is said to be passive if there exists a storage function such that the increase in storage function is bounded by the energy supplied to it at every time step. This definition may be overly restrictive. Therefore, we consider a periodically controlled system to be passive as long as the increase in storage function at the end of a period as compared to the beginning of the period is bounded by the accumulated energy supplied to it within this period. Because the input/output structure switches periodically, we need to exploit the zero dynamics of the closed-loop system and relate it to the passivity of the periodically controlled system. We prove that: 1) There exists a maximum allowable transmission ratio (MATR) between the time steps at which the system evolves open and closed-loop within a period such that the generalized passivity of system zero dynamics is guaranteed; and 2) If the system zero dynamics is passive, the original periodically controlled nonlinear system is also locally passive.

I. Introduction

This paper investigates the passivity properties of discrete-time periodically controlled nonlinear systems that are nonpassive in open loops and feedback passive in closed-loops. In the classical passivity theory, a system is said to be passive if there exists a storage function such that the increase in storage function is bounded by the energy supplied to it at every time step. The supplied power is the product of system input and output. For the periodically controlled system, since it is nonpassive in open loop, the increase in storage function might be greater than the energy supplied to it at these time steps. Thus, the system is considered to be nonpassive according to the classical definition of passivity. The extension of passivity to switched systems as in [1] is not helpful either since that work assumes each of the individual modes to be

The authors are with the Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN-46556, USA. {ywang18,vgupta2,antsaklis.1}@nd.edu.

passive. In this paper, we generalize this classical definition and consider such a periodically controlled system to be passive if the increase in storage function at the end of a period as compared to the beginning of the period is bounded by the total energy supplied to it within this period. Furthermore, due to the switching between open and closed-loop in each period, the structure of system input/output varies periodically. Exploiting the zero dynamics of the closed-loop process, we prove the existence of a maximum allowable transmission ratio (MATR) between the open and closed-loop within a period such that the system consisting of the open loops and the zero dynamics of the closed-loops is passive under the generalized definition. For the sake of simplicity, we will call this system as the zero dynamics of the original periodically controlled system. Given a passive zero dynamics, we prove that the periodically controlled nonlinear system is locally passive using feedback.

We first review some related literature. The study of periodic systems has received considerable attention over the last few decades due to its abundance in control and signal processing [2]. One of the main categories uses a lifting technique to represent periodically time-varying systems by the Linear Time-Invariant (LTI) ones (see [3]–[5] and references therein). Other popular approaches for periodic system control include Riccati equation based [6], Linear Matrix Inequality (LMI) based [7], and Model Predictive Control (MPC) based [8], [9] stability approaches.

In the literature, most of the work on periodic systems has focused on stability analysis. While stability is a fundamental property of dynamical systems, passivity ([10]–[13]) is also desirable because of several additional properties that passivity guarantees i) the free dynamics and zero dynamics of a passive system is passive, ii) both the negative feedback and parallel interconnections of passive systems are passive, and iii) a passive system can achieve asymptotical stability using feedback if it is zero state detectable (ZSD) [13]. Under the classical definition, passivity has been applied to the analysis of many dynamical systems, e.g., Cyber-Physical Systems (CPS) [14], switched systems [15], event-triggered systems [16], and hybrid systems [17].

Although passivity has been widely used as an input-output property for systems analysis, the classical definition may be overly restrictive. Specially, we generalize the definition to periodically controlled nonlinear systems that are open loop nonpassive and closed-loop feedback passive. It is consistent with the generalized passivity definition for networked nonlinear systems with nonconsecutive packet drops in [18]. However, in this paper, we consider systems that evolve in consecutive closed-loop processes followed by consecutive open loop processes. Feedback control is applied periodically to keep system passivity under the generalized definition. Also related is the generalized asymptotic stability analysis of discrete-time nonlinear time varying systems in [19], where the Lyapunov function is non-increasing only on certain unbounded time sets.

Different from the stability analysis in [19], the passivity analysis is complicated by the fact that both the input and the output are periodically time-varying. Due to this difficulty, we analyze the passivity of the periodically controlled system based on its zero dynamics, which is the internal dynamics of the system that is consistent with constraining the system output to zero [12]. We prove that given a passive zero dynamics, there exists a feedback control law to passivate the closed-loop system ([12], [20], [21], and in particular, [22]) such that the periodically controlled nonlinear system is passive under the generalized passivity definition.

The remainder of the paper is organized as follows. In Section II, we formulate the discrete-time periodically controlled nonlinear system and give a generalized passivity definition. Section III analyzes the passivity properties of the system zero dynamics. In Section IV, we investigate the passivity of the original periodically controlled system given a passive zero dynamics. A numerical example is provided in Section V. Section VI gives the concluding remarks. Some background on the classical passivity theory in discrete-time setting is provided in the Appendix.

II. Problem Formulation

Consider the following discrete-time nonlinear system

$$\begin{cases} \mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}, \tag{1}$$

with state space $\mathbf{X} = \mathbb{R}^n$, set of input values $\mathbf{U} = \mathbb{R}^m$ and set of output values $\mathbf{Y} = \mathbb{R}^m$. $\mathbf{x}(t) \in \mathbf{X}$, $\mathbf{u}(t) \in \mathbf{U}$, and $\mathbf{y}(t) \in \mathbf{Y}$ are the state, input and output variables, respectively. Here, $\mathbf{u}(t)$ is a nonlinear feedback controller. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be smooth (i.e., C^∞) vector fields. The system is assumed to have relative degree zero, i.e., $\frac{\partial h(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}$ is non-singular. This is a reasonable assumption because it is shown respectively in [23] and [21] that, a nonlinear system can be rendered lossless/passivity if and only if it has relative degree zero and lossless/passive zero dynamics. Therefore, we are not concerned with the passivity of discrete-time systems with outputs independent of inputs. The open loop of system (1) is given as follows

$$\begin{cases} \mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{0}) \\ \mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{0}) \end{cases}. \tag{2}$$

Let a system evolve in closed-loop (1) at time instants $k_1^c, k_2^c, \cdots$ and open loops (2) at time instants $k_1^o, k_2^o, \cdots$. We assume the open loop is nonpassive and the closed-loop is feedback passive according to Definition A.3 in the Appendix. The generalized passivity definition of a system $\mathcal{S}$ consisted of both nonpassive open loops and feedback passive closed-loops is given as follows.

Definition 2.1: [18] A nonlinear system $\mathcal{S}$ is said to be *locally passive* if there exists a positive semidefinite storage function $\tilde{\tilde{V}}(\mathbf{x}(.)) \geq 0$ ($\tilde{\tilde{V}}(\mathbf{x}(.)) = 0$ if and only if $\mathbf{x}(.) = \mathbf{0}$) such that for any $\mathbf{x}(k) \in \mathbb{R}^n$, $\mathbf{u}(k)\mathbb{R}^m$, and any given $t \in \mathbb{Z}^+$, the following passivity inequality holds in a neighborhood of the equilibrium point $(\mathbf{x}^*(k), \mathbf{u}^*(k))$:

$$\tilde{\tilde{V}}(\mathbf{x}(t)) - \tilde{\tilde{V}}(\mathbf{x}(1)) \leq \sum_{\theta=1}^{t-1} \mathbf{u}^{\mathrm{T}}(\theta)\mathbf{y}(\theta). \tag{3}$$

Now assume a system which evolves in closed-loop (1) for τ consecutive time instants followed by $T - \tau$ consecutive open loops (2). Repeat the process periodically, we have the following periodically controlled system:

$$\begin{aligned} &\begin{cases} \mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}, \ kT + 1 \leq t \leq kT + \tau - 1, \\ &\text{and} \\ &\begin{cases} \mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{0}) \\ \mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{0}) \end{cases}, \ kT + \tau \leq t \leq (k+1)T - 1, \end{aligned} \tag{4}$$

where T is the period of the periodic system. The quantity τ is the number of time steps at which system evolves in closed-loop feedback passive configuration and $T - \tau$ is the number of time steps at which system evolves in open loop. The ratio between the time steps at which the system evolves in an uncontrolled and controlled fashion is $r = \frac{T-\tau}{\tau}$. Figure 1 shows the framework of such a periodically controlled system.

Given Definition 2.1, the generalized passivity definition for the periodically controlled system (4) is as follows.

Definition 2.2: The discrete-time periodically controlled nonlinear system (4) is said to be *locally*

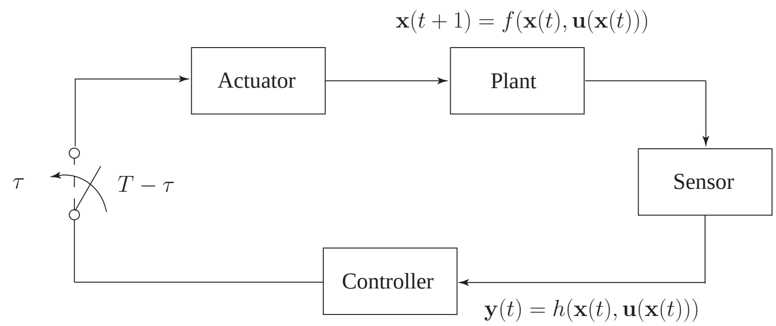

Fig. 1. Basic framework of a periodically controlled system.

passive with respect to the supply rate $\mathbf{u}^{\mathrm{T}}\mathbf{y}$ if there exists a positive semidefinite storage function $\tilde{\tilde{V}}(\mathbf{x})$ ($\tilde{\tilde{V}}(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$) such that the following passivity inequality holds in a neighborhood of the equilibrium point $(\mathbf{x}^*, \mathbf{u}^*)$

$$\tilde{\tilde{V}}(\mathbf{x}((k+1)T)) - \tilde{\tilde{V}}(\mathbf{x}(kT+1)) \leq \sum_{t=kT+1}^{(k+1)T-1} \mathbf{u}^{\mathrm{T}}(t)\mathbf{y}(t), \ \forall T \in \mathbb{Z}^+, \ k \in \{0\} \bigcup \mathbb{Z}^+. \tag{5}$$

Remark 2.1: Assume that system (4) starts with closed-loop (1) and continues for τ consecutive time steps. According to Definition 2.2, $T-\tau$ consecutive open loops (2) are allowed to follow as long as a feedback control is applied periodically such that system passivity is preserved for each period.

Remark 2.2: Definition 2.2 is consistent with the classical passivity definition if the we have both passive open loop and closed-loop systems (see inequality (23) in the Appendix).

III. Passivity Analysis for Zero Dynamics

In this section, we investigate the passivity property of the system zero dynamics. In the closed-loop, because the system has relative degree zero, by the implication function theory ([22], [24]), it is guaranteed that there exists a feedback control $\mathbf{u}(\mathbf{x}, \mathbf{v})$ such that $h(\mathbf{x}, \mathbf{u}(\mathbf{x}, \mathbf{v})) = \mathbf{v}$. Therefore, the periodically controlled system (4) can be rewritten as the following transformed system with the open loop dynamics remaining the same

$$\begin{aligned} &\begin{cases} \mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t), \mathbf{v}(t))) = \tilde{f}(\mathbf{x}(t), \mathbf{v}(t)) \\ \mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t), \mathbf{v}(t))) = \mathbf{v}(t) \end{cases}, \ kT+1 \leq t \leq kT+\tau-1, \\ &\text{and} \\ &\begin{cases} \mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{0}) \\ \mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{0}) \end{cases}, \ kT+\tau \leq t \leq (k+1)T-1. \end{aligned} \tag{6}$$

Without loss of generality, by shifting the system coordinates, we can consider the origin as the equilibrium state, i.e., $\tilde{f}(\mathbf{0},\mathbf{0}) = \mathbf{0}$. Here $\mathbf{v}(t)$ is chosen such that the closed-loop of system (4) is feedback passive according to Definition A.3 in the Appendix.

The open loop system and the zero dynamics of the transformed closed-loop system form the following periodic system

$$\begin{cases} \mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{u}^*(\mathbf{x}(t),\mathbf{0})) = \bar{f}(\mathbf{x}(t)) \\ \mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}^*(\mathbf{x}(t),\mathbf{0})) = \mathbf{0} \end{cases}, \; kT+1 \le t \le kT+\tau-1,$$
and
$$\begin{cases} \mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{0}) \\ \mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{0}) \end{cases}, \; kT+\tau \le t \le (k+1)T-1. \tag{7}$$

For simplicity, we call (7) the zero dynamics of the transformed system (6).

Following Definition 2.2, we give the following generalized passivity definition for the zero dynamics.

Definition 3.1: The zero dynamics (7) is said to be *locally passive* if there exists a positive semidefinite storage function V ($V(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$) such that the following inequality is satisfied in a neighborhood of the equilibrium point $\mathbf{x}^*$ for all $k \in \{0\} \bigcup \mathbb{Z}^+$

$$V(\mathbf{x}((k+1)T)) - V(\mathbf{x}(kT+1)) \le 0, \; \forall T \in \mathbb{Z}^+. \tag{8}$$

Remark 3.1: Note that the storage function for the periodically controlled system $\tilde{\tilde{V}}$ is not necessarily the same as the storage function of the zero dynamics V.

We now determine the maximum allowable transmission ratio (MATR) between the open loop non-passive time steps and the closed-loop feedback passive time steps to ensure a passive zero dynamics satisfying the inequality (8).

Lemma 3.1: Consider the zero dynamics given by Equation (7) and define the maximum allowable transmission ratio (MATR) as

$$r^* = \frac{(T-1)\ln\sigma}{\ln\sigma - T\ln\zeta}, \; \forall T \in \mathbb{Z}^+, \tag{9}$$

where $0 < \sigma < 1$ and $\zeta > 1$ are some bounded constants. If there exists a positive semidefinite storage function $V(\mathbf{x}),\ V(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$ such that

$$\begin{aligned} &V(\bar{f}(\mathbf{x}(t))) \leq \sigma V(\mathbf{x}(t)),\ 0 < \sigma < 1,\ kT+1 \leq t \leq kT+\tau-1 \\ &V(f(\mathbf{x}(t),\mathbf{0})) \leq \zeta V(\mathbf{x}(t)),\ \zeta > 1,\ kT+\tau \leq t \leq (k+1)T-1, \end{aligned} \tag{10}$$

with $\forall T \in \mathbb{Z}^+,\ k \in \{0\} \bigcup \mathbb{Z}^+$, then the zero dynamics (7) is passive.

Proof: Under condition (10), we have

$$V(\mathbf{x}((k+1)T)) \leq \sigma^{\tau-1}\zeta^{T-\tau} V(\mathbf{x}(kT+1)).$$

According to the inequality (8), the zero dynamics (7) is passive if $\sigma^{\tau-1}\zeta^{T-\tau} \leq 1$. This is satisfied with $r \leq r^*$. ∎

Remark 3.2: Note that r^* gives a bound, instead of the actual value, of MATR. With this understanding, we abuse the nomenclature and refer the bound as MATR throughout the paper.

Remark 3.3: The choice of ζ and σ determines how close the ratio r^* is to the actual value of MATR. The minimum ζ and σ that satisfy the inequality (10) will result in a least conservative r^*.

Lemma 3.2: Given r^* in Equation (9) such that the zero dynamics (7) is passive under the generalized passivity definition, if there exists a storage function that satisfies the inequality (10), then for any $r < r^*$, system (7) is also passive.

Proof: We consider two cases. i) Let $r = \frac{T^*-\tau}{\tau} < r^* = \frac{T^*-\tau^*}{\tau^*}$, where $\tau > \tau^*$. It follows that $\sigma^{\tau-1}\zeta^{T^*-\tau} \leq \sigma^{\tau^*-1}\zeta^{T^*-\tau^*} \leq 1$. ii) Let $r = \frac{T-\tau^*}{\tau^*} < r^* = \frac{T^*-\tau^*}{\tau^*}$, where $T < T^*$. It follows that $\sigma^{\tau^*-1}\zeta^{T-\tau^*} \leq \sigma^{\tau^*-1}\zeta^{T^*-\tau^*} \leq 1$. That is to say, the passivity of the zero dynamics is preserved by either increasing the number of closed-loops τ given period T^* or decreasing the length of the period T given τ^* closed-loops. ∎

Remark 3.4: Same proof holds for the case when the controlled and uncontrolled time steps are not consecutive [18].

IV. Passivity Analysis for Periodically Controlled System

In Section III, we analyzed the passivity properties of the zero dynamics (7). In this section, we prove that given a passive zero dynamics there exists a storage function such that the original periodically controlled system (4) is passive using feedback. Let us first define the generalized passivity inequality for the transformed periodically controlled system (6) following Definition 2.2.

Definition 4.1: The transformed periodically controlled system (6) is said to be *locally passive* with respect to the supply rate $\mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t)$ if there exists a positive semidefinite storage function $\tilde{V}$ ($\tilde{V}(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$) such that the following passivity inequality holds

$$\tilde{V}(\mathbf{x}((k+1)T)) - \tilde{V}(\mathbf{x}(kT+1)) \leq \sum_{t=kT+\tau}^{(k+1)T} \mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t),\ \forall T \in \mathbb{Z}^{+},\ k \in \{0\} \bigcup \mathbb{Z}^{+}. \tag{11}$$

Theorem 4.1: If the zero dynamics (7) is *passive* according to Definition 3.1 with a storage function V such that the determinant of Hessian matrix of $V(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$ is non-zero and there exists a storage function $\tilde{V}$ such that the inequality (10) holds, then the transformed system (6) is *locally passive* using feedback.

Proof: Consider the storage function $\tilde{V}(\mathbf{x}(.)) = aV(\mathbf{x}(.))$ with a constant $a > 0$. We first prove that with a suitable choice of the constant a, this storage function guarantees that, for every vector sequence $\{\mathbf{v}(t)\}$, the following inequality holds during closed-loops

$$\tilde{V}(\tilde{f}(\mathbf{x}(t), \mathbf{v}(t))) - \tilde{V}(\mathbf{x}(t)) \leq \mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t) \tag{12}$$

for $kT + 1 \leq t \leq kT + \tau - 1,\ \forall T \in \mathbb{Z}^{+},\ k \in \{0\} \bigcup \mathbb{Z}^{+}$.

This is equivalent to prove that

$$\phi(\mathbf{x}(t), \mathbf{v}(t)) = \sum_{i=1}^{m} v_i^2(t) + \tilde{V}(\mathbf{x}(t)) - \tilde{V}(\tilde{f}(\mathbf{x}(t), \mathbf{v}(t)), \tag{13}$$

$kT + 1 \leq t \leq kT + \tau - 1$ has a local minimum at $(\mathbf{x}(t), \mathbf{v}(t)) = (\mathbf{0}, \mathbf{0})$. For notational convenience, we denote this pair by $(\mathbf{0}, \mathbf{0})$ and suppress the dependence on t of the terms in (13). Thus, consider the first order derivatives of $\phi(\mathbf{x}, \mathbf{v})$ at $(\mathbf{0}, \mathbf{0})$. We have for $i = 1, \cdots, n$, $r = 1, \cdots, m$,

$$\begin{aligned} \left.\frac{\partial \phi(\mathbf{x}, \mathbf{v})}{\partial x_i}\right|_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}} &= \left[\frac{\partial \tilde{V}}{\partial x_i} - \sum_{h=1}^{n} \frac{\partial \tilde{V}}{\partial \tilde{f}_h} \frac{\partial \tilde{f}_h(\mathbf{x}, \mathbf{v})}{\partial x_i}\right]_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}} \\ \left.\frac{\partial \phi(\mathbf{x}, \mathbf{v})}{\partial v_r}\right|_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}} &= \left[2v_r - \sum_{h=1}^{n} \frac{\partial \tilde{V}}{\partial \tilde{f}_h} \frac{\partial \tilde{f}_h(\mathbf{x}, \mathbf{v})}{\partial v_r}\right]_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}}. \end{aligned}$$

The storage function $V(\mathbf{x}(t))$, and hence the function $\tilde{V}(\mathbf{x}(t)) = aV(\mathbf{x}(t))$ has a local minimum at $\mathbf{x}(t) = \mathbf{0}$ because V is positive semidefinite with $V(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$. Moreover, origin is a local equilibrium of the system; thus, at $\mathbf{x}(t) = \mathbf{v}(t) = \mathbf{0}$, $\tilde{f}(\mathbf{x}(t), \mathbf{v}(t)) = \mathbf{0}$. Combining these facts, we see that

$$\left.\frac{\partial\phi(\mathbf{x},\mathbf{v})}{\partial x_i}\right|_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}} = 0, \qquad i = 1,\cdots,n$$
$$\left.\frac{\partial\phi(\mathbf{x},\mathbf{v})}{\partial v_r}\right|_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}} = 0, \qquad r = 1,\cdots,m.$$

Next, we check the elements of the Hessian matrix of $\phi(\mathbf{x},\mathbf{v})$ at $(\mathbf{0},\mathbf{0})$. We have for $i,j = 1,\cdots,n$ and $r,s = 1,\cdots,m$,

$$\left.\frac{\partial^2\phi(\mathbf{x},\mathbf{v})}{\partial x_j\partial x_i}\right|_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}} = a\left[\frac{\partial^2 V}{\partial x_j\partial x_i} - \sum_{h,l=1}^{n}\frac{\partial^2 V}{\partial\tilde{f}_h\partial\tilde{f}_l}\frac{\partial\tilde{f}_h}{\partial x_i}\frac{\partial\tilde{f}_l}{\partial x_j}\right]_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}}$$
$$\left.\frac{\partial^2\phi(\mathbf{x},\mathbf{v})}{\partial v_r\partial x_i}\right|_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}} = -a\left[\sum_{h,l=1}^{n}\frac{\partial^2 V}{\partial\tilde{f}_h\partial\tilde{f}_l}\frac{\partial\tilde{f}_h}{\partial x_i}\frac{\partial\tilde{f}_l}{\partial v_r}\right]_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}}$$
$$\left.\frac{\partial^2\phi(\mathbf{x},\mathbf{v})}{\partial v_s\partial v_r}\right|_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}} = 2\delta_{rs} - a\left[\sum_{h,l=1}^{n}\frac{\partial^2 V}{\partial\tilde{f}_h\partial\tilde{f}_l}\frac{\partial\tilde{f}_h}{\partial v_r}\frac{\partial\tilde{f}_l}{\partial v_s}\right]_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}}.$$

Denote $\tilde{\phi}(\mathbf{x}(t)) = \phi(\mathbf{x}(t),\mathbf{0}) = a\Big(V(\mathbf{x}(t)) - V(\tilde{f}(\mathbf{x}(k),\mathbf{0})\Big)$, so that

$$\left.\frac{\partial^2\phi(\mathbf{x},\mathbf{v})}{\partial x_j\partial x_i}\right|_{\mathbf{x}=\mathbf{0},\mathbf{v}=\mathbf{0}} = \left.\frac{\partial^2\tilde{\phi}(\mathbf{x})}{\partial x_j\partial x_i}\right|_{\mathbf{x}=\mathbf{0}}. \tag{14}$$

Because the closed-loop is passive and hence has a passive zero dynamics, $\tilde{\phi}(\mathbf{x})$ has a local minimum at $\mathbf{x}=\mathbf{0}$, and by assumption, the determinant of Hessian matrix of the storage function $V(\mathbf{x})$ at $\mathbf{x}=\mathbf{0}$ is non-zero, we obtain that the eigenvalues of the Hessian matrix of $\tilde{\phi}(\mathbf{x})$ at $\mathbf{x}=\mathbf{0}$ are all positive. Denote these eigenvalues by λ_i, $\forall i = 1,2,\cdots,n$. Furthermore, the Hessian matrix of $\tilde{\phi}(\mathbf{x})$ at $\mathbf{x}=\mathbf{0}$ is symmetric and can be diagonalized. Thus, with an appropriate choice of coordinates, the Hessian matrix of $\phi(\mathbf{x},\mathbf{v})$ at $(\mathbf{0},\mathbf{0})$ can be evaluated to be of the form

$$\begin{bmatrix} a\lambda_1 & \cdots & 0 & ab_{11} & \cdots & ab_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a\lambda_n & ab_{n1} & \cdots & ab_{nm} \\ ab_{11} & \cdots & ab_{n1} & 2+ac_{11} & \cdots & ac_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ ab_{1m} & \cdots & ab_{nm} & ac_{m1} & \cdots & 2+ac_{mm} \end{bmatrix}. \tag{15}$$

Now, we apply [22, Lemma 12] which states that for $\lambda_i > 0$ and $\forall a = (0, \hat{a}),\ \hat{a} = \min_j a_j^u$ where

$$a_j^u = \min\left\{1, \frac{2^j \lambda_1 \cdots \lambda_n - \epsilon}{|\alpha_1| + \cdots + |\alpha_j|}\right\},\ j = 1, \cdots, m \tag{16}$$

with $0 < \epsilon \ll 1$ and $\alpha_l,\ l = 1, \cdots, j$ being some constants related to λ_i, b_{il} and $c_{rl},\ i = 1, \cdots, n$, $r = 1, \cdots, j$, $l = 1, \cdots, j$, the determinant of matrix (15) is greater than zero. Sylester's criterion now readily yields that the Hessian matrix of $\phi(\mathbf{x}, \mathbf{v})$ at $(\mathbf{0}, \mathbf{0})$ as evaluated in (15) is positive definite. Therefore, $\phi(\mathbf{x}, \mathbf{v})$ has a local minimum at $(\mathbf{0}, \mathbf{0})$. Thus, during closed-loops, the relation (12) holds. Summing (12) for all the time steps t in closed-loops, we then obtain the following inequality

$$\tilde{V}(\mathbf{x}(kT+\tau)) - \tilde{V}(\mathbf{x}(kT+1)) \leq \sum_{t=kT+1}^{kT+\tau-1} \mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t),\ \forall T \in \mathbb{Z}^+,\ k \in \{0\} \bigcup \mathbb{Z}^+. \tag{17}$$

with the equality holds at $(\mathbf{0}, \mathbf{0})$.

During open loops $kT + \tau \leq t \leq (k+1)T$, since the corresponding zero dynamics is nonpassive, according to (10), we have

$$\begin{aligned} \tilde{V}(f(\mathbf{x}(t), \mathbf{0})) - \tilde{V}(\mathbf{x}(t)) &= a\left(V(f(\mathbf{x}(t), \mathbf{0})) - V(\mathbf{x}(t))\right) \\ &\leq a(\zeta - 1)V(\mathbf{x}(t)). \end{aligned} \tag{18}$$

We now choose a in the interval $(0, \tilde{a})$ where

$$\tilde{a} = \min_T \frac{\sum_{t=kT+1}^{kT+\tau-1} \phi(\mathbf{x}(t), \mathbf{v}(t))}{(\zeta - 1)\sum_{t=kT+\tau}^{(k+1)T-1} V(\mathbf{x}(t))}$$

for all $T \in \mathbb{Z}^+$, then the following inequality is satisfied,

$$a(\zeta - 1)\sum_{t=kT+\tau}^{(k+1)T-1} V(\mathbf{x}(t)) + \sum_{t=kT+1}^{kT+\tau-1}\left[\tilde{V}(\tilde{f}(\mathbf{x}(t), \mathbf{v}(t)) - \tilde{V}(\mathbf{x}(t))\right] \leq \sum_{t=kT+1}^{kT+\tau-1} \mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t). \tag{19}$$

Since

$$\begin{aligned} &\sum_{t=kT+\tau}^{(k+1)T-1}\left[\tilde{V}(f(\mathbf{x}(t), \mathbf{0})) - \tilde{V}(\mathbf{x}(t))\right] + \tilde{V}(\mathbf{x}(kT+\tau)) - \tilde{V}(\mathbf{x}(kT+1)) \\ &= \tilde{V}(\mathbf{x}((k+1)T)) - \tilde{V}(\mathbf{x}(kT+1)), \end{aligned}$$

according to the inequalities (18) and (19), there exists $a \in (0, \min(\hat{a}, \tilde{a}))$, such that the inequality (11) holds with the equality holding if and only if $(\mathbf{x}, \mathbf{v}) = (\mathbf{0}, \mathbf{0})$. ■

Given this result, we can now establish that local passivity of the transformed system (6) implies local passivity of the original periodically controlled system (4).

Theorem 4.2: If the transformed system (6) is passive such that the inequalities (11) hold, under a feedback control law $\mathbf{u}(t)$, then the original periodically controlled system (4) is locally passive.

Proof: According to Theorem 4.1, for the closed-loop of the transformed system there exists a positive semidefinite storage function $\tilde{V}(\mathbf{x}(t)) \geq 0$ with $\tilde{V}(\mathbf{x}(t)) = 0$ if and only if $\mathbf{x}(t) = \mathbf{0}$, such that for any $T \in \mathbb{Z}^{+}$, $\tilde{V}(\mathbf{x}(kT+\tau)) - \tilde{V}(\mathbf{x}(kT+1)) \leq \sum_{t=kT+1}^{kT+\tau-1} \mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t)$. For the periodically controlled system (4), consider the storage function $\tilde{\tilde{V}} = \rho\tilde{V}$ where $\rho > 0$ is a constant to be suitably designed. Also, define the term $\eta(k) = \sum_{t=kT+1}^{(k+1)T-1} \mathbf{u}^{\mathrm{T}}(t)\mathbf{y}(t),\ \forall k \in \{0\} \bigcup \mathbb{Z}^{*}$. Since both $\mathbf{u}(t)$ and $\mathbf{y}(t)$ are bounded in the neighborhood of $\mathbf{x}(t) = \mathbf{0}$ and $\mathbf{v}(t) = \mathbf{0}$, we see that $\eta(k)$ is also bounded. Now, there are two cases.

1) If $\eta(k) \geq 0$, we have $\tilde{\tilde{V}}(\mathbf{x}((k+1)T)) - \tilde{\tilde{V}}(\mathbf{x}(kT+1)) = \rho(\tilde{V}(\mathbf{x}((k+1)T)) - \tilde{V}(\mathbf{x}(kT+1))) \leq \rho \sum_{t=kT+1}^{kT+\tau-1} \mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t)$. The inequality (5) holds if $\rho \leq \min_k \frac{\eta(k)}{\sum_{t=kT+1}^{kT+\tau-1} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k)}$.
2) If $\eta(k) < 0$, this corresponds to the case when the periodically controlled system (4) is Lyapunov stable as well. Because $\eta(k)$ is bounded, we can guarantee that with a sufficiently large choice of ρ, the following inequality holds: $\tilde{\tilde{V}}(\mathbf{x}((k+1)T)) - \tilde{\tilde{V}}(\mathbf{x}(kT+1)) = \rho(\tilde{V}(\mathbf{x}((k+1)T)) - \tilde{V}(\mathbf{x}(kT+1))) \leq \eta(k) = \sum_{t=kT+1}^{(k+1)T-1} \mathbf{u}^{\mathrm{T}}(t)\mathbf{y}(t) \leq 0,\ \forall \rho > 0$, where we choose

$$\rho \geq \max_k \frac{\eta(k)}{\tilde{V}(\mathbf{x}((k+1)T)) - \tilde{V}(\mathbf{x}(kT+1))}. \tag{20}$$

Thus, we can design the constant $\rho > 0$ and the corresponding storage function $\tilde{\tilde{V}} = \rho\tilde{V},\ \rho > 0$ such that the periodically controlled system (4) is locally passive in the neighborhood of $\mathbf{x}(k) = \mathbf{0}$ and $\mathbf{v}(k) = \mathbf{0}$. ■

Definition 4.2: A system is said to be locally zero state detectable (ZSD) ([25]) if there exists a neighborhood $\mathbf{N}$ of the origin such that $\forall \mathbf{x}(0) = \mathbf{x}_0 \in \mathbf{N}$,

$$\mathbf{y}(k)|_{\mathbf{u}(k)=\mathbf{0}} = h(\phi(k;\mathbf{x}_0)) = \mathbf{0},\ \forall k \in \mathbb{Z}_{+}$$

implies

$$\lim_{k\to+\infty} \phi(k;\mathbf{x}_0) = \mathbf{0},$$

where $\phi(k;\mathbf{x}_0)$ is a trajectory of the uncontrolled system $\mathbf{x}(k+1) = f(\mathbf{x}(k),\mathbf{0})$ from $\mathbf{x}(0) = \mathbf{x}_0$.

Theorem 4.3: If the system (4) is passive and locally zero state detectable (ZSD) [25], under a feedback control law of the form $\mathbf{u}(t) = -\psi(\mathbf{y}(t))$ where $\psi(\mathbf{0}) = \mathbf{0}$ and $\mathbf{y}^{\mathrm{T}}(t)\psi(\mathbf{y}(t)) > 0,\ \forall \mathbf{y} \neq \mathbf{0}$, then the equilibrium $(\mathbf{0}, \mathbf{0})$ is locally asymptotically stable.

Proof: According to the passivity definition, for every time step k in the closed-loop, we have

$$\tilde{\tilde{V}}(f(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t)))) - \tilde{\tilde{V}}(\mathbf{x}(t)) \leq \mathbf{u}^{\mathrm{T}}(t)\mathbf{y}(t) = -\mathbf{y}^{\mathrm{T}}(t)\psi(\mathbf{y}(t)) \leq 0$$

with equality holding if and only if $\mathbf{y}(t) = \mathbf{0}$. For every time step t in open loop, the storage function may increase. However, the increase is always bounded (conditions (10)). Therefore, with the finite period T, we have

$$V(\mathbf{x}((k+1)T)) - V(\mathbf{x}(kT+1)) \leq 0.$$

According to Theorem 2 in [19], the equilibrium point $\mathbf{x}^* = \mathbf{0}$ is Lyapunov stable. The asymptotic stability follows from ZSD. Observe that all the trajectories of the closed-loop system eventually approach the invariant set

$$I = \{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}(t+1)) = V(\mathbf{x}(t))\},$$

which implies

$$\mathbf{y}^{\mathrm{T}}(t)\psi(\mathbf{y}(t)) = 0,\ \forall t \in \{0\} \bigcup \mathbb{Z}_+.$$

Hence $\mathbf{y}(t) = 0$ and $\mathbf{w}(t) = -\psi(\mathbf{y}(t)) = \mathbf{0}\ \forall t \in \{0\} \bigcup \mathbb{Z}_+$. Thus by ZSD $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$. ■

Theorem 4.1: If two periodically controlled systems $\mathcal{S}_1$ and $\mathcal{S}_2$ of the form (4) are both passive, then their parallel and negative feedback interconnections (as defined in Figure 2) are both passive.

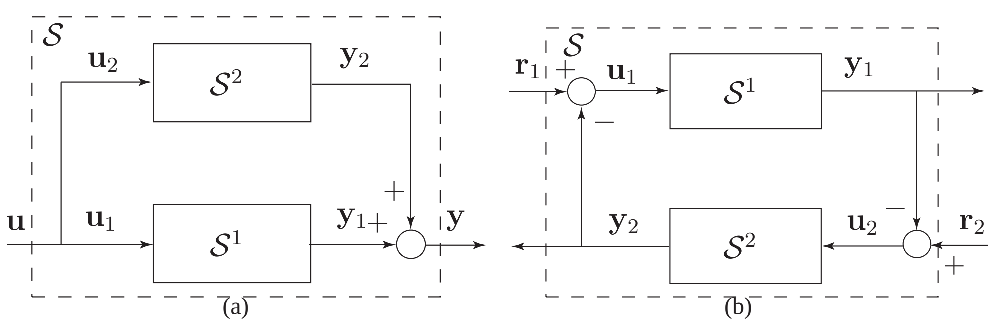

Fig. 2. (a) Parallel, and (b) negative feedback interconnections for two passive switched nonlinear systems $\mathcal{S}^1$ and $\mathcal{S}^2$.

Proof: Let the control inputs for $\mathcal{S}^i$ be $\mathbf{u}_i(t)$, the corresponding output be $\mathbf{y}_i(t)$ and the storage function be $\tilde{\tilde{V}}_i(t)$. For the parallel interconnection, we have for the interconnected system $\mathcal{S}$, the control input $\mathbf{u}(t) = \mathbf{u}_1(t) = \mathbf{u}_2(t)$ and the output $\mathbf{y}(t) = \mathbf{y}_1(t) + \mathbf{y}_2(t)$. For $\mathcal{S}$, consider the storage function $\tilde{\tilde{V}}(t) = \tilde{\tilde{V}}_1(t) + \tilde{\tilde{V}}_2(t)$. For any time $T \in \mathbb{Z}^+,\ k \in \{0\} \bigcup \mathbb{Z}^+$, we have

$$\begin{aligned}
&\tilde{\tilde{V}}(\mathbf{x}((k+1)T)) - \tilde{\tilde{V}}(\mathbf{x}(kT+1)) \\
&= (\tilde{\tilde{V}}_1(\mathbf{x}((k+1)T)) - \tilde{\tilde{V}}_1(\mathbf{x}(kT+1))) + (\tilde{\tilde{V}}_2(\mathbf{x}((k+1)T)) - \tilde{\tilde{V}}_2(\mathbf{x}(kT+1))) \\
&\le \sum_{t=kT+1}^{(k+1)T-1} \mathbf{u}_1^{\mathrm{T}}(t)\mathbf{y}_1(t) + \sum_{t=kT+1}^{(k+1)T-1} \mathbf{u}_2^{\mathrm{T}}(t)\mathbf{y}_2(t) \\
&\le \sum_{t=kT+1}^{(k+1)T-1} \mathbf{u}^{\mathrm{T}}(t)\mathbf{y}(t).
\end{aligned} \tag{21}$$

Similarly, for the negative feedback interconnection, we have for the interconnected system $\mathcal{S}$, the control inputs and outputs as $\mathbf{r}_1(t) = \mathbf{u}_1(t) + \mathbf{y}_2(t)$ and $\mathbf{r}_2(t) = \mathbf{u}_2(t) + \mathbf{y}_1(t)$. Consider the storage function $\tilde{\tilde{V}}(t) = \tilde{\tilde{V}}_1(t) + \tilde{\tilde{V}}_2(t)$. For any time $T \in \mathbb{Z}^+,\ k \in \{0\} \bigcup \mathbb{Z}^+$, we have $\tilde{\tilde{V}}(\mathbf{x}((k+1)T)) - \tilde{\tilde{V}}(\mathbf{x}(kT+1)) \le \sum_{t=kT+1}^{(k+1)T-1} (\mathbf{r}_1^{\mathrm{T}}(t)\mathbf{y}_1(t) + \mathbf{r}_2^{\mathrm{T}}(t)\mathbf{y}_2(t))$. ■

V. Numerical Example

In this section, we provide a numerical example to illustrate the major concept presented in the above sections. Let us consider the following periodically controlled nonlinear system

$$\begin{cases} x_1(t+1) = -0.3x_1^2(t)x_2(t) + 1.5x_2(t) + u(k) \\ x_2(t+1) = x_1(t) - u^2(t) \\ y(t) = 2x_2(t) + u(t) \end{cases},\ kT+1 \le t \le kT+\tau-1,$$

$$\begin{cases} x_1(t+1) = -0.3x_1^2(t)x_2(t) + 1.5x_2(t) \\ x_2(t+1) = x_1(t) \\ y(t) = 2x_2(t) \end{cases},\ kT+\tau \le t \le (k+1)T.$$

The transformed system is given as:

$$\begin{cases} x_1(t+1) = -0.3x_1^2(t)x_2(t) - 0.5x_2(t) + v(t) \\ x_2(t+1) = x_1(t) - (v(t) - 2x_2(t))^2 \\ y(t) = v(t) \end{cases}, \ kT+1 \leq t \leq kT+\tau-1,$$

$$\begin{cases} x_1(t+1) = -0.3x_1^2(t)x_2(t) + 1.5x_2(t) \\ x_2(t+1) = x_1(t) \\ y(t) = 2x_2(t) \end{cases}, \ kT+\tau \leq t \leq (k+1)T.$$

The zero dynamics of the above system is

$$\begin{cases} x_1(t+1) = -0.3x_1^2(t)x_2(t) - 0.5x_2(t) \\ x_2(t+1) = x_1(t) - 4x_2^2(t) \\ y(t) = v(t) \end{cases}, \ kT+1 \leq t \leq kT+\tau-1$$

$$\begin{cases} x_1(t+1) = -0.3x_1^2(t)x_2(t) + 1.5x_2(t) \\ x_2(t+1) = x_1(t) \\ y(t) = 2x_2(t) \end{cases}, \ kT+\tau \leq t \leq (k+1)T.$$

Note that the closed-loop of system (22) is locally ZSD and has relative degree zero. The control $u(t) = -y(t) = -x_2(t)$ is used. For the zero dynamics (22), we choose a quadratic storage function $V = \mathbf{x}^{\mathrm{T}}P\mathbf{x} = x_1^2 + 0.5x_2^2$ with the positive definite matrix $P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$. One can easily verify that the determinant of the Hessian matrix of V at $\mathbf{x} = [0\ 0]^{\mathrm{T}}$ is not zero. We choose the parameters $\zeta = 1.908$ and $\sigma = 0.5594$. According to Equation (9), it follows that $r^* = \frac{0.5809(T-1)}{0.5809+0.6461T}$. We fix the period $T = 8$ and obtain $r^* = 0.7072$ and $\tau = \lceil 4.6860 \rceil = 5$. Next, we calculate the storage function V at each time step to check condition (10). Table I shows the maximum ratio of $\frac{V(f(\mathbf{x}(t),\mathbf{0})}{V(\mathbf{x}(t))}$ for the uncontrolled and controlled time steps within three periods, i.e., $k = 1, 2, 3$. From the table, we can verify that the inequality (10) is satisfied at every time step with $\zeta = 1.908$ and $\sigma = 0.5594$. The storage function $\tilde{V}$ for the transformed periodically controlled system (22) is chosen as $0.48V$ with $\hat{a} = 0.48$ and $\tilde{a} = 5.6915$. The storage function for the original periodically controlled system can be chosen as $12V$ with $\rho = 12$ satisfies inequality (20) under the case when $\eta < 0$.

As shown in Figure 3(a), the periodically controlled system (22) is passive during time steps $[8k + 1, 8k + 5]$ and nonpassive (i.e., the increase in storage function is not bounded by the supplied energy at

$\max V(\mathbf{x}(t+1))/V(\mathbf{x}(t))$	$kT+1 \leq t \leq kT+\tau-1$	$kT+\tau \leq t \leq (k+1)T$
$k=0$	0.5594	1.908
$k=1$	0.5308	1.907
$k=2$	0.5166	1.8897

TABLE I
CHECK OF EQUATION (10) WITHIN THREE PERIODS.

each time step) during time steps $[8k+6, 8k+8]$. Figure 3(b) shows the generalized passivity check for the system. We can see that the generalized passivity inequalities (5) hold for each period. Figure 3(c) shows the evolution of the state dynamics of the periodically controlled system in 6 periods. Both states are locally asymptotically stable at the origin.

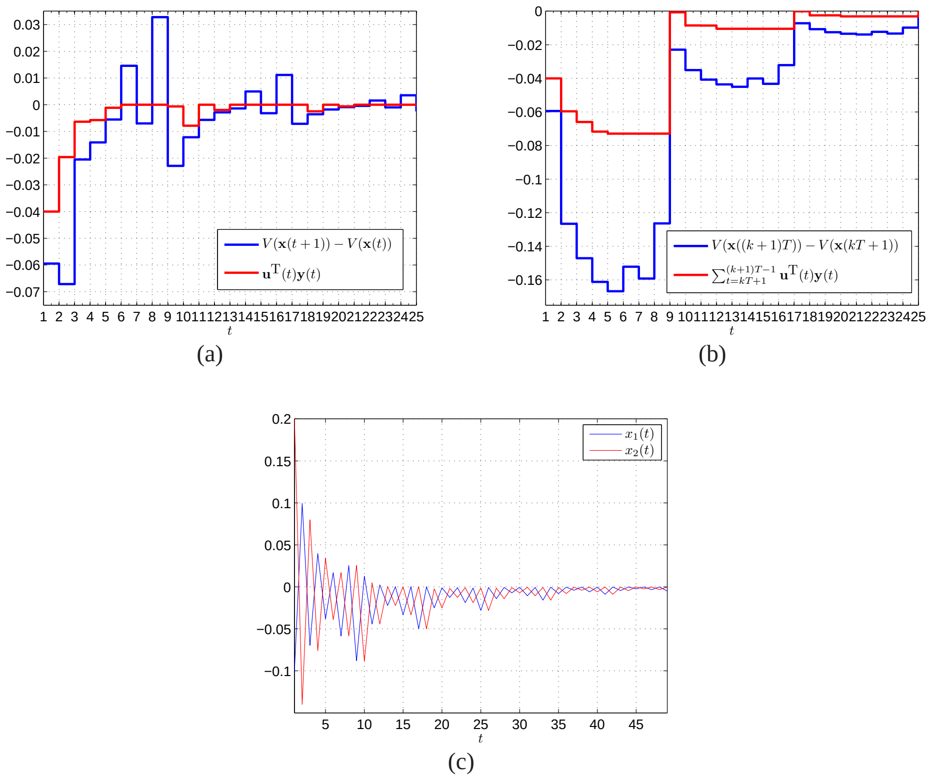

Fig. 3. (a) Passivity check for the periodically controlled system (22) according to classical passivity definition in 3 periods, (b) Passivity check for system (22) according to the generalized passivity definition (5) in 3 periods, and (c) State dynamics of system (22) after 6 periods.

VI. CONCLUSION AND FUTURE WORK

In this paper, we propose a generalized concept of passivity for discrete-time periodically controlled systems with consecutive feedback passive closed-loops followed by consecutive nonpassive open loops. Different from the classical passivity theory, the increase in storage function is not required to be bounded by the supplied energy at each time step during a period. The system is said to be passive if the increase in storage function at the end of a period as compared to the beginning of the period is bounded by the accumulated energy supplied to it within the period. A feedback control is applied periodically to guarantee system passivity under the generalized definition. We study the passivity properties of the periodically controlled systems based on its zero dynamics. We prove that given a maximum allowable transmission ratio, the system zero dynamics is guaranteed to be passive. Given a passive zero dynamics, it is proved that the periodically controlled system is also locally passive. Further research will focus on the extension of the results presented in this paper to event-triggered control systems. The generalized concept of passivity will also be investigated in the stochastic settings.

VII. Acknowledgement

The support of the National Science Foundation under Grant No. CNS-1035655 is gratefully acknowledged. The authors would also like to thank Michael McCourt in the Electrical Engineering Department at the University of Notre Dame for his suggestive discussions.

Appendix

Background on Passivity

Consider a system of the form

$$\begin{cases} \mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}, \tag{22}$$

where $\mathbf{x} \in \mathbf{X} = \mathbb{R}^n$, $\mathbf{u} \in \mathbf{U} = \mathbb{R}^m$ and $\mathbf{y} \in \mathbf{Y} = \mathbb{R}^m$ are the state, input, and output variables, respectively. $\mathbf{X}, \mathbf{U}$ and $\mathbf{Y}$ are the state, input, and output spaces, respectively. $t \in \{0\} \bigcup \mathbb{Z}^+$, f and h are smooth. All considerations are restricted to an open set of $\mathbf{X} \times \mathbf{U}$ containing $(\mathbf{x}^*, \mathbf{u}^*)$ having $\mathbf{x}^* = f(\mathbf{x}^*, \mathbf{u}^*)$. Without loss of generality, it is assumed that $(\mathbf{x}^*, \mathbf{u}^*) = (\mathbf{0}, \mathbf{0})$ and $h(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

Definition A.1: [23] A system of the form (22) is said to be *dissipative* with respect to the *supply rate* $w \in \mathbf{U} \times \mathbf{Y} \to \mathbb{R}$ if there exists a positive semidefinite function $V : \mathbf{X} \to \mathbb{R}^+$, $V(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$, called the *storage function*, such that $\forall (\mathbf{x}(t), \mathbf{u}(t)) \in \mathbf{X} \times \mathbf{U}$, $\forall t$

$$V(f(\mathbf{x}(t), \mathbf{u}(t))) - V(\mathbf{x}(t)) \leq w(\mathbf{y}(t), \mathbf{u}(t)).$$

Note that the above inequality holds if and only if $\forall(\mathbf{x}(t), \mathbf{u}(t)) \in \mathbf{X} \times \mathbf{U},\ \forall t$

$$V(f(\mathbf{x}(t), \mathbf{u}(t))) - V(\mathbf{x}(0)) \leq \sum_{\theta=0}^{t} w(\mathbf{y}(\theta), \mathbf{u}(\theta)). \tag{23}$$

Definition A.2: [23] A system of the form (22) is said to be *passive* if it is dissipative with respect to the supply rate $w(\mathbf{y}(t), \mathbf{u}(t)) = \mathbf{u}^{\mathrm{T}}(t)\mathbf{y}(t)$. That is, $\forall(\mathbf{x}(t), \mathbf{u}(t)) \in \mathbf{X} \times \mathbf{U},\ \forall t$

$$V(f(\mathbf{x}(t), \mathbf{u}(t))) - V(\mathbf{x}(t)) \leq \mathbf{u}^{\mathrm{T}}(t)\mathbf{y}(t).$$

Let $\mathbf{u}(\mathbf{x}, \mathbf{v}) : \mathbf{X} \times \mathbf{U} \to \mathbf{U}$ denote a nonlinear feedback control law. If $\mathbf{u}(\mathbf{x}, \mathbf{v})$ is locally regular, i.e., $\frac{\partial \mathbf{u}(\mathbf{x}(k),\mathbf{v}(k))}{\partial \mathbf{v}(k)} \neq \mathbf{0}$ for all $(\mathbf{x}, \mathbf{v}) \in \mathbf{X} \times \mathbf{U}$, the system

$$\begin{cases} \mathbf{x}(t+1) = f(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t), \mathbf{v}(t))) = \tilde{f}(\mathbf{x}(t), \mathbf{v}(t)) \\ \mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(\mathbf{x}(t), \mathbf{v}(t))) = \mathbf{v}(t) \end{cases}$$

is referred as the feedback transformed system.

Definition A.3: [21] Consider system (22) with a positive semidefinite storage function $V(\mathbf{x})$ with $V(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$ and supply rate $\mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t)$. The system is said to be *feedback passive* with respect to $V(\mathbf{x})$ and $\mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t)$ if there exists a regular feedback control law $\mathbf{u}(\mathbf{x}(t), \mathbf{v}(t))$ with $\mathbf{v}(t)$ as the new input such that

$$V(\tilde{f}(\mathbf{x}(t), \mathbf{v}(t))) - V(\mathbf{x}(t)) \leq \mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t).$$

REFERENCES

[1] J. Zhao and D. J. Hill, "Dissipativity theory for switched systems," *IEEE Transactions on Automatic Control*, vol. 53, pp. 941–953, May 2008.

[2] S. Bittanti, *Deterministic and Stochastic Linear Periodic Systems*. Time Series and Linear systems, Springer, 1986.

[3] P. P. Khargonekar, K. Poolla, and A. Tannenbaum, "Robust Control of Linear Time-Invariant Plants using Periodic Compensation," *IEEE Transactions on Automatic Control*, vol. AC-30, pp. 1088–1096, November 1985.

[4] B. A. Francis and T. T. Georgiou, "Stability Theory for Linear Time-Invariant Plants with Periodic Digital Controllers," *IEEE Transactions on Automatic Control*, vol. 33, pp. 820–832, September 1988.

[5] P. G. Voulgaris, M. A. Dahleh, and L. S. Valavani, "$\mathcal{H}^{\infty}$ and $\mathcal{H}^{2}$ Optimal Controllers for Periodic and Multi-Rate Systems," *Automatica*, vol. 30, pp. 251–263, February 1994.

[6] S. Bittanti, P. Colaneri, and G. De Nicolao, "An Algebraic Riccati Equation for the Discrete-Time Periodic Prediction Problem," *Systems and Control Letters*, vol. 14, pp. 71–78, January 1990.

[7] C. E. De Souza and A. Trofino, "An LMI Approach to Stabilization of Linear Discrete-Time Periodic Systems," *International Journal of Control*, vol. 73, no. 8, pp. 696–703, 2000.

[8] K. B. Kim, J. W. Lee, and W. H. Kwon, “Intervalwise Receding Horizon $\mathcal{H}^{\infty}$ Tracking Control for Discrete Linear Periodic Systems,” *IEEE Transactions on Automatic Control*, vol. 45, no. 4, pp. 747–752, 2000.

[9] M. Reble, C. Böhm, and F. Allgöwer, “Nonlinear Model Predictive Control for Periodic Systems using LMIs,” *Proceedings of the European Control Conference*, 2009.

[10] J. C. Willems, “Dissipative Dynamical Systems Part I: General Theory,” *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321–351, 1972.

[11] J. C. Willems, “Dissipative Dynamical Systems Part II: Linear Systems with Quadratic Supply Rates,” *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 352–393, 1972.

[12] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, 2002.

[13] J. Bao and P. L. Lee, *Process Control: the Passive Systems Approach*. Springer, 2007.

[14] X. Koutsoukos, N. Kottenstette, J. Hall, P. J. Antsaklis, and J. Sztipanovits, “Passivity-Based Control Design for Cyber-Physical Systems,” *International Workshop on Cyber-Physical Systems Challenges and Applications*, June 2008.

[15] M. J. McCourt and P. J. Antsaklis, “Stability of Networked Passive Switched Systems,” *49th IEEE Conference on Decision and Control*, pp. 1263–1268, December 2010.

[16] H. Yu and P. J. Antsaklis, “Event-Triggered Real-Time Scheduling for Stabilization of Passive/Output Feedback Passive Systems,” *American Control Conference*, pp. 1674–1679, 2011.

[17] A. Bemporad, G. Bianchini, and F. Brogi, “Passivity Analysis andd Passification of Discrete-time Hybrid Systems,” *IEEE Transactions on Automatic Control*, vol. 53, pp. 1004–1009, May 2008.

[18] Y. Wang, V. Gupta, and P. J. Antsaklis, “On Passivity of Networked Nonlinear Systems with Packet Drops,” *IEEE Transactions on Automatic Control*, 2012. under review. http://www.nd.edu/∼isis/techreports/isis-2012-001.pdf.

[19] D. Aeyels and J. Peuteman, “A New Asymptotic Stability Criterion for Nonlinear Time-Variant Differential Equations,” *IEEE Transactions on Automatic Control*, vol. 43, no. 7, pp. 968–971, 1998.

[20] C. I. Byrnes, A. Isidiri, and J. C. Willems, “Passivity, Feedback Equivalence, and the Global Stabilization of Minimum Phase Nonlinear Systems,” *IEEE Transactions on Automatic Control*, vol. 36, pp. 1228–1240, November 1991.

[21] E. M. Eavarro-López, *Dissipativity and Passivity-Related Properties in Nonlinear Discrete-Time Systems*. PhD thesis, Universitat Politècnica de Catalunya, 2002.

[22] E. M. Eavarro-López and E. Foddas-Colet, “Feedback Passivity of Nonlinear Discrete-Time Systems with Direct Input-Output Link,” *Automatica*, vol. 40, no. 8, pp. 1423–1428, 2004.

[23] C. I. Byrnes and W. Lin, “Lossless, Feedback Equivalence, and the Global Stabilization of Discrete-Time Nonlinear Systems,” *IEEE Transactions on Automatic Control*, vol. 39, pp. 83–98, January 1994.

[24] R. C. James and G. James, *Mathematics Dictionary*. Springer, 1992.

[25] W. Lin and C. I. Byrness, “Passivity and Absolute Stabilization of a Class of Discrete-time Nonlinear Systems,” *Automatica*, vol. 31, no. 2, pp. 263–267, 1995.