Multiagent Compositional Stability Exploiting System Symmetries

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Abstract

This paper considers nonlinear symmetric control systems. By exploiting the symmetric structure of the system stability results are derived that are independent of the number of components in the system. This work contributes to the fields of research directed toward compositionality and composability of large-scale system in that a system can be "built-up" by adding components while maintaining system stability. The modeling framework developed in this paper is a generalization of many existing results which focus on interconnected systems with specific dynamics. The main utility of the stability result is one of scalability or compositionality. If the system is stable for a given number of components, under appropriate conditions stability is then guaranteed for a larger system composed of the same type of components which are interconnected in a manner consistent with the smaller system. The results are general and applicable to a wide class of problems. The examples in this paper focus on the formation control problems for multi-agent robotic systems.

Keywords: symmetric systems, multiagent coordination, nonlinear systems, compositionality

1 Introduction

Recent research efforts have been directed toward the analysis of *composability* and *compositionality* of control systems [17, 2]. These concepts are not equivalent, but each do relate to the nature in which system components affect overall system properties. In this paper conditions are determined under which a stable symmetric system remains stable if additional components are added in a structured manner, particularly, in a manner which maintains the symmetric aspects of the system. While the results in this paper are general, one important application, which is the focus of the examples, is the mobile robot formation control problem.

Control of multi-agent systems is an important area of engineering research that has been the focus of much research attention for several decades, but most intensively since approximately the mid-1990s. Formation control for multiple mobile robotic systems is a prototypical application and simiarly has a long history, with the main focus being on the use of potential functions for coordination (see for example [15, 3, 13] and the citations therein). The use of potential functions has an obvious appeal in that they facilitate stability analyses using Lyapunov functions. The drawbacks are well-known also, which include among other things, the existence of multiple local minima in complex environments, the fact that realistic potential functions representing the realities of sensor ranges introduce mathematical limitations on the potential functions which complicate and limit the stability analysis *etc.* As observed in [12], many of the prior efforts have assumed specific dynamics with the correct observation that they probably generalize; however, our approach in this paper is intended to be much more general. Perhaps the work closest to this present work be that of [12] wherein a control Lyapunov function is assumed to exist for each agent, from which formation functions and bounds on formation speed can be derived to ensure stability. The added benefit of the results in this paper is that our formulation provides the type of cases and underlying structure for systems to which the results in [12] will apply. Furthermore, our results here apply to a broader class of systems, such as fully distributed ones, to which the previous results do not necessarily apply.

The main contributions of the present paper are:

1. a nonlinear extension of the model and results in [1] and [14] with a simpler representation of system symmetries than our previous work;

2. the presentation of a theoretical framework that is underlying many of the formation control algorithms in the literature;

3. general stability theorems that are applicable to such systems regardless of the number of components (compositionality); and,

4. robustness results that ensure stability even under certain types of component failures.

These results will allow a control design engineer to focus the analysis on a smaller, more tractable system with a guarantee that stability will hold for a much larger system. This paper essentially extends the previous work of one of the authors related to the properties of symmetric systems [9, 7, 8, 10, 11] to consider nonlinear system stability.

The previous work cited considers system symmetries that are defined by a group action on the configuration manifold for a distributed system that was induced by the action of a permutation group. The main drawback of such an approach is that, in the general case, identifying such symmetries can be problematic. However, in the case of most engineering and robotics systems, where the individual robots are the components that are symmetric, symmetry identification is much less of a problem. Rather than using this prior approach, this paper will introduce a more straight-forward approach which is a nonlinear extension of the approach used in [1] and [14]. However, it is emphasized that the prior approaches [6, 9, 7, 8, 10, 11] and [5] offer a general approach to the problem that can be used in cases more general than the ones addressed here.

This rest of this paper is organized as follows. Section 2 defines a symmetric system, equivalence relations among different symmetric systems and equivalence classes of symmetric systems. Section 3 presents the nonlinear stability results for symmetric systems. Section 4 presents some examples of the application of these results. Section 5 presents an extension of the results from Section 3 to the case of robust stability in the case where an agent or agents in a symmetric system fail. Finally, Section 6 outline conclusions and future work.

2 Symmetric Systems

This section defines symmetric systems and the relationship among symmetric systems with different numbers of components. As a motivational example, consider a formation of large number of identical mobile robots where each robot has a control law that attempts to control it so that it maintains a desired distance from its neighbors. Intuitively if more of the same type of robots with the same control law are added to the formation, or some are removed, the properties of the formation as a whole should not drastically change, or at least sometimes should not drastically change. As a step toward formalizing and determining conditions when this holds, we must formulate definitions for systems when more agents are added or some are removed in structured manner. Toward this end, we define *symmetric systems* and *equivalent symmetric systems*.

The first step is to extend the basic system component description from the linear case in [1] to the nonlinear case. The "basic building block" in one spatial dimension (more general interconnection topologies will be considered subsequently) is illustrated in Figure 1. The outputs from the component are $w^-(t)$ and $w^+(t)$, and the inputs are u, $v^-(t)$ and $v^+(t)$. In this paper the signals $v^\pm$ will represent the effects of the coupling with the other components and u are the usual control inputs which need to be designed for stability, performance, robustness, *etc.* If it is necessary to distinguish between them, the $v^\pm$ signals will be called *coupling inputs*, the u will be called *control inputs* and collectively they will be called the *inputs*. When interconnected in one spatial dimension, a system comprised of a collection of these building blocks is as illustrated in Figure 2.

We wish to express component-by-component, the usual dynamics of a nonlinear control system expressed

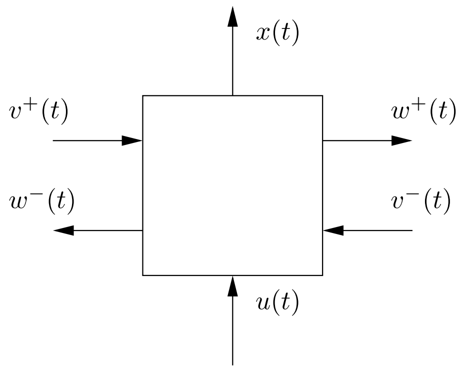

Figure 1: System building block in one spatial dimension.

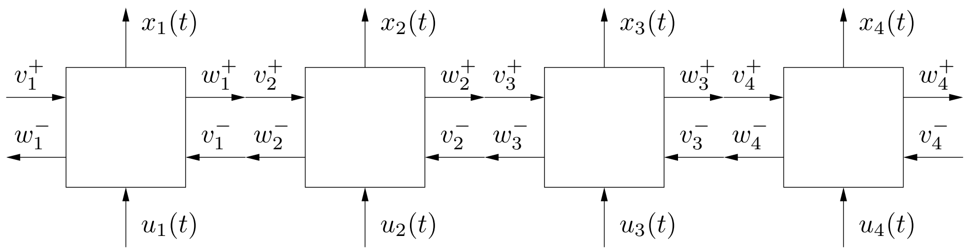

Figure 2: System interconnected in one spatial dimension.

by

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j.$$

For the ith component, we write

$$\dot{x}_i = f_i(x) + \sum_{j=1}^{m_i} g_{i,j}(x) u_{i,j},$$

where $x \in \mathbb{R}^n$, the vector fields $f(x), g_j(x) \in T\mathbb{R}^n$ and m_i is the number of inputs for the ith component.

In order to define a symmetric system that has structure that will prove to be useful, we will consider, in order, the following aspects of a system comprised of many interacting components:

- the nature of the relationship between the nonlinear dynamics of a component and its coupling inputs;
- the nature of the structure of how the components are interconnected;
- the nature of the dynamics of individual components; and,
- the nature nature of the individual control laws in each component.

In the most general case, the vector fields, f_i and $g_{i,j}$ in the equation of motion for the ith component and the outputs w_i^+ and w_i^- for the component may depend on the state of the component, x_i *as well as* the coupling inputs, $v_i^{\pm}$, so the the dynamics of component i are given by

$$\begin{aligned}
\dot{x}_i(t) &= f_i\left(x_i(t), v_i^+(t), v_i^-(t)\right) + \sum_{j=1}^{m_i} g_{i,j}\left(x_i(t), v_i^+(t), v_i^-(t)\right) u_{i,j}(t) \\
w_i^-(t) &= w_i^-\left(x_i(t), v_i^+(t), v_i^-(t)\right) \\
w_i^+(t) &= w_i^+\left(x_i(t), v_i^+(t), v_i^-(t)\right).
\end{aligned}$$

We will consider how the system is interconnected shortly, but for now observe that for a system of interconnected components where the incoming signals, $v^{\pm}(t)$ are from the outgoing signals from the component's neighbors, since the vector fields f_i and $g_{i,j}$ arise from the physical dynamics of the component, if these vector fields can depend on the outputs from the neighbors, this would reflect a change in the physical dynamics of the system due to the coupling between components. The class of the types of coupling that could be represented by this formulation is very broad and could include, for example, when there is a physical joining of agents, as with reconfigurable, modular robots.

For a very large class of problems, including formation control for mobile robots, there is no physical contact between the robots and hence the nature of the coupling between the robots is simplified. In particular, it is only through the control inputs that the output from the other components affects the dynamics of an agent, which is expressed by

$$\begin{aligned}
\dot{x}_i(t) &= f_i\left(x_i(t)\right) + \sum_{j=1}^{m_i} g_{i,j}\left(x_i(t)\right) u_{i,j}\left(x_i(t), v_i^+(t), v_i^-(t)\right) \\
w_i^-(t) &= w_i^-\left(x_i(t)\right) \\
w_i^+(t) &= w_i^+\left(x_i(t)\right).
\end{aligned} \tag{1}$$

For the rest of this paper, we will restrict our attention to systems of this type.

Now we consider the nature of the interconnections in the system. For a system with N components, a subset of the components have *periodic interconnections in one dimension* if the inputs and outputs of adjacent components are related by

$$w_i^+(t) = v_{i+1}^+(t), \quad w_i^-(t) = v_{i-1}^-(t), \quad v_i^+(t) = w_{i-1}^+(t), \quad v_i^-(t) = w_{i+1}^-(t), \tag{2}$$

for all i in some subset $\mathcal{I} \subset \{1, \ldots, N\}$. A set of components that have periodic interconnections is called a *orbit of periodically interconnected components.* The subset of the component index set corresponding to the orbit is called the *orbit index.* Of course, a system may have multiple orbits of periodically interconnected components, and in such a case there will be multiple orbit index sets.

The system illustrated in Figure 2 is of this type for $\mathcal{I} = \{2, 3\}$. It is possible for the entire system to have periodic interconnections in one dimension if Equation 2 holds for all $i \in \{1, \ldots, N\}$ and for $\mathrm{mod}(N)$, or if the system has an infinite number of components on a one-dimenaional integer lattice. For the system in Figure 2, if component 4 is connected to component 1 in the same manner that the other components are connected; namely $v_1^+ = w_4^+$ and $v_4^- = w_1^-$ then the whole system has periodic interconnections.

For the set of components with periodic interconnections if the dynamics of the system are further resticted in that feedback can be expressed in terms of the outputs from the neighbors then the control inputs for component i in Equation 1 can be written as

$$u_{i,j}(t) = u_{i,j}\left(x_i(t), w_{i-1}^+(x_{i-1}(t)), w_{i+1}^-(x_{i+1}(t))\right), \quad i \in \mathcal{I}. \tag{3}$$

Now we consider the case when the components in an orbit of periodically interconnected components have identical dynamics. An *orbit of symmetric components* is an orbit of periodically interconnected components in one dimension if

$$f_i(x) = f_k(x), \quad g_{i,j}(x) = g_{k,j}(x), \quad w_i^-(x) = w_k^-(x), \quad w_i^+(x) = w_k^+(x), \quad m_i = m_k = m$$

for $x \in \mathbb{R}^n$, for all $i, k = \mathcal{I}$ and for each $j = 1, \ldots, m$. Finally, when the components in an orbit of symmetric components have identical control laws, we have a *symmetry orbit* which requires

$$u_{i,j}(x_1, w_{i-1}^+(x_2), w_{i+1}^-(x_3)) = u_{k,j}(x_1, w_{k-1}^+(x_2), w_{k+1}^-(x_3))$$

for $(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, for all $i, k = \mathcal{I}$ and for each $j = 1, \ldots, m$.

The idea behind a symmetry orbit is that the agents in the orbit are identical, have identical control laws and furthermore are identically interconnected. We observe that, in general, it is only necessary for the dynamics of each system to be "identical" in the sense that they are diffeomorphically related, in which case under a coordinate transformation they are identical. Identifying nonlinear coordinate transformations under which systems are equal is a difficult problem beyond the scope of this paper. In this paper we will restrict our attention to systems with components with identical dynamics with the recognition that the results apply to a broader set of problems.

Of course, systems may be spatially interconnected in dimensions greater than one or with a different type of periodicity, as is illustrated in Figures 3 and 4, respectively. With respect to the latter notion, interconnetions are not necessarily limited to connections with only two neighbors in each dimension, as is illustrated for the one-dimensional case in Figure 4. For clarity of presentation, in both figures the control input is not illustrated. Additionally, in Figure 4 the two directed edges connecting each component are represented by one arrow, *i.e.*, all four signals are represented by one edge.

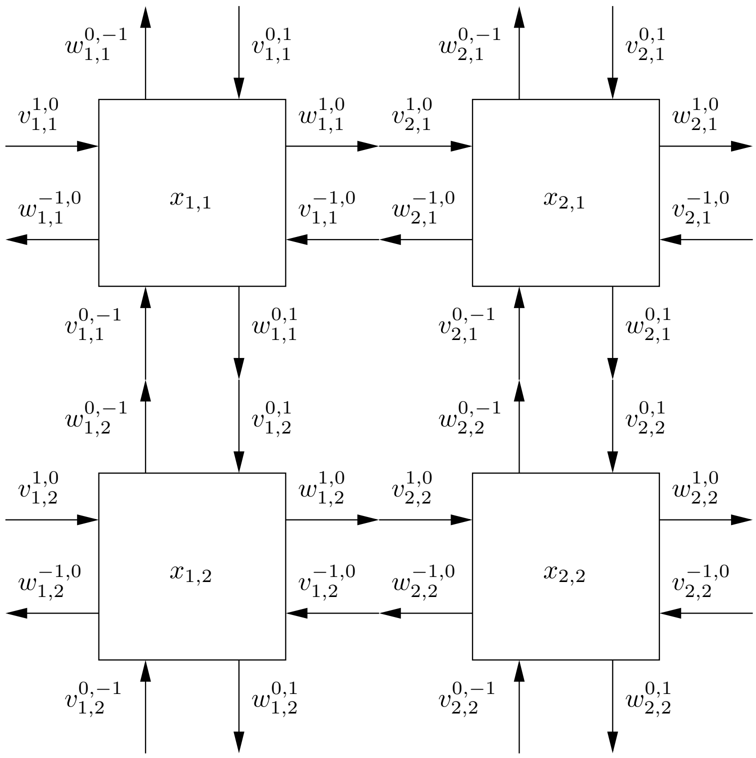

Figure 3: System with periodic interconnections in two dimensions.

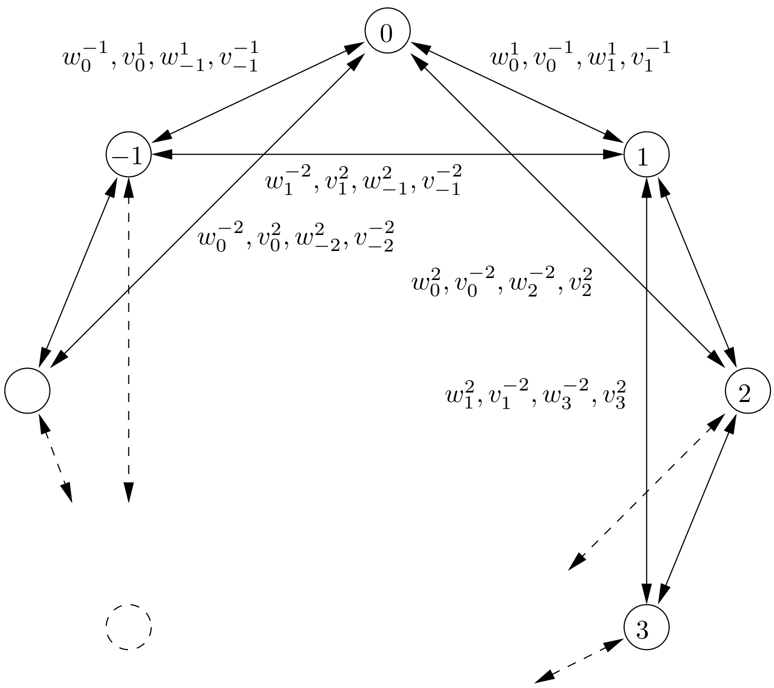

Figure 4: System topology for Example 2.3.

In order to handle these more general cases, we consider the nature of the groups generated by the manner in which components are interconnected. The types of systems considered in this paper will have components that either are members of groups, or subsets of groups. Recall that a *group* is nonempty set, G with

1. a binary associative operation, $\sigma : G \times G \to G$,
2. an identity element e such that $\sigma(e, g) = \sigma(g, e) = g$ for all $g \in G$, and
3. for every $g \in G$ there exists an element $g^{-1} \in G$ such that $\sigma(g, g^{-1}) = \sigma(g^{-1}, g) = e$.

We use the notation $|G|$ to denote the number of elements in a set G. The rest of this section will consider systems defined on groups.

A subgroup is a subset of a group that is itself a group. Of particular importance in this paper are elements of a group that *generate* a subgroup. If X is a subset of a group G, then the smallest subgroup of G containing X is called the *subgroup generated by* X. The idea is that the (sub)group generated by X can be "built up" from the elements of S operating on each other until finally the set is closed. We will typically use a "multiplication" notation instead of σ for the operation, *i.e.*, $g_1g_2 = \sigma(g_1, g_2)$. Constraints among the generators are given by *relations* of the form $s_1s_2 \ldots s_m = e$ for $s_1, \ldots, s_m \in S$. Finally, we will represent systems by a *Cayley graph*, which is a directed graph with vertices that are the elements of a group, G, generated by the subset X, with directed edges from g_1 to g_2 only if $g_2 = sg_1$ for some $s \in X$. A directed edge from node g_1 to g_2 represents that a coupling input to g_2 is equal to an output from g_1. The edges are directed, an edge from g_1 to g_2 does not necessarily imply an edge is directed from g_2 to g_1. See [16] for a more extensive exposition.

Example 2.1 Consider the ring of components illustrated in Figure 4. Each vertex has edges connecting to four other vertices and hence the system is generated by four elements. Let g denote a vertex, *i.e.*, $g \in \{-2, -1, 0, 1, \ldots, N-3\} = G$. Consider the subset of generators $X = \{-2, -1, 1, 2\}$, the group operation to be addition and the relation $s^N = e$. This relation makes the group operation of addition to be mod N, and hence the group is the quotient of $\mathbb{Z}$ where elements of $\mathbb{Z}$ that differ by a multiple of N are equivalent. The Cayley graph is illustrated in Figure 4. A vertex is only adajacent to four neighbors because the set of generators has four elements.

For the system illustrated in Figure 3, let $G = \mathbb{Z} \times \mathbb{Z}$ and for $g = (n_1, n_2) \in G$, define the group operation by addition, *i.e.*, for $g_1 = (n_1, n_2)$ and $g_2 = (m_1, m_2)$, $g_1g_2 = (n_1 + m_1, n_2 + m_2)$. For the set of generators $s_{1,0} = (1, 0)$ $s_{-1,0} = (-1, 0)$, $s_{0,1} = (0, 1)$ and $s_{0,-1} = (0, -1)$ the Cayley graph is illustrated in Figure 3. With no relation on the generators, the group would be an infinite integer lattice.

For a system on the group G with the set of generators $X = \{s_1, s_2, \ldots, s_{|X|}\}$, denote the state variable corresponding to $g \in G$ by x_g, the set of neighbors for component $g \in G$ by $Xg = \{s_1g, s_2g, \ldots, s_{|X|}g\}$ and the states of the neighbors by x_{Xg}. For component g, denote the set of outputs to be $\{w_g^{s_1}, w_g^{s_2}, \ldots, w_g^{s_{|X|}}\}$ and similary the set of inputs $\{v_g^{s_1}, v_g^{s_2}, \ldots, v_g^{s_{|X|}}\}$. A general system can have any number of coupling inputs and outputs, but in the present case we will define them in such a way to have the same number of each. Subsequently when we define periodic interconnections, we will impose the structure that w_g^s is the output from g that is taken as an input to sg.

The dynamics of a component, $g \in G$ are represented by

$$\begin{aligned}\dot{x}_g(t) &= f_g\left(x_g(t)\right) + \sum_{j=1}^{m_g} g_{g,j}\left(x_g(t)\right) u_{g,j}\left(x_g(t), v_g^{s_1}(t), v_g^{s_2}(t), \ldots, v_g^{s_{|X|}}(t)\right) \\ w_g^s(t) &= w_g^s\left(x_g(t)\right), \quad \text{for all } s \in X.\end{aligned} \tag{4}$$

Note that the symbol g will be used in two ways, both as the vector field in $\dot{x} = f(x) + g(x)u$ and also in the sense of $g \in G$, where the distinction should be clear from the context.

Periodic interconnections and a symmetry orbit are defined in a manner similar to the case of one spatial dimension, leading to the following main definition in this paper.

DEFINITION 2.2: (

def:symmetric Let G be a group with a set X of generators. A system with components $g \in \mathcal{I} \subset G$ with dynamics given by Equation 4 has *periodic interconnections on* $\mathcal{I}$ if

$$v_g^s(t) = w_{s^{-1}g}^s\left(x_{s^{-1}g}(t)\right), \quad w_g^s(t) = v_{sg}^s\left(x_g(t)\right) \tag{5}$$

for all $g \in \mathcal{I}$ and $s \in X$. Furthermore, if

$$f_{g_1}(x) = f_{g_2}(x), \quad g_{g_1,j}(x) = g_{g_2,j}(x), \quad w_{g_1}^s(x) = w_{g_2}^s(x), \quad m_{g_1} = m_{g_2} = m \tag{6}$$

for all $s \in X$, $g_1, g_2 \in \mathcal{I}$, $x \in \mathbb{R}^n$ and $j \in \{1, \ldots, m\}$, then $\mathcal{I}$ forms an *orbit of symmetric compoments.* Finally, if the control laws also satisfy

$$\begin{aligned}u_{g_1,j}&\left(x_1(t), w_{s_1^{-1}g_1}^{s_1}(x_2(t)), w_{s_2^{-1}g_1}^{s_2}(x_3(t)), \ldots, w_{s_{|X|}^{-1}g_1}^{s_{|X|}}(x_{|X|+1}(t))\right) = \\ &u_{g_2,j}\left(x_1(t), w_{s_1^{-1}g_2}^{s_1}(x_2(t)), w_{s_2^{-1}g_2}^{s_2}(x_3(t)), \ldots, w_{s_{|X|}^{-1}g_2}^{s_{|X|}}(x_{|X|+1}(t))\right)\end{aligned} \tag{7}$$

for all $g_1, g_2 \in \mathcal{I}$, $j \in \{1, \ldots, m\}$, $s \in X$ and $(x_1, x_2, \ldots, x_{|X|+1}) \in \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ then the elements of $\mathcal{I}$ form a *symmetry orbit.* Such a system with a symmetry orbit is called a *symmetric system on* $\mathcal{I}$. If $\mathcal{I} = G$ it is a symmetric system on G.

EXAMPLE 2.3 A recurring example in this paper will be system of N planar agents with second order dynamics used in [13]. We will first show this specific example fits within the general framework we are devloping. Each robot has a position and velocity in $\mathbb{R}^2 \times \mathbb{R}^2$, with equations of motion for the ith robot given by

$$\frac{d}{dt}\begin{bmatrix} x_i \\ \dot{x}_i \\ y_i \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} \dot{x}_i \\ 0 \\ \dot{y}_i \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_{i,1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_{i,2}. \tag{8}$$

All computations are mod $(N+1)$. The goal formation is a regular $(N+1)$-polygon centered at the origin, hence the desired formation distance between components i and j is

$$d_{ij} = \begin{cases} 1, & |i-j| = 1 \\ \frac{\sin\left(\frac{2\pi}{N+1}\right)}{\sin\left(\frac{\pi}{N+1}\right)}, & |i-j| = 2 \end{cases}$$

and the desired distance of robot i to the origin is

$$r_i = \frac{1}{2\sin\frac{\pi}{N}}.$$

Take the control law to be

$$\begin{bmatrix} u_{i,1} \\ u_{i,2} \end{bmatrix} = -\sum_j \begin{bmatrix} \frac{\left(\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-d_{ij}\right)}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}(x_i - x_j) \\ \frac{\left(\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-d_{ij}\right)}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}(y_i - y_j) \end{bmatrix} - k_d \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} - \begin{bmatrix} \frac{\sqrt{x_i^2+y_i^2}-r_i}{\sqrt{x_i^2+y_i^2}} x_i \\ \frac{\sqrt{x_i^2+y_i^2}-r_i}{\sqrt{x_i^2+y_i^2}} y_i \end{bmatrix} \tag{9}$$

where k_d is a positive constant damping gain and $j \in \{i-2, i-1, i+1, i+2\}$.

To show that this system has a symmetry orbit where the orbit contains all the robots in the system, we need to show it satisfies all the elements of Definition **??**. First, observe that this system can be represented by the graph illustrated in Figure 4 with $G = \{-2, -1, 0, 1, 2, \ldots, N-3\}$, the group operation to be addition and let $X = \{-2, -1, 1, 2\}$ with the relation $s^N = 0$. With these definitions, the Cayley graph for the system is as illustrated in Figure 4. Also, observe from the control law in Equation 9, the control for robot i depends on the states for robots $i-2$, $i-1$, $i+1$ and $i+2$, which are equivalent to the four generators. Hence, define each of the outputs for robot i to be the vector of the robot's position, *i.e.*,

$$w_i^s = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \tag{10}$$

where $s \in X = \{-2, -1, 1, 2, \}$. Define the inputs to component $i \in \{-2, -1, 0, 1, \ldots, N-3\}$ to be

$$v_i^s = \begin{bmatrix} x_{i-s} \\ y_{i-s} \end{bmatrix}, \quad s \in \{-2, -1, 1, 2\},$$

which satisfies Equation 5. The dynamics, as given in Equation 8 satisfy Equation 6. Finally, the feedback law given in Equation 9 satisfies Equation 7. Because these hold for all $i \in \{-2, -1, 0, \ldots, N-3\}$ the system has an orbit of symmetric components which contains all the components in the system.

The utility of the definition of a symmetric system is that it is possible to "build up" an equivalent system by adding components to it and requiring that they be interconnected in a manner equivalent to the original system. We will define two systems to be equivalent if they have symmetry orbits with identical components which are interconnected in the same manner, but they possibly have a different number of components in the symmetry orbit. The means by which this can be done is to have the systems related by having the same generators, but possibly different relations which can result in a different group.

Definition 2.4: (

def:equivalentsystems Two symmetric systems on the finite groups G_1 and G_2 are *equivalent* if G_1 and G_2 are generated by the same set of generators, X,

$$f_{g_1}(x) = f_{g_2}(x), \quad g_{g_1,j}(x) = g_{g_2,j}(x), \quad w^s_{s^{-1}g_1}(x) = w^s_{s^{-1}g_2}(x) \tag{11}$$

and

$$u_{g_1,j}\left(x_1(t), w^{s_1}_{s_1^{-1}g_1}(x_2(t)), w^{s_2}_{s_2^{-1}g_1}(x_3(t)), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g_1}(x_{|X|+1}(t))\right) = u_{g_2,j}\left(x_1(t), w^{s_1}_{s_1^{-1}g_2}(x_2(t)), w^{s_2}_{s_2^{-1}g_2}(x_3(t)), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g_2}(x_{|X|+1}(t))\right)$$

for all $g_1 \in G_1$, $g_2 \in G_2$, $s \in X$, $x \in \mathbb{R}^n$, $\left(x_1, x_2, \ldots, x_{|X|+1}\right) \in \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and $j \in \{1, \ldots, m\}$ where $m = m_{g_1} = m_{g_2}$.

Example 2.5 Returning to Example 2.3, consider two systems with components that satisfy Equation 8 and components belonging to two groups, $G_1 = \{-2, -1, 0, 1, 2, \ldots, N-3\}$ and $G_2 = \{-2, -1, 0, 1, 2, \ldots, M-3\}$ where $M > N$. These systems are equivalent because the dynamics of all the components are identical, the feedback definitions are identical. Both groups are generated by $X = \{-2, -1, 1, 2\}$. The only difference is the relation for G_1 is $s^N = 0$ and the relation for G_2 is $s^M = 0$.

For notational convenience, we will concatenate all the states and vector fields from each component into one system description, $\dot{x} = f(x) + g(x)u(t)$ where

$$x_G = \begin{bmatrix} x_{g_1} \\ x_{g_2} \\ \vdots \\ x_{g_{|G|}} \end{bmatrix}, \quad u_G = \begin{bmatrix} u_{g_1} \\ u_{g_2} \\ \vdots \\ u_{g_{|G|}} \end{bmatrix}, \quad f_G(x_G) = \begin{bmatrix} f_{g_1}(x_{g_1}) \\ f_{g_2}(x_{g_2}) \\ \vdots \\ f_{g_{|G|}}(x_{g_{|G|}}) \end{bmatrix}, \quad g_G(x_G) = \begin{bmatrix} g_{g_1}(x_{g_1}) \\ g_{g_2}(x_{g_2}) \\ \vdots \\ g_{g_{|G|}}(x_{g_{|G|}}) \end{bmatrix}$$

symmetry orbit with N components. The $x_{g_i} \in \mathbb{R}^n$ are the states of the g_ith component in the symmetry orbit.

3 Stability of Symmetric Systems

This section presents the compositionality stability results. The results are directed toward being able to infer stability of a whole equivalence class of systems based on the stability of one of the members of the class and exploit the symmetric nature of the systems we are considering. The results are Lyapunov-based and the first result, Proposition 3.1 concerns negative (semi)definiteness of the derivative of a Lyapunov function for each member of an equivalence class of symmetric systems. Following it is Proposition 3.3 builds on it for Lyapunov stability results as does Proposition 3.4 for stability based on LaSalle's invariance principle.

Proposition 3.1 *Given a symmetric system on the finite group G with generators X, assume $V_G : \mathcal{D}_G \to \mathbb{R}$ is continuously differentiable on some open domain $\mathcal{D}_G \subset \mathbb{R}^{n\times|G|}$ containing the origin and that*

$$V_G(x_G) = \sum_{g\in G} V_g\left(x_g, x_{Xg}\right) = \sum_{g\in G} V_g\left(x_g, w^{s_1}_{s_1^{-1}g}(x_{s_1^{-1}g}), w^{s_2}_{s_2^{-1}g}(x_{s_2^{-1}g}), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g}(x_{s_{|X|}^{-1}g})\right), \tag{12}$$

with $\dot{V}_G < 0$ (resp. $\dot{V}_G \leq 0$) for $x \in \mathcal{D}_G$. Assume furthermore that the V_g are symmetric in the sense that

$$V_{g_1}\left(x_1, w^{s_1}_{s_1^{-1}g_1}(x_2), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g_1}(x_{|X|+1})\right) = V_{g_2}\left(x_1, w^{s_1}_{s_1^{-1}g_2}(x_2), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g_2}(x_{|X|+1})\right) \tag{13}$$

for all $g_1, g_2 \in G$ *and* $\left(x_1, x_2, \ldots, x_{|X|+1}\right) \in \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$.

Then an equivalent system on $\hat{G}$ *is such that for*

$$V_{\hat{G}}(x_{\hat{G}}) = \sum_{g \in \hat{G}} V_g \left(x_g, w^{s_1}_{s_1^{-1}g}(x_{s_1 g}), w^{s_2}_{s_2^{-1}g}(x_{s_2 g}), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g}(x_{s_{|X|}g}) \right), \tag{14}$$

$\dot{V}_{\hat{G}} < 0$ *(resp.* $\dot{V}_{\hat{G}} \leq 0$*) for* $x \in \mathcal{D}_{\hat{G}}$ *where* $\mathcal{D}_{\hat{G}}$ *is a set that is open and contains the origin.*

Proof: By direct computation,

$$\begin{aligned}
\dot{V}_G(x_G) &= \dot{V}_G\left(x_1, x_2, \ldots, x_{|G|}\right) \\
&= \sum_{g \in G} \dot{V}_g(x_g, x_{Xg}) \\
&= \sum_{g \in G} \sum_{s \in X} \frac{\partial V_g}{\partial x_{sg}} \left(f_{sg}(x_{sg}) + \sum_{j=1}^{m} g_{sg,j}(x_{sg}) u_{sg,j}(x_{sg}, w^{s_1^{-1}}_{s_1 sg}(x_{s_1 sg}), \ldots, w^{s_{|X|}^{-1}}_{s_{|X|}sg}(x_{s_{|X|}sg})) \right).
\end{aligned}$$

Let $\mathcal{D}_{\mathbb{R}^n}$ be an open subset of $\mathbb{R}^n$ containing the origin and consider the subset of $\mathbb{R}^{n \times |G|}$, $\hat{\mathcal{D}}_G = \{(x, x, \ldots, x)\,|x \in \mathcal{D}_{\mathbb{R}^n}\}$. Because $\mathcal{D}_G$ is open and contains the origin, there exists a $\mathcal{D}_{\mathbb{R}^n}$ such that $\hat{\mathcal{D}}_G \subset \mathcal{D}_G$ and thus $\dot{V}(x_G) < 0$ (resp. $\dot{V}(x_G) \leq 0$) for $x_G \in \hat{\mathcal{D}}_G$. Furthermore, due to the symmetry of V required by Equation 13, the fact that the system is a symmetric system and by the continuity of $\dot{V}$, each term in the series $\dot{V}_G\left(x_G\right) = \sum_{g \in G} \dot{V}_g\left(x_g, x_{Xg}\right)$ is less than zero (resp. less than or equal to zero) in some open set containing $\hat{\mathcal{D}}_G$. By continuity of $\dot{V}$, this holds in the union of some neighborhoods of each of those points as well.

Let $\hat{\mathcal{D}}_{\hat{G}} = \{(x, x, \ldots, x)\,|x \in \mathbb{R}^n\} \subset \mathbb{R}^{n \times |\hat{G}|}$. By the symmetry of the system and definition of equivalent symmetric systems,

$$\dot{V} = \sum_{g \in \hat{G}} \sum_{s \in X} \frac{\partial V_g}{\partial x_{sg}} \left(f_{sg}(x_{sg}) + \sum_{j=1}^{m} g_{sg,j}(x_{sg}) u_{sg,j}(x_{sg}, w^{s_1^{-1}}_{s_1 sg}(x_{s_1 sg}), \ldots, w^{s_{|X|}^{-1}}_{s_{|X|}sg}(x_{s_{|X|}sg})) \right)$$

must be less than zero for $x \in \hat{\mathcal{D}}_{\hat{G}}$ because each of the terms in the sum must also be negative. Finally, by continuity of $\dot{V}$, this holds for some open set $\mathcal{D}_{\hat{G}}$ containing $\hat{\mathcal{D}}_{\hat{G}}$ ■

Remark 3.2 The utility of this Proposition is that if $\dot{V} \leq 0$ for a symmetric system, then we can conclude that $\dot{V} \leq 0$ for any equivalent system. This is consistent with the intuitive notion that we should be able to add or remove identical components as long as they interact similarly with their neighbors. The "similar" interaction is enforced by the requirement that the group structure of equivalent symmetric systems be generated by the same set of generators.

Proposition 3.1 only concerns the properties of the function $V(x)$. The following two propositions complete the picture with respect to Lyapunov stability (Proposition 3.3) and LaSalle's invariance principle (Proposition 3.4).

Proposition 3.3 *Let* $x = 0 \in \mathcal{D}_G \subset \mathbb{R}^{n \times |G|}$ *be an equilibrium point for the system on* G*. Assume* $V_G(0) = 0$*,* $V_G(x) > 0$ *for* $x \in \mathcal{D}_G - \{0\}$ *and* $\dot{V}_G(x) \leq 0$ *for* $x \in \mathcal{D}_G$*. Then the origin is stable for an equivalent system* *on $\hat{G}$. Moreover, if $\dot{V}_G(x) < 0$ for $x \in \mathcal{D}_G - \{0\}$, then the origin is asymptotically stable for an equivalent system on $\hat{G}$.*

Proof: Using the notation from the proof to Proposition 3.1, it follows from Equation 11 that if the origin is an equilibriumm for the system on G, it must be an equilibrium for any equivalent system. Also, if $0 \in \hat{\mathcal{D}}_G \subset \mathcal{D}_G$, then $0 \in \mathcal{D}_{\hat{G}}$ because $0 \in \mathcal{D}_{\mathbb{R}^n}$. If $V_G(0) = 0$, then $V_g(0) = 0$ by Equation 13. Then by Equation 14, $V_{\hat{G}}(0) = 0$ and by the same reasoning $V_{\hat{G}}(x) > 0$ for $x \in \mathcal{D}_{\hat{G}} - \{0\}$. Also by Equations 13 and 14 $\dot{V}(x) \leq 0$ for $x \in \mathcal{D}_{\hat{G}}$, which implies stability in the sense of Lyapunov for an equivalent system. Furthermore, if $\dot{V}(x) < 0$ for $x \in \mathcal{D}_{\hat{G}} - \{0\}$, then $x = 0 \in \mathcal{D}_{\hat{G}}$ is asymptotically stable. ■

The utility of Proposition 3.3 is that if we can prove with a Lyapunov function that the origin of a symmetric system is stable, then it follows that the origin of any equivalent system is also stable. Furthermore it is stable in the same sense, *i.e.*, stable or asymptotically stable.

PROPOSITION 3.4 *Let $V_G : \mathbb{R}^{n\times|G|} \to \mathbb{R}$ be a continuously differentiable radially unbounded function and suppose that $\dot{V}_G \leq 0$ for all $x \in \left\{x \in \mathbb{R}^{n\times|G|} | V_G(x) \leq c\right\}$. Then for an equivalent system on $\hat{G}$ there exists a $\hat{c}$ such that $\dot{V}_{\hat{G}}(x) \leq 0$ for $x \in \Omega^{\hat{c}}_{\hat{G}} = \left\{x \in \mathbb{R}^{n\times|\hat{G}|} | V_{\hat{G}} \leq \hat{c}\right\}$ and any solution in $\Omega^{\hat{c}}_{\hat{G}}$ will approach the largest invariant subset contained in the set $S_{\hat{G}} = \left\{x \in \Omega^{\hat{c}}_{\hat{G}} | \dot{V} = 0\right\}$.*

Proof: By the same reasoning as in the proof of Proposition 3.1, there exists an open set containing the origin, $\mathcal{D}_{\hat{G}}$ in which $\dot{V}_{\hat{G}} \leq 0$. Let $\hat{c}$ be such that $\left\{x \in \mathbb{R}^{n\times|\hat{G}|} | V_{\hat{G}} \leq \hat{c}\right\} \subset \mathcal{D}_{\hat{G}}$. The rest of the propositon follows from LaSalle's invariance principle. ■

COROLLARY 3.5 *Let $x = 0$ be an equilibrium point for the system on G. Let $V_G : \mathcal{D}_G \to \mathbb{R}$ be a continuously differentiable positive definite function on the domain $\mathcal{D}_G$ containing the origin such that $\dot{V}_G(x) \leq 0$ for $x \in \mathcal{D}_G$. Suppose no solution can stay in the set $\left\{x \in \mathcal{D}_G | \dot{V}(x) = 0\right\}$ other than the trivial solution. Then the origin is asymptotically stable for the system on $\hat{G}$.*

Proposition 3.4 and Corollary 3.5 make use of the usual conditions for LaSalle's invariance principle, and extend that to equivalent symmetric systems.

These results allow us to infer stability in various forms for an entire equivalence class of systems based on the stability of one member of the class. It explicitly makes use of the fact that the system is on a group; hence, it is limited to systems on groups and does not apply, for example, to systems such as a line formation of robots with components on the "end." The strength of the Proposition is that it is not necessary to check any stability properties of individual elements but rather only the stability of the entire formation corresponding to one member of the equivalence class must be determined.

4 Examples

This section will complete Example 2.3 and present an additional example.

EXAMPLE 4.1 Continuing Example 2.3, for a fleet of 5 agents, define a Lyapunov function as

$$V = \sum_{i=1}^{5} V_i = \sum_{i=1}^{5} \frac{1}{2}\left[(\dot{x}_i^2 + \dot{y}_i^2) + \left(\sqrt{x_i^2 + y_i^2} - r_i\right)^2 + \sum_j \left(\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} - d_{ij}\right)^2\right], \tag{15}$$

where $j \in \mathcal{W}_i = \mathcal{V}_i = \{i-2, i-i, i+1, i+2\}$, d_{ij} is the desired distance between robots and r_i is the desired distance of robot i from the origin, as defined previously. We will show that $\dot{V} \leq 0$ for a five-robot system, and hence from Proposition 3.1, $\dot{V} \leq 0$ for any equivalent system. Note that the origin is not an equilibrium for this system, so we must resort to Proposition 3.4 for a stability-type property.

By construction, this Lyapunov function satisfies the hypothesis of Proposition 3.1. Computing $\dot{V}$ gives

$$\begin{aligned}
\dot{V} &= \nabla V \cdot (f + gu) \\
&= \sum_{i=1}^{5} \begin{bmatrix} \frac{\sqrt{x_i^2+y_i^2}-r_i}{\sqrt{x_i^2+y_i^2}} x_i + \sum_j \left(\frac{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-d_{ij}}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}} (x_i - x_j) \right) \\ \dot{x}_i \\ \frac{\sqrt{x_i^2+y_i^2}-r_i}{\sqrt{x_i^2+y_i^2}} y_i \sum_j \left(\frac{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-d_{ij}}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}} (y_i - y_j) \right) \\ \dot{y}_i \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} \dot{x}_i \\ -\frac{\sqrt{x_i^2+y_i^2}-r_i}{\sqrt{x_i^2+y_i^2}} x_i - \sum_j \frac{\left(\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-d_{ij}\right)(x_i-x_j)}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}} - k_d \dot{x}_i \\ \dot{y}_i \\ -\frac{\sqrt{x_i^2+y_i^2}-r_i}{\sqrt{x_i^2+y_i^2}} x_i - \sum_j \frac{\left(\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-d_{ij}\right)(y_i-y_j)}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}} - k_d \dot{y}_i \end{bmatrix} \\
&= \sum_{i=1}^{5} -k_d \left(\dot{x}_i^2 + \dot{y}_i^2 \right).
\end{aligned}$$

Proposition 3.1 ensures that there will exist a domain in which $\dot{V} \leq 0$ for any equivalent system as well. Because the origin is not an equilibrium for the system, Proposition 3.3 does not apply. However, LaSalle's principle does apply to the five-agent system. Clearly $\dot{V} = 0$ when there is no velocity. By construction of the control inputs given in Equation 9 are only zero when the agents have converged to the desired formation centered at the origin. Thus, the largest invariant set with $\dot{V} = 0$ is the desired formation. However, because there is a rotational symmetry about the origin, there are an infinite number of configurations satisfying the formation objective. LaSalle's principle implys the system will converge to the desired formation. Also, Proposition 3.4 implys convergence of the formations to the desired configurations for any equivalent system as well.

Simulation results for a five-agent system are illustrated in Figures 5 and 6 with $k_d = 0.25$. Figure 5 shows the trajectories for the individual agents, and Figure 6 shows the final configuration. Simulation results for a 17-agent system are illustrated in Figures 7 and 8 with $k_d = 0.5$. Figure 7 shows the trajectories for the individual agents, and Figure 8 shows the final configuration, illustrating convergence to the desired formation for the system independent of the number of agents.

Example 4.2 This example considers formation control of a fleet of unicycle-like vehicles. Rather than being a distributed algorithm, a global formation function is minimized. This example is motivated by the results in [12] and illustrates the application of Corollary 3.5 because the dynamics are expressed in terms of an *error* function, which goes to zero if the robots achieve the desired formation.

Each of the robots has dynamics given by

$$\dot{x}_i = u_{1,i} \cos\theta_i, \quad \dot{y}_i = u_{1,i} \sin\theta_i, \quad \dot{\theta}_i = u_{2,i},$$

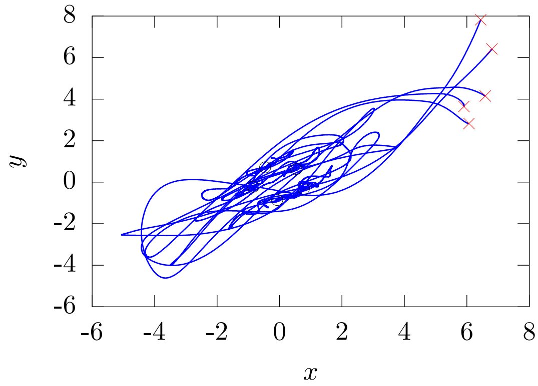

Figure 5: Trajectories for distributed control for a five-vehicle system.

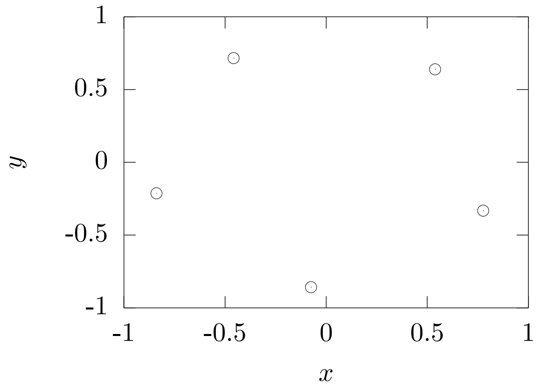

Figure 6: Final formation for distributed control for a five-vehicle system.

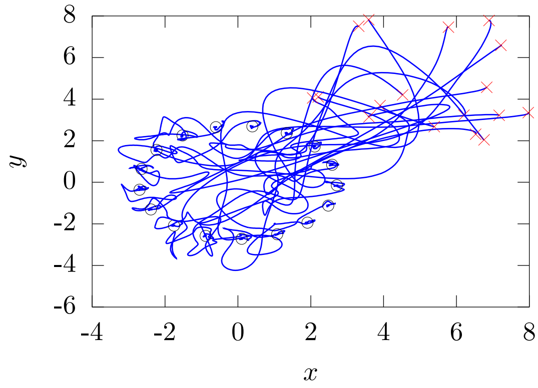

Figure 7: Trajectories for distributed control for a 17-vehicle system.

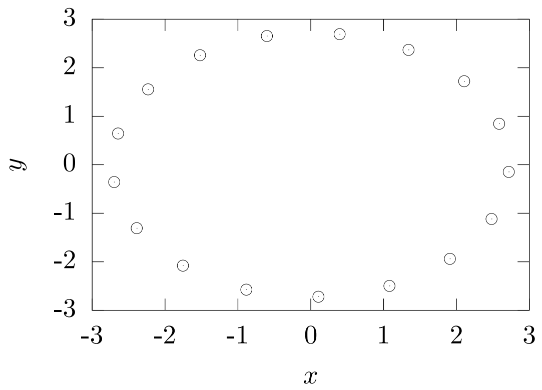

Figure 8: Final formation for distributed control for a 17-vehicle system.

where $u_{1,i}$ and $u_{2,i}$ are the inputs, which are the kinematic linear and angular velocities of the unicycle, respectively. It is well known that this model is dynamic feedback linearizable [4]. Defining

$$\begin{aligned}\dot{\xi}_i &= v_{1,i}\cos\theta + v_{2,i}\sin\theta\\ u_{1,i} &= \xi_i\\ u_{2,i} &= \frac{-v_{1,i}\sin\theta_i + v_{2,i}\cos\theta_i}{\xi_i}\end{aligned}$$

which gives the system

$$\begin{aligned}\dot{x}_i &= \xi_i\cos\theta_i, & \dot{\theta}_i &= \frac{1}{\xi_i}\left(-v_{1,i}\sin\theta_i + v_{2,i}\sin\theta_i\right),\\ \dot{y}_i &= \xi_i\sin\theta_i, & \dot{\xi}_i &= v_{1,i}\cos\theta_i + v_{2,i}\sin\theta_i,\end{aligned}$$

which clearly has a singularity at $\xi_i = 0$, which corresponds to zero velocity. If the desired trajectory is given by $\left(x_i^d(t), y_i^d(t)\right)$, then the inputs

$$\begin{aligned}u_{1,i} &= \ddot{x}_i^d - \left(x_i - x_i^d\right) - \left(\dot{x}_i - \dot{x}_i^d\right)\\ u_{2,i} &= \ddot{y}_i^d - \left(y_i - y_i^d\right) - \left(\dot{y}_i - \dot{y}_i^d\right)\end{aligned}$$

achieve asymptotic tracking. To see this, define

$$e_{x,i} = x_i - x_i^d, \quad e_{y,i} = y_i - y_i^d$$

from which the error dynamics using those inputs are

$$\frac{d}{dt}\begin{bmatrix} e_{x,i}\\ \dot{e}_{x,i}\\ e_{y,i}\\ \dot{e}_{y,i}\end{bmatrix} = \begin{bmatrix} \dot{e}_{x,i}\\ -e_{x,i} - \dot{e}_{x,i}\\ \dot{e}_{y,i}\\ -e_{y,i} - \dot{e}_{y,i}\end{bmatrix}.$$

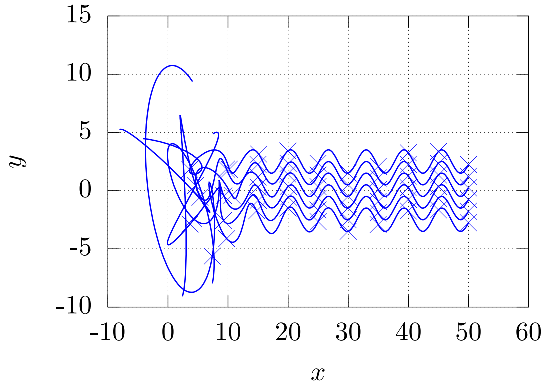

Figure 9: Trajectories for six unicycle robots.

Defining $V_i = \frac{1}{2}\left(e_{x,i}^2 + \dot{e}_{x,i}^2 + e_{y,i}^2 + \dot{e}_{y,i}^2\right)$ gives

$$\dot{V}_i = \begin{bmatrix} e_{x,i} \\ \dot{e}_{x,i} \\ e_{y,i} \\ \dot{e}_{y,i} \end{bmatrix} \cdot \begin{bmatrix} \dot{e}_{x,i} \\ -e_{x,i} - \dot{e}_{x,i} \\ \dot{e}_{y,i} \\ -e_{y,i} - \dot{e}_{y,i} \end{bmatrix} = -\dot{e}_{x,i}^2 - \dot{e}_{y,i}^2.$$

Since V is positive definite, radially unbounded and continuously differentiable, from LaSalle's invariance principle we can conclude global asymptotic stability.

So, for this system the Lyapunov function $V = \sum_{i=1}^{N} V_i$ can be defined, and by Proposition 3.1, $\dot{V} \leq 0$ for all N since it was true for $N = 1$. Simulation results are illustrated in Figures 9 and 10 for six and 13 unicycles respectively. In each case the desired trajectory is given by $x_i^d = t$ and $y_i^d = \sin(t) + i - \frac{N}{2}$. Each x mark on the figures represent a specific times, which illustrate that not only do the robots track the desired trajectories in space, they also are doing so at the desired time. Corollary 3.5 guarantees convergence to zero error dynamics for any equivalent system.

5 Formation Robustness under Agent Failures

The results in the previous sections may be used to formulate some robustness results. First these results are motivated by an example which illustrates the type of system behavior we want to prove.

Example 5.1 Consider the system from Examples 2.3 and 4.1 with five agents and assume that agent 5 fails in a manner that it has zero velocity and is completely unresponsive to any control input. One would hope that the rest of the formation will converge to a formation that accommodates such a failure. In fact, this does happen, as is illustrated in Figures 11 and 12. Figure 11, illustrates the trajectories of the agents when agent five fails and remains stationary. Figure 12 illustrates the initial and final configurations for that system.

Clearly it is not *a priori* necessary that stability will be preserved when an agent fails. In fact, in general it would not be expected because the system being controlled is not the same one for which the controller

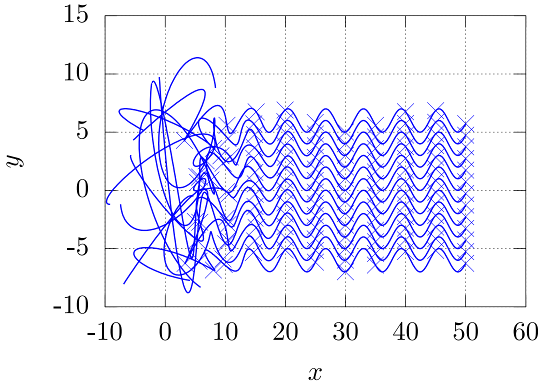

Figure 10: Trajectories for 13 unicycle robots.

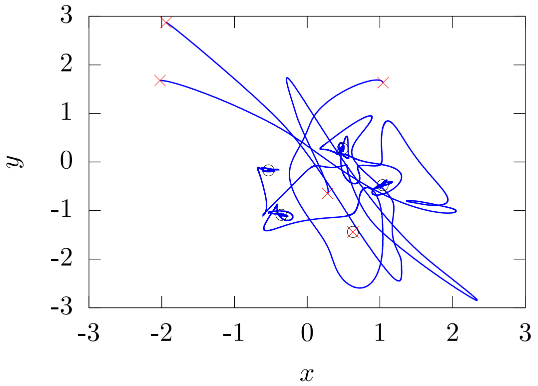

Figure 11: Robust formation control for a five-agent system.

was designed. Also, consistent with the theme of this paper, we would like results to apply to an entire equivalence class of systems as well.

The following corollary to Proposition 3.4 provides the desired result.

COROLLARY 5.2 *If a symmetric distributed system on G satisfies the conditions of Proposition 3.4 on a set Ω_G^c with the origin in the interior of Ω_G^c, then if any number of agents fail with zero velocity then there exists a $\hat{c}$ such that if the system starts in $\Omega_G^{\hat{c}}$ all solutions stay in $\Omega_G^{\hat{c}}$ for all time.*

Proof: Let $\mathcal{D}_{\mathbb{R}^n}$ be an open subset of $\mathbb{R}^n$ containing the origin and consider the subset of $\mathbb{R}^{n\times|G|}$, $\hat{\mathcal{D}}_G = \{(x, x, \ldots, x)\,|x \in \mathcal{D}_{\mathbb{R}^n}\}$. Because the origin is in the interior of Ω_G^c, there exists a $\mathcal{D}_{\mathbb{R}^n}$ such that $\hat{\mathcal{D}}_g \subset \mathcal{D}_G$ and thus $\dot{V}(x_G) \leq 0$ for $x_G \in \Omega_G^c$. Furthermore, due to the symmetry of V required by Equation 12, the fact that the system is a symmetric system and by the continuity of $\dot{V}$, each term in $\dot{V}_G(x_G) = \sum_{g\in G} \dot{V}_g\,(x_g, x_{Xg})$

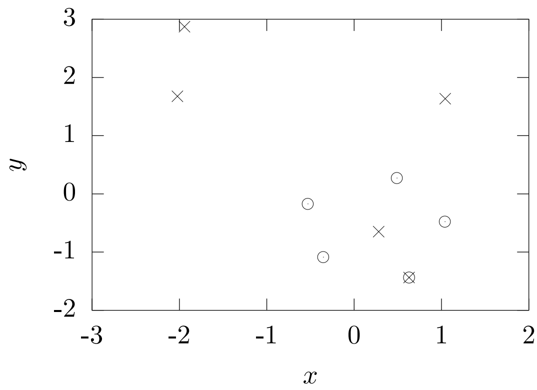

Figure 12: Robust formation control for a five-agent system. Initial conditions are indicated by a $\times$ and final configurations by a $\circ$.

is less than or equal to zero in some open set containing $\hat{\mathcal{D}}_G$. Because points with zero velocity are contained in $\Omega_G^{\hat{c}}$ the result follows from LaSalle's Principle. ■

6 Conclusions

This paper considers stability of coordinated and distributed systems, with an application focus on coordinated control of systems of mobile robots. The model used is a nonlinear extension of the work in [1, 14], which was directed toward spatially periodic systems "built-up" from periodically interconnected components. Observing that many of the formation control algorithms in the literature are not limited by the number of components, but often are limited by assuming specific dynamics, the main contribution was to formulate a theoretical framework in which stability of many distributed systems can be considered. The result was demonstrated on two systems, one of which was fully distributed and the other of which was no decentralized.

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