Passivity and Dissipativity of a Nonlinear System and its Linearization

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Abstract

In this paper, we investigate when passivity for a nonlinear system can be inferred from its linearization. The nonlinear system considered here are affine in control and with feedthrough terms, in both continuous-time domain and discrete-time domain. Our main results demonstrate when the linearized system is simultaneously strict passive and strict input passive (SSIP), the nonlinear system will be SSIP as well within a neighborhood of the equilibrium point. We establish algebraic conditions under which a linear system is SSIP based on the positive real lemma. The results are extended to dissipative systems and in particular, passivity indices can be derived for a nonlinear system from the linearized system within a neighborhood of the equilibrium.

Index Terms

Passivity, Dissipativity, Linearization, Nonlinear Systems, Passivity Indices.

I. Introduction

Energy dissipation is a fundamental concept in the study of the behavior of a dynamical system [1]. Passivity, and its generalization dissipativity, characterizes the energy consumption of a system and is used in a variety of applications, e.g. electrical, mechanical, chemical and communication systems [2], [3]. The compositional property (for instance, negative feedback connection of two passive systems remains passive) makes passivity a powerful tool to analyze complicated, coupled systems, such as Cyber-physical systems [4].

In this paper, we pursue passivity and dissipativity properties of a nonlinear system from its first-order approximation. Nonlinear behaviors (including saturation, backlash and dead zone) abound in most physical systems [5], [6]. Although nonlinear models are more accurate to characterize the dynamical systems, analysis and control design methods are more available for linear systems. Therefore, using its first-order approximation is one effective approach to provide a local description of the nonlinear system [5], [6]. The approach of *linearization* has been used in nonlinear optimal control, model matching and input-output decoupling and so on [7], [1]. A well-known result regarding stability is that a nonlinear system will be stable in a neighborhood of the equilibrium point if its linearization is asymptotically stable. Passivity is closely related to stability [7], [3] and this paper is to pursue this line of research by exploring passivity for a nonlinear system from its linearized system. Passivity theory for linear systems (e.g. [8]) is well established, for instance, the well known KYP lemma relates passivity with the algebraic structure of the system. It is also of practical importance to analyze the linearized system in order to further control the nonlinear system [1], [9].

A similar problem has been investigated in e.g. [1] for a continuous-time nonlinear system that affine in control and without feedthrough term, i.e. of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x). \end{aligned} \tag{1}$$

In [1], *strict passivity* for the linearized system is required to ensure passivity for the nonlinear system (1) under some rank and integrability conditions. In discrete-time domain, however, this is not the case. For Σ_d of the form

$$\begin{aligned} \Sigma_d : x(k+1) &= f(x(k)) + g(x(k))u(k), \\ y(k) &= h(x(k)) + J(x(k))u(k), \end{aligned} \tag{2}$$

it does not make sense to study passivity when $J(x) = 0$ even in the corresponding linear case [10]. This is one difference from the continuous-time domain. In this paper, we consider a nonlinear system with feedthrough term in both continuous-time domain, i.e.

$$\begin{aligned} \Sigma_c : \dot{x} &= f(x) + g(x)u, \\ y &= h(x) + J(x)u, \end{aligned} \tag{3}$$

and discrete-time domain given by (2). For Σ_c and Σ_d, *strict passivity alone* for the linearized system may not be sufficient to show local passivity as for (1).

The problem we are interested in is stated as follows: *in order to investigate passivity or dissipativity for the nonlinear system Σ_c (or Σ_d) around the equilibrium, what passivity properties (or other conditions) of the linearized model are required?* Our main results show that if the linearized system is simultaneously *strict passive* and *strict input passive* (SSIP), the nonlinear system Σ_c or Σ_d will be SSIP in a neighborhood of the equilibrium. Conditions are established under which the linearization of Σ_c or Σ_d is SSIP. In continuous-time domain, the condition is nothing but *strongly positive real* [11] or *extended strictly positive real* [12]. The results can be extended to systems that may not be affine in control as well if the linearized system is shown to be SSIP.

Dissipativity (a generalization of passivity) of a nonlinear system and its linearization is studied in [7], [13]. The results demonstrate that if a nonlinear system Σ_c or Σ_d is QSR dissipative [14], then its linearization is QSR dissipative as well. [7] also studies when dissipativity of a continuous-time nonlinear system can be inferred from its linearization based on a Hamiltonian matrix. In this paper, we investigate the problem based on a series of algebraic conditions inspired from the essentiality of SSIP, in both continuous-time and discrete-time domain. As a particular case of QSR dissipativity, we relate passivity indices for a nonlinear system and its linearized system. The passivity indices (characterize how passive the system is) for linear system can be easily calculated from the transfer function and can be further used to control the nonlinear system [15], [4].

The rest of the paper is organized as follows. Section II provides some background material on passivity theory and dissipativity. Section III presents preliminary results relating passivity and QSR dissipativity for a nonlinear system and its linearization about an equilibrium. The main results are given in Section IV to show local passivity and local QSR dissipativity for a nonlinear system from its strict passive and strict input passive linearized system. Section VI provides some concluding remarks.

Notation: $\mathbb{R}^m$ denotes the Euclidean space of dimension m. I denotes the identity matrix of appropriate dimensions. For a dynamical system, its states, control input and output are denoted by $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, and $y \in \mathcal{Y} \subseteq \mathbb{R}^m$, respectively. For state-space models (3) and (2), f, g, h, J are smooth mappings of appropriate dimensions and we assume $f(0) = 0, h(0) = 0$ without loss of generality. A linear version of Σ_c is given by

$$\begin{aligned} \mathcal{G}_c : \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{4}$$

and $\mathcal{G}_d$ as a linear version of Σ_d is given by

$$\begin{aligned} \mathcal{G}_d : x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k). \end{aligned} \tag{5}$$

In the following analysis, we assume for linear systems (4) and (5), $\{A, B\}$ is controllable and $\{A, C\}$ is observable.

For a matrix $P \in \mathbb{R}^{m\times n}$, its transpose it denoted by P^T. For a symmetric matrix where $P = P^T$, $P > 0$ denotes it is positive-definite. The maximum eigenvalue of P is denoted by $\overline{\lambda}(P)$ and its minimum eigenvalue is denoted by $\underline{\lambda}(P)$. The norm of a vector or a matrix is given by $\|P\|$.

II. BACKGROUND MATERIAL

Definition 1 (CT-Dissipative [7], [14]). A state-space system Σ_c given by (3) is said to be *dissipative* with respect to supply rate $w(u(t), y(t))$, if there exists a nonnegative function $V(x)$ satisfying $V(0) = 0$ such that for all $x_0 \in \mathcal{X}$, all $t_1 \geq t_0$, and all $u \in \mathbb{R}^m$,

$$V(x(t_1)) \leq V(x(t_0)) + \int_{t_0}^{t_1} w(u(t), y(t))dt, \tag{6}$$

where $x(t_0) = x_0$, and $x(t_1)$ is the state at time t_1 resulting from initial condition x_0 and input u. If (6) holds with strict inequality, Σ_c is called *strict dissipative* (SD). If (6) holds with equality, Σ_c is called *conservative*.

Definition 2 (DT-Dissipative [10], [13]). A state-space system Σ_d given by (2) is said to be *dissipative* with respect to $W(u(k), y(k))$, if there exists a nonnegative function $V(x)$ satisfying $V(0) = 0$ such that for all $x_0 \in \mathcal{X}$, all $k \geq k_0$, and all $u \in \mathbb{R}^m$,

$$V(x(k)) - V(x(k_0)) \leq \sum_{i=k_0}^{k-1} W(y(i), u(i)). \tag{7}$$

If (7) holds with strict inequality, Σ_d is called *strict dissipative*. If (7) holds with equality, Σ_d is called *conservative*.

The nonnegative function $V(x)$ in the above definitions are called *storage function*. In this paper, we assume the storage function $V(x)$ is analytic and thus the Taylor series expansion about $x = 0$ yields [1], [13]

$$V(x) = x^T P x + V_h(x), \tag{8}$$

where $P = P^T > 0$ and $V_h(x)$ contains the higher order terms of $V(x)$. In this case, (6) is equivalent to

$$\dot{V}(x) \triangleq \frac{\partial V}{\partial x}(f(x) + g(x)u) \leq w(u, y). \tag{9}$$

It has also been shown in [10] that (7) is equivalent to

$$V(x(k+1)) - V(x(k)) \leq W(u(k), y(k)) \tag{10}$$

The two inequalities (9) and (10) are sometimes used to define dissipative systems in the literature.

A quadratic supply rate for dynamical systems Σ_c and Σ_d is of particular interest and given as

$$r(u, y) = u^T R u + 2y^T S u + y^T Q y, \tag{11}$$

where Q, S, R are matrices and Q, R are symmetric. If Σ_c or Σ_d is dissipative with respect to supply rate (11), it is called *QSR-dissipative*, see e.g. [16], [14]. Some special cases of QSR-dissipative systems are given as follows.

Definition 3 ([11], [3]). Suppose Σ_c given by (3) or Σ_d given by (2) is QSR-dissipative. It is called:

1) *passive* if $Q = R = 0, S = \frac{1}{2}I$; In particular, if (6) or (7) holds with strict inequality for $r(u, y) = u^T y$, the system is called *strict passive* (SP). If equality holds, the system is called *lossless*.
2) *strict input passive* (SIP) if $Q = 0, R = -\nu I, S = \frac{1}{2}I$ for some $\nu > 0$;
3) *strict output passive* (SOP) if $Q = -\mu I, R = 0, S = \frac{1}{2}I$ for some $\mu > 0$.

Note that SP and SIP do not imply each other in general. For instance, a continuous-time (CT) system whose transfer function given by $G(s) = \frac{1}{s+1}$ is SP but not SIP. Also, a discrete-time (DT) system whose transfer function given by $H(z) = \frac{z+1}{z}$ is SIP but not SP. A stronger property than SP or SIP alone is called SSIP defined as *simultaneously strict passive and strict input passive*. Consider a linear system

$$\begin{aligned} \dot{x} &= -x + 0.5u, \\ y &= x + 0.5u. \end{aligned}$$

With a storage function $V(x) = \frac{1}{2}x^2$, we obtain

$$\begin{aligned} \dot{V} - u^T y &= x(-x + 0.5u) - u(x + 0.5u) \\ &= -\frac{1}{4}u^2 - \frac{3}{4}x^2 - \frac{1}{4}(u + x)^2 \leq -\frac{1}{4}u^2 - \frac{3}{4}x^2. \end{aligned}$$

Thus, the linear system is SSIP.

Definition 4 ([17]). If any of the property for Σ_c or Σ_d defined above holds in a neighborhood of $(\hat{x}=0,\hat{u}=0)\in\mathcal{X}\times\mathcal{U}$, it is called a *local* property for Σ_c or Σ_d.

Remark 1: In [1], *local passivity* is defined in a ball around $\hat{x}=0$ and all control u that "does not drive the state to far from the equilibrium point". Sobolev space is used in [18] to define *local passivity*, where the magnitudes of control u and the derivative of u are bounded. In this paper, we consider local passivity or dissipativity in a neighborhood of $(\hat{x}=0,\hat{u}=0)\in\mathcal{X}\times\mathcal{U}$ as in [17]. Note that all the definitions are essentially equivalent such that local is "*both in terms of small-gain inputs and local internal stability regions*" (see [19] for local dissipativity).

We need the following results to justify (strict) passivity and (strict) QSR-dissipativity for linear systems $\mathcal{G}_c$ given by (4) or $\mathcal{G}_d$ given by (5).

Lemma 1 ([5]): $\mathcal{G}_c$ is SP if and only if there exist matrices $P=P^T>0,L,W$ and $\varepsilon>0$, such that

$$\begin{aligned} PA+A^TP&=-L^TL-\varepsilon P,\\ PB&=C^T-L^TW,\\ W^TW&=D+D^T. \end{aligned}$$

Lemma 2 ([13]): $\mathcal{G}_d$ is SP if and only if there exist matrices $P=P^T>0,L,W$ and $\rho>0$, such that

$$\begin{aligned} -P+A^TPA&=-L^TL-\rho P,\\ A^TPB&=C^T-L^TW,\\ W^TW&=D+D^T-B^TPB. \end{aligned}$$

Lemma 3 ([20]): $\mathcal{G}_c$ is SD if and only if there exist matrices $P=P^T>0,L,W$ and $\varepsilon>0$ such that

$$\begin{aligned} 0&=PA+A^TP+\varepsilon P-C^TQC+L^TL,\\ 0&=PB-C^T(QD+S)+L^TW,\\ 0&=R+S^TD+D^TS+D^TQD-W^TW. \end{aligned}$$

Lemma 4 ([13]): $\mathcal{G}_d$ is SD if and only if there exist matrices $P=P^T>0,L,W$ and $\rho>0$ such that

$$\begin{aligned} 0&=A^TPA-P+\rho P-C^TQC+L^TL,\\ 0&=A^TPB-C^T(QD+S)+L^TW,\\ 0&=R+S^TD+D^TS+D^TQD-B^TPB-W^TW. \end{aligned}$$

Remark 2: 1). If $\varepsilon=0$ in Lemma 1 (or resp. $\rho=0$ in Lemma 2), the system is *passive*. For a linear system, (strict) passivity is equivalent to (strict) positive realness [21].

2). If $\varepsilon=0$ in Lemma 3 (or resp. $\rho=0$ in Lemma 4), the system is *QSR-dissipative*. Strict dissipative linear system is referred to *exponentially dissipative* in CT domain and *geometrically dissipative* in DT domain [13].

III. Preliminary Results

A. Linearization about an Equilibrium

We assume the pair $(\hat{x}=0,\hat{u}=0)$ is an equilibrium for nonlinear systems Σ_c given by (3) or Σ_d given by (2), without loss of generality. The linearization of Σ_c (resp. Σ_d) about the equilibrium $(0,0)$ is given by $\mathcal{G}_c$ (resp. $\mathcal{G}_d$) with

$$A=\frac{\partial f}{\partial x}|_{x=0},B=g(0),C=\frac{\partial h}{\partial x}|_{x=0},D=J(0). \tag{12}$$

The linearized model is accurate up to first order and called first-order approximation [6]. Taylor series expansion for f, g, h, J about $x = 0$ are given as

$$
\begin{aligned}
&f(x) = Ax + F(x), h(x) = Cx + H(x), \\
&g(x) = B + G(x), J(x) = D + M(x),
\end{aligned} \tag{13}
$$

where $F(x), H(x), G(x), M(x)$ contains higher-order terms of $f(x), h(x), g(x), J(x)$, respectively. We say (12) is the linearzation of Σ_c or Σ_d if there is no confusion (in CT or DT domain) in the context.

Remark 3: In this paper, we focus on linearization about an equilibrium by (12). Another situation is linearization about a trajectory (usually prescribed) which often results in a linear time-varying (LTV) system [6]. Passivity theory for LTV systems has been studied in e.g. [22], [23], [24].

The following result is given in [13] that relates dissipativity of a nonlinear system and its linearization.

Theorem 1 ([13]): Assume system (3) (resp. (2)) is QSR dissipative, then its linearized system (4) (resp. (5)) together with (12) is QSR dissipative with the *same* supply rate.

A few remarks about this result.

1) The assumption of complete reachability of the nonlinear system (3) or (2) in [13] is not used here since the existence of a storage function is implied from Definition 1 or Definition 2.
2) If the nonlinear system (3) or (2) is QSR dissipative with storage function $V(x)$ in (8), then $x^T P x$, the quadratic terms in $V(x)$, is a storage function for the linearized system (12) w.r.t the same supply rate.
3) As a particular case of QSR dissipativity, passivity of a nonlinear system implies passivity of its linearized system as well, but this is not true for SP.

B. An example: from Linearity to Nonlinearity

To this end, we know passivity (resp. QSR dissipativity) of a nonlinear system implies passivity (resp. QSR dissipativity) of its linearized system. However, does passivity (resp. QSR dissipativity) of its linearization imply passivity (resp. QSR dissipativity) of the nonlinear system? In general, this is not true. To see this, let us consider an example from [7].

Example 1 ([7]): Consider a nonlinear system

$$
\begin{aligned}
\dot{x_1} &= x_2, \\
\dot{x_2} &= -\sin(x_1) + u, \\
y &= \cos(x_1) x_2.
\end{aligned}
$$

Its linearized system is given through the following matrices

$$
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 0.
$$

This linearized system is *lossless* with storage function $V = \frac{1}{2}(x_1^2 + x_2^2)$. To show the nonlinear system is passive (or lossless) in a neighborhood of origin, we need to find a storage function V_2, such that for x close enough to origin,

$$
\frac{\partial V_2}{\partial x_1} x_2 + \frac{\partial V_2}{\partial x_2}(-\sin x_1) \leq 0, \frac{\partial V_2}{\partial x_2} = \cos(x_1) x_2.
$$

It has been shown in [7], however, such storage function does not exist and therefore the nonlinear system is not passive (or lossless) even for x close enough to the origin.

This example demonstrates in order to show local passivity for a nonlinear system, a *stronger* condition than passivity (losslessness in this example) for the linearized system, such as SP and/or SIP may be required. In the following sections, we will see that *SSIP* for the linearized system is required to show Σ_c or Σ_d is locally passive. This is analogous to *asymptotic stability* of the linearization that required to show Lyaponuv stability of a nonlinear system.

C. Strict Passive Linearized System

In this section, we consider when the linearized system of (3) and (2) is strict passive (SP). It turns out additional conditions may be needed besides strict passivity.

Theorem 2 (CT-SP): Consider a nonlinear system Σ_c given by (3) and its linearized system $\mathcal{G}_c$ given by (12). Suppose $\mathcal{G}_c$ is controllable and observable. If $\mathcal{G}_c$ is SP and there exist a constant $l \geq 0$ such that

$$\lim_{\|x\|^2 \to 0} \frac{\|J(x) - D\|}{\|x\|^2} \leq l, \tag{14}$$

then Σ_c is locally strict passive (LSP).

Proof: From Lemma 1, there exists $V(x) = \frac{1}{2}x^T Px$ as a storage function for $\mathcal{G}_c$ such that [5]

$$\begin{aligned} u^T y - \frac{\partial V}{\partial x}(Ax + Bu) &= u^T(Cx + Du) - x^T P(Ax + Bu) \\ &\geq \frac{1}{2}\varepsilon x^T Px. \end{aligned}$$

Apply $V(x) = \frac{1}{2}x^T Px$ as a locally defined storage function for the nonlinear system Σ_c, we have from (13),

$$\begin{aligned} & u^T(h(x) + J(x)u) - \frac{\partial V}{\partial x}(f(x) + g(x)u) \\ =& u^T(Cx + Du) + u^T(H(x) + M(x)u) \\ & - x^T P(Ax + Bu) - x^T P(F(x) + G(x)u) \\ \geq& \frac{1}{2}\varepsilon x^T Px + u^T(H(x) + M(x)u) - x^T P(F(x) + G(x)u) \\ \triangleq& \Upsilon(x, u). \end{aligned}$$

To make the proof work, we would like to make sure $\Upsilon(x, u) \geq 0$ for a neighborhood of $x = 0, u = 0$. To achieve this, we use the following relation

$$\begin{aligned} \Upsilon(x, u) \geq& \frac{1}{2}\varepsilon x^T Px - \|x^T PF(x)\| \\ & - \|u\|\|(H(x) - G^T(x)Px)\| - \|u^T M(x)u\|. \end{aligned}$$

From the fact that $x^T Px \geq \underline{\lambda}(P)\|x\|^2$, where $\underline{\lambda}(P) > 0$ denotes the minimum eigenvalue of P, we obtain

$$\frac{1}{2}\varepsilon x^T Px \geq (1 - \theta)\delta\|x\|^2 + \theta\delta\|x\|^2,$$

for some $\theta \in (0, 1)$ and $\delta \triangleq \frac{1}{2}\varepsilon\underline{\lambda}(P) > 0$. From Taylor's theorem and assumption (14), there exist a ball around $x = 0$ and a ball around $u = 0$ for which

$$\begin{aligned} & \theta\delta\|x\|^2 - \|x^T PF(x)\| \\ & - \|u\|\|(H(x) - G^T(x)Px)\| - \|u^T M(x)u\| \\ \geq& \|x\|^2(\theta\delta - \xi_1\|x\| - \xi_2\|u\| - l\|u\|^2) \geq 0, \end{aligned}$$

where $\xi_1 > 0$ and $\xi_2 > 0$ are constants. Therefore we have the following inequality

$$\Upsilon(x, u) \geq (1 - \theta)\delta\|x\|^2,$$

for $\theta \in (0, 1)$ and a neighborhood of $x = 0, u = 0$. This implies LSP of the nonlinear system Σ_c. ■

Note that if $J(x) \equiv 0$, the system Σ_c is reduced to (1). It is obvious $D \equiv 0$ in its linearization (12) and thus $J(x) - D = 0$. Therefore, (14) is necessarily satisfied with $l = 0$.

Corollary 1: Consider a nonlinear system (1) and its linearized system (12) where $D = 0$. Suppose the linearized system is controllable and observable. If its linearization is SP, then system (1) is LSP.

Remark 4: A similar result for system (1) is present in [1] under some rank and integrability conditions.

Next, we present a simple example.

Example 2: A nonlinear system is given by

$$\begin{aligned}\dot{x} &= -x + x^3 + (-x+1)u, \\ y &= x - x^2 + (ax^2+1)u,\end{aligned}$$

where $|a| \leq 1$. Its linearized system is characterized by $A = -1, B = 1, C = 1, D = 1$. It is simple to verify that the linearized system is SP with storage function $V(x) = \frac{1}{2}x^2$. For the nonlinear system, we can derive

$$\begin{aligned}\dot{V} - uy &= x(-x + x^3 + (-x+1)u) \\ &\quad - u(x - x^2 + (ax^2+1)u) \\ &= -x^2(1-x^2) - (ax^2+1)u^2 \\ &\leq -(x^2+u^2)(1-x^2).\end{aligned}$$

Thus, $\dot{V} - uy \leq 0$ if $x \in [-1, 1]$, i.e. the nonlinear system is LP for $u \in \mathbb{R}$ such that $x \in [-1, 1]$. Furthermore, for $u \in \mathbb{R}$ such that $x \in [-0.5, 0.5]$, the nonlinear system is LSP.

Analogously, in the discrete-time domain, we can derive the following result for system (2) when its liniearization is strict passive.

Theorem 3 (DT-SP): Consider a nonlinear system Σ_d given by (2) and its linearized system $\mathcal{G}_d$ given by (12). Suppose $\mathcal{G}_d$ is controllable and observable. If $\mathcal{G}_d$ is SP and there exist $l_1 \geq 0, l_2 \geq 0$ such that

$$\lim_{\|x\|^2 \to 0} \frac{\|J(x) - D\|}{\|x\|^2} \leq l_1, \lim_{\|x\|^2 \to 0} \frac{\|g(x) - B\|}{\|x\|^2} \leq l_2, \tag{15}$$

then the nonlinear system Σ_d is LSP.

Proof: From Lemma 2, we know $V(x) = x^T P x$ is a storage function for $\mathcal{G}_d$ such that $\forall k$,

$$\begin{aligned}&V(x(k+1)) - V(x(k)) - u^T(k)y(k) \\ &= \frac{1}{2}(Ax(k) + Bu(k))^T P (Ax(k) + Bu(k)) \\ &\quad - \frac{1}{2}x(k)^T P x(k) - u(k)^T (Cx(k) + Du(k)) \\ &\leq -\frac{1}{2}\rho x^T(k) P x(k).\end{aligned}$$

For notational convenience, we omit the time index k in the following proof. Apply $V(x)$ as a locally defined storage function for the nonlinear system Σ_d. From (13), we have

$$\begin{aligned}\Psi(x,u) &\triangleq \frac{1}{2}\left[f(x) + g(x)u\right]^T P \left[f(x) + g(x)u\right] - \frac{1}{2}x^T P x \\ &\quad - u^T \left[h(x) + J(x)u\right] \\ &\leq -\frac{1}{2}\rho x^T P x + \phi(x,u),\end{aligned}$$

and $\phi(x,u)$ is given by

$$\begin{aligned}\phi(x,u) &= -u^T (H(x) + M(x)u) \\ &\quad + (F(x) + G(x)u)^T P (Ax + Bu) \\ &\quad + \frac{1}{2}(F(x) + G(x)u)^T P (F(x) + G(x)u).\end{aligned} \tag{16}$$

To make the proof work, we would like to make sure $-\frac{1}{2}\rho x^T P x + \phi(x,u) \leq 0$ for a neighborhood of $x = 0, u = 0$. To achieve this, we use the following relation

$$-\frac{1}{2}\rho x^T P x \leq -(1-\theta)\delta\|x\|^2 - \theta\delta\|x\|^2,$$

for some $\theta \in (0,1)$, where $\delta \triangleq \frac{1}{2}\rho\underline{\lambda}(P) > 0$ and $\underline{\lambda}(P) > 0$ denotes the minimum eigenvalue of P. From Taylor's theorem and (15), there exist a ball around $x = 0$ and a ball around $u = 0$ for which

$$-\theta\delta\|x\|^2 + \phi(x,u) \leq -\theta\delta\|x\|^2 + \|\phi(x,u)\| \leq 0,$$

and the following inequality holds

$$\Psi(x,u) \leq -\frac{1}{2}\rho x^T P x + \phi(x,u) \leq -(1-\theta)\delta\|x\|^2,$$

for some $\theta \in (0,1)$, i.e. the nonlinear system Σ_d is LSP. ∎

Again, we consider a simple example as follows.

Example 3: A discrete-time nonlinear system is given by

$$\begin{aligned} x(k+1) &= 0.5x(k) - (2x^2(k) - 1)u(k), \\ y(k) &= x(k) - x^3(k) + (x^2(k)+1)u(k). \end{aligned}$$

Its linearized system is characterized by $A = 0.5, B = 1, C = 1, D = 1$. This linearized system is SP with storage function $V(x) = \frac{1}{2}x^2$. For the nonlinear system, we have

$$\begin{aligned} &V(x(k+1)) - V(x(k)) - u(k)y(k) \\ =&\frac{1}{2}\left[0.5x(k) - (2x^2(k)-1)u(k)\right]^2 - \frac{1}{2}x^2(k) \\ &- u(k)\left[x(k) - x^3(k) + (x^2(k)+1)u(k)\right] \\ =&\frac{1}{2}\left(0.5x(k) + u(k)\right)^2 - \frac{1}{2}x^2(k) - u(k)\left(x(k)+u(k)\right) \\ &+ \chi(x(k),u(k)) \\ \leq& -\frac{1}{4}x^2(k) + \chi(x(k),u(k)), \end{aligned}$$

where $\chi(x,u) = -u^2x^2(3-2x^2) \leq 0$ for $|x|^2 \leq \frac{3}{2}$. Thus, the nonlinear system is LSP for $u \in \mathbb{R}$ such that $x \in [-\sqrt{1.5}, \sqrt{1.5}]$.

Remark 5: The conditions (14) and (15) are *sufficient but not necessary*. The two theorems only captures a sub-class of nonlinear systems that we are interested in. Next, we will show (14) and (15) are not required if SP for the linearized system is replaced by a stronger condition SSIP.

IV. Main Results

In this section, we establish conditions under which a linear system is SSIP and demonstrate a SSIP linearized system implies SSIP of the nonlinear system (3) and (2) within a neighborhood of $(\hat{x} = 0, \hat{u} = 0)$.

A. SSIP Linear Systems

Algebraic conditions are established as follows based on Lemma 1 (resp. Lemma 2) for a linear system (4) (resp. (5)) to be *simultaneously* SP and SIP (SSIP).

Lemma 5: If (4) is strict passive and $D + D^T > 0$, then the system is SSIP.

Proof: From Lemma 1, we know there exists a storage function $V(x) = \frac{1}{2}x^T Px$ for (4) such that for $\varepsilon > 0$ [5],

$$\dot{V} - u^T y = -\frac{1}{2}(Lx + Wu)^T (Lx + Wu) - \frac{1}{2}\varepsilon x^T Px.$$

The following relation holds for b such that $0 < b^2 < 1$,

$$\begin{aligned}
&\dot{V} - u^T y \\
=& -\frac{1}{2}(\frac{1}{b}Lx + bWu)^T(\frac{1}{b}Lx + bWu) \\
&- \frac{1}{2}x^T\left(\varepsilon P - (\frac{1}{b^2} - 1)L^T L\right)x - \frac{1}{2}(1 - b^2)u^T W^T Wu \\
\leq& -\frac{1}{2}(1 - b^2)u^T W^T Wu - \frac{1}{2}x^T\left(\varepsilon P - (\frac{1}{b^2} - 1)L^T L\right)x \\
\triangleq& -\frac{1}{2}u^T Q_1 u - \frac{1}{2}x^T Q_2 x,
\end{aligned}$$

where $Q_1 = (1 - b^2)W^T W, Q_2 = \varepsilon P - \left(\frac{1}{b^2} - 1\right) L^T L$.

From $W^T W = D + D^T > 0$ and $0 < b^2 < 1$, we obtain $Q_1 > 0$. Next, we prove $Q_2 > 0$ or equivalently

$$\underline{\lambda}(P)\varepsilon - \left(\frac{1}{b^2} - 1\right)\overline{\lambda}(L^T L) > 0.$$

If $\overline{\lambda}(L^T L) > 0$, choose b to satisfy

$$0 < \frac{\overline{\lambda}(L^T L)}{\varepsilon\underline{\lambda}(P) + \overline{\lambda}(L^T L)} < b^2 < 1, \tag{17}$$

then we have $Q_2 > 0$. It is obvious that $Q_2 > 0$ if $\overline{\lambda}(L^T L) = 0$. As a result, $Q_2 > 0$ for appropriate choice of b. Thus, $Q_1 > 0, Q_2 > 0$ and there exist constants $\varepsilon_1 > 0, \varepsilon_2 > 0$ (in fact $\underline{\lambda}(Q_1)$, $\underline{\lambda}(Q_2)$ respectively), such that

$$\dot{V} - u^T y \leq -\varepsilon_1 u^T u - \varepsilon_2 x^T x.$$

Therefore, the linear system (4) is SSIP. ∎

Remark 6: To ensure SSIP, it is required that the linear system (4) is SP and $D + D^T > 0$, this is nothing but the definition for *strongly positive real* system [11], [20] or *extended strictly positive real* system [12], [3].

The concept of strongly positive realness does *not* apply to (5) in the discrete-time domain [11]. In fact, SP for (5) implies SIP. To show SSIP, we need the following result.

Lemma 6: If (5) is SP with a storage function $V(x) = \frac{1}{2}x^T Px$ and $D + D^T - B^T PB > 0$, then it is SSIP.

Proof: From Lemma 2, there exists a storage function $V(x) = \frac{1}{2}x^T Px$ such that for $\forall k$ and some $\eta^2 > 1$,

$$\begin{aligned}
&\Xi(x,u) \triangleq V(x(k+1)) - V(x(k)) - u(k)y(k) \\
&= -\frac{1}{2}(\eta Lx + \frac{1}{\eta}Wu)^T(\eta Lx + \frac{1}{\eta}Wu) \\
&\quad - \frac{1}{2}(1 - \frac{1}{\eta^2})u^T W^T Wu - \frac{1}{2}x^T \left[\rho P - (\eta^2 - 1)L^T L\right] x \\
&\leq -\frac{1}{2}(1 - \frac{1}{\eta^2})u^T W^T Wu - \frac{1}{2}x^T \left[\rho P - \left(\eta^2 - 1\right) L^T L\right] x.
\end{aligned}$$

Define $Q_3 = (1 - \frac{1}{\eta^2})W^T W, Q_4 = \rho P - \left(\eta^2 - 1\right) L^T L$. In the following, we prove $Q_3 > 0, Q_4 > 0$. First, $Q_3 > 0$ because $W^T W = D + D^T - B^T PB > 0$ and $\eta^2 > 1$. Next, $Q_4 > 0$ is equivalent to

$$\rho\underline{\lambda}(P) - (\eta^2 - 1)\overline{\lambda}(L^T L) > 0.$$

It is obvious that $Q_4 > 0$ if $\overline{\lambda}(L^T L) = 0$. If $\overline{\lambda}(L^T L) > 0$, choose η such that

$$1 < \eta^2 < \frac{\overline{\lambda}(L^T L) + \rho\underline{\lambda}(P)}{\overline{\lambda}(L^T L)}, \tag{18}$$

then $Q_4 > 0$ for η that satisfies (18). As a result, $Q_3 > 0, Q_4 > 0$ for appropriate choice of η. Thus there exist $\varepsilon_3 > 0, \varepsilon_4 > 0$ (in fact $\underline{\lambda}(Q_3)$, $\underline{\lambda}(Q_4)$ respectively), such that

$$\Xi(x,u) \leq -\varepsilon_3 u^T u - \varepsilon_4 x^T x.$$

Therefore, the system (5) is SSIP. ■

Remark 7: Note that $D + D^T - B^T PB > 0$ for some $P > 0$ is only sufficient to show SSIP. If $D + D^T - B^T PB = 0$ for some P, what we can do is to use another P.

B. SSIP: from Linearity to Nonlinearity

Next, we are going to show SSIP of a linearized system implies *local* SSIP for the nonlinear system, in both continuous-time domain *and* discrete-time domain.

Theorem 4 (CT-SSIP): Consider a nonlinear system Σ_c given by (3) and its linearized system $\mathcal{G}_c$ given by (12). If $\mathcal{G}_c$ is SP and $D + D^T > 0$, then Σ_c is locally SSIP.

Proof: From Lemma 5, there exists a storage function $V = \frac{1}{2}x^T Px$ for (4), such that for $\varepsilon_1 > 0, \varepsilon_2 > 0$,

$$u^T y - \dot{V} \geq \varepsilon_1 x^T x + \varepsilon_2 u^T u.$$

Apply $V(x)$ as a locally defined storage function for (3) and we obtain from (13) that

$$\begin{aligned}
&u^T(h(x) + J(x)u) - \frac{\partial V}{\partial x}(f(x) + g(x)u) \\
&= u^T(Cx + Du) + u^T(H(x) + M(x)u) \\
&\quad - x^T P(Ax + Bu) - x^T P(F(x) + G(x)u) \\
&\geq \left(\varepsilon_1 x^T x - \|u^T H(x)\| - \|x^T PF(x)\| - \|x^T PG(x)u\|\right) \\
&+ \left(\varepsilon_2 u^T u - \|u^T M(x)u\|\right).
\end{aligned}$$

From Taylor's theorem, we obtain for (x,u) close enough to $(0,0)$ and some constants $\alpha_1,\alpha_2,\alpha_3>0$,

$$\begin{aligned}&\frac{1}{2}\varepsilon_1 x^T x-\|u^T H(x)\|-\|x^T PF(x)\|-\|x^T PG(x)u\|\\ \geq & x^T x(\frac{1}{2}\varepsilon_1-\alpha_1\|u\|-\alpha_2\|x\|)\geq 0,\\ &\frac{1}{2}\varepsilon_2 u^T u-\|u^T M(x)u\|\geq u^T u(\frac{1}{2}\varepsilon_2-\alpha_3\|x\|)\geq 0,\end{aligned}$$

and therefore for the nonlinear system (3)

$$u^T y-\dot{V}\geq \frac{1}{2}\varepsilon_1 x^T x+\frac{1}{2}\varepsilon_2 u^T u,$$

which implies local SSIP for (3). ■

Theorem 5 (DT-SSIP): Consider a nonlinear system Σ_d given by (2) and its linearized system $\mathcal{G}_d$ given by (12). If $\mathcal{G}_d$ is SP with a storage function $V(x)=\frac{1}{2}x^T Px$ and $D+D^T-B^T PB>0$, then Σ_d is locally SSIP.

Proof: From Lemma 6, there exists a storage function $V(x)=\frac{1}{2}x^T Px$ for (5) such that for $\varepsilon_3>0,\varepsilon_4>0$,

$$\Xi(x,u)\leq -\varepsilon_3 x^T x-\varepsilon_4 u^T u.$$

Apply $V(x)=\frac{1}{2}x^T Px$ for the nonlinear system Σ_d, we obtain from (13) that for $\forall k$,

$$\begin{aligned}&V(k+1)-V(k)-u(k)^T y(k)=\Xi(x,u)+\phi(x,u)\\ &\leq -\varepsilon_3 x^T x-\varepsilon_4 u^T u+\|\phi(x,u)\|,\end{aligned}$$

where $\phi(x,u)$ is given by (16). Rearrange the terms in $\phi(x,u)$, when x and u close to the origin, we obtain

$$\begin{aligned}&V(k+1)-V(k)-u(k)^T y(k)\\ \leq &-\frac{1}{2}\varepsilon_3 x^T x-\frac{1}{2}\varepsilon_4 u^T u\\ &-x^T x(\frac{1}{2}\varepsilon_3-\beta_1\|x\|-\beta_2\|x\|^2-\beta_3\|u\|-\beta_4\|u\|\|x\|)\\ &-u^T u(\frac{1}{2}\varepsilon_4-\beta_5\|x\|-\beta_6\|x\|^2)\\ \leq &-\frac{1}{2}\varepsilon_3 x^T x-\frac{1}{2}\varepsilon_4 u^T u,\end{aligned}$$

where $\beta_i>0, i=1,2,\ldots 6$. Thus, Σ_d is locally SSIP. ■

To illustrate these results, let us consider two numerical examples, one in continuous-time domain and the other in discrete-time domain.

Example 4: Consider the following nonlinear system

$$\begin{aligned}\dot{x_1}&=-x_1^2+x_2,\\ \dot{x_2}&=-x_1-x_2+(ax_1+1)u,\\ y&=x_1+2x_2+(bx_2+1)u,\end{aligned}$$

where $a\neq 0, b\neq 0$. Its linearized system is given by (4) together with

$$A=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}, B=\begin{bmatrix}0\\ 1\end{bmatrix}, C=\begin{bmatrix}1 & 2\end{bmatrix}, D=1.$$

This linear system is SSIP with storage function $V(x) = x_1^2 + x_1x_2 + x_2^2$, because

$$\begin{aligned}\dot{V} - u^T y &= -(x_1^2 + x_1x_2 + x_2^2) - u^2 \\ &\leq -\frac{1}{2}(x_1^2 + x_2^2) - u^2.\end{aligned}$$

Apply $V(x)$ as a locally defined storage function for the nonlinear system and we obtain

$$\begin{aligned}\dot{V} - u^T y = &-(x_1^2 + x_1x_2 + x_2^2) - u^2 \\ &- 2x_1^3 - x_1^2x_2 + ax_1^2u + 2ax_1x_2u - bx_2u^2 \\ \leq &-\frac{1}{2}(x_1^2 + x_2^2) - u^2(1 - |bx_2|) \\ &- x_1^2(2x_1 + x_2 - au + |au|) + |au|x_2^2 \\ \leq &-x_1^2(\frac{1}{2} - |2x_1| - |x_2|) - u^2(1 - |bx_2|) \\ &- x_2^2(\frac{1}{2} - |au|).\end{aligned}$$

In a neighborhood of $x = 0$ and $u = 0$, where $|u| < \frac{1}{3|a|}, |x_1| < \frac{1}{8}, |x_2| < \min\{\frac{1}{12}, \frac{1}{2|b|}\}$, we have

$$\dot{V} - u^T y \leq -\frac{1}{6}(x_1^2 + x_2^2) - \frac{1}{2}u^2,$$

thus the nonlinear system is locally SSIP.

Example 5: Consider a discrete time nonlinear system

$$\begin{aligned}x_1(k+1) &= \frac{\alpha x_2(k)}{1 + x_1^2(k)} + (1 + x_2(k))u(k), \\ x_2(k+1) &= \frac{\beta x_1(k)}{1 + x_2^2(k)}, \\ y(k) &= \alpha x_2(k) + (\sin(x_1) + 1)u(k),\end{aligned}$$

where for $\alpha^2 + \beta^2 < 1$ and $\alpha, \beta \neq 0$, its linearized system is given by (5) together with

$$A = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & \alpha \end{bmatrix}, D = 1.$$

This linear system is SSIP with a storage function $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ because

$$\begin{aligned}&V(k+1) - V(k) - u^T(k)y(k) \\ =&\frac{1}{2}(\alpha^2 - 1)x_2^2(k) + \frac{1}{2}(\beta^2 - 1)x_1^2(k) - \frac{1}{2}u^2(k) \\ \leq& -\zeta(x_1^2(k) + x_2^2(k)) - \frac{1}{2}u^2(k),\end{aligned}$$

where $\zeta \triangleq \frac{1}{2}\min\{1 - \beta^2, 1 - \alpha^2\} > 0$.

Apply V as a locally defined storage function for the nonlinear system and we obtain

$$\begin{aligned}&V(k+1)-V(k)-u^T(k)y(k)\\ \leq&-\zeta(x_1^2(k)+x_2^2(k))-\frac{1}{2}u^2(k)\\ &+\left(\frac{1}{2}x_2^2(k)+x_2(k)-\sin(x_1(k))\right)u^2(k)\\ &+x_2^2(k)|\alpha u(k)|+\frac{1}{2}(x_1^4(k)+x_2^4(k))\\ \leq&-\left(\zeta-\frac{1}{2}x_1^2(k)\right)x_1^2(k)-(\zeta-|\alpha u(k)|)\,x_2^2(k)\\ &-\left(\frac{1}{2}-\frac{1}{2}x_2^2(k)-|x_2(k)|-|\sin(x_1(k))|\right)u^2(k)\\ \leq&-c_1(x_1^2(k)+x_2^2(k))-c_2u^2(k),\end{aligned}$$

for (x,u) close enough to $(0,0)$ and some $c_1,c_2>0$. Therefore the nonlinear system is locally SSIP.

C. QSR Dissipativity: from Linearity to Nonlinearity

The results in the previous sections focus on passivity based on positive-real lemmas. In this section, we extend to QSR dissipative systems based on Lemma 3 or 4.

Analogously, we can establish conditions under which a strict dissipative linear system is SSIP. In the continuous-time domain, for (4), the condition is given by

$$R+S^TD+D^TS+D^TQD>0, \tag{19}$$

and in the discrete-time domain, for (5), it is required that

$$R+S^TD+D^TS+D^TQD-B^TPB>0. \tag{20}$$

The proof is quite similar to Lemma 5 or 6 by "completing the square" and manipulating the coefficients of the quadratic terms in x or u. If the linearized system (4) or (5) is shown to be SSIP, it can be derived that the nonlinear system (3) or (2) will be locally QSR-dissipative as follows.

Theorem 6 (CT-QSR): Consider a nonlinear system Σ_c given by (3) and its linearized system $\mathcal{G}_c$ given by (12). Suppose $\mathcal{G}_c$ is controllable and observable. If $\mathcal{G}_c$ is strict QSR-dissipative and (19) is satisfied, then Σ_c is locally strictly QSR-dissipative with the same supply rate.

Proof: The linearized system is strict dissipative w.r.t (11), from Lemma 3, there exists a storage function $V(x)=x^TPx$ for $\mathcal{G}_c$ such that for some $\varepsilon>0$,

$$\begin{aligned}&\dot{V}-r(u,y)\\ =&2x^TP(Ax+Bu)-(u^TRu+2y^TSu+y^TQy)\\ =&-(Lx+Wu)^T(Lx+Wu)-\varepsilon x^TPx\\ \leq&-\varepsilon x^TPx.\end{aligned}$$

By assuming $R+S^TD+D^TS+D^TQD>0$, we obtain $W^TW>0$. Therefore, $\mathcal{G}_c$ is SSIP via the techniques in the proof of Lemma 5 and thus for some $\kappa_1>0,\kappa_2>0$,

$$\dot{V}-r(u,y)\leq-\kappa_1x^Tx-\kappa_2u^Tu.$$

Apply $V(x)=x^TPx$ as a locally defined storage function for the nonlinear system Σ_c. Denote $\Lambda(x,u)$ as the function of x and u by substituting (3) into $\dot{V}-r(u,y)$ and we obtain

$$\Lambda(x,u)\leq-\kappa_1x^Tx-\kappa_2u^Tu+\Theta(x,u),$$

and $\Theta(x,u)$ given as follows contains the higher-order (order≥ 2) terms in either x or u.

$$\begin{aligned}\Theta(x,u) &= 2x^T P(F(x)+G(x)u) - 2u^T S(H(x)+M(x)u)\\ &- (H(x)+M(x)u)^T Q(H(x)+M(x)u)\\ &- 2(H(x)+M(x)u)^T Q(Cx+Du).\end{aligned}$$

From Taylor's theorem, we can derive that for $d_1 \geq 0, d_2 \geq 0$ (and at least one $d_i > 0$), there exist a ball around $x=0$ and a ball around $u=0$, for which

$$\|\Theta(x,u)\| \leq d_1\|x\|^2\|u\| + d_2\|u\|^2\|x\|.$$

Thus, when (x,u) close to the origin $(0,0)$, we obtain

$$\Lambda(x,u) \leq -\frac{1}{2}\kappa_1 x^T x - \frac{1}{2}\kappa_2 u^T u,$$

Therefore, Σ_c is locally strict dissipative w.r.t the same supply rate given by matrices Q,S,R. ■

Remark 8: The problem of studying linearization of a QSR-dissipative systems has been studied in [7] (p. 211-213). One of the sufficient conditions is the same as (19) for $D \neq 0$ and $R > 0$ for $D = 0$. The result in [7] relies on solvability of Halmiltonian-Jacobi inequalities, however, our result depends on the algebraic conditions for strict dissipativity of a linear system given in [13].

The arguments can be developed for Σ_d in the discrete-time domain as well.

Theorem 7 (DT-QSR): Consider a nonlinear system Σ_d given by (2) and its linearized system $\mathcal{G}_d$ given by (12). Suppose $\mathcal{G}_d$ is controllable and observable. If $\mathcal{G}_d$ is strict QSR-dissipative and (20) is satisfied, then Σ_d is locally strictly QSR-dissipative with the same supply rate.

A direct application of these two theorems is for a particular quadratic supply rate by setting $S = \frac{1}{2}(1+\rho\sigma)I, R = -\sigma I, Q = -\rho I$. In this case, (11) is reduced to

$$\varpi(u,y) = (1+\rho\sigma)u^T y - \sigma u^T u - \rho y^T y, \tag{21}$$

where σ is called the input feed-forward passivity index (denoted by IFP(σ)) and ρ is called the output feedback passivity index (denoted by OFP(ρ)). If Σ_c or Σ_d is dissipative w.r.t $\varpi(u,y)$, it is said to have IFP(σ) and OFP(ρ). The two passivity indices characterize the level of passivity for a given dynamical system and can be used in control designs and system stability analysis [25], [15].

The following result is immediate from Theorem 1.

Corollary 2: Suppose Σ_c given by (3) (resp. Σ_d given by (2)) has IFP(σ) and OFP(ρ), then its linearized system (12) has IFP(σ) and OFP(ρ) as well.

To determine the passivity indices for a nonlinear system from its linearized system, we can use Theorem 6 and 7 by replacing (19) with

$$-\sigma I + \frac{1}{2}(1+\rho\sigma)(D+D^T) - \rho D^T D > 0,$$

in continuous-time domain and (20) with

$$-\sigma I + \frac{1}{2}(1+\rho\sigma)(D+D^T) - \rho D^T D - B^T P B > 0,$$

in discrete-time domain.

V. Discussions and Extensions

In this section, we first compare passivity analysis and stability analysis based on linearization from a Lyaponuov approach point of view. Then, we extend the results in Section IV to nonlinear systems of a more general form.

A. *Passivity & Stability: a Lyapunov Approach*

From the analysis in Section IV, it can been seen there is a strong connection between stability and passivity analysis by using a Lyapunov approach as shown in the following.

1) If a nonlinear system is stable, then its linearized system is stable. This corresponds to a passive nonlinear system has a passive linearized system.
2) If the linearized system is marginally stable, we cannot say whether the nonlinear system is locally stable or not. This corresponds to the case that a lossless linearized system does not tell us whether the nonlinear system is passive or not.
3) If the linearized system is asymptotically stable, then the nonlinear system is locally stable. This corresponds to the case that a SSIP linearized system implies local passivity of the nonlinear system.
4) If a linearized system is unstable, then the nonlinear system is unstable from Chetayev's instability theorem. If the linearized system is not passive, the nonlinear system is not (globally) passive, however, for local passivity, we do not have theoretic guarantee in general.

B. *Passivity: Extension to General Nonlinear Systems*

The results in Section IV can be extended to a general nonlinear system which may not be affine in control input. The results claim that if the linearized system is SSIP, the nonlinear system will be locally SSIP.

Consider a continuous-time nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x,u), \end{aligned} \tag{22}$$

or a discrete-time nonlinear system of the form

$$\begin{aligned} x(k+1) &= f(x(k),u(k)), \\ y(k) &= h(x(k),u(k)), \end{aligned} \tag{23}$$

with $f(0,0)=0$ and $h(0,0)=0$ without loss of generality. The linearization is given through (4) or (5) with

$$A=\frac{\partial f}{\partial x}(0,0), B=\frac{\partial f}{\partial u}(0,0), C=\frac{\partial h}{\partial x}(0,0), D=\frac{\partial h}{\partial u}(0,0). \tag{24}$$

It follows by using Taylor series expansion that f and h in (22) or (23) can be rewritten as

$$\begin{aligned} f(x,u) &= Ax+Bu+F(x,u), \\ h(x,u) &= Cx+Du+H(x,u), \end{aligned}$$

where $F(x,u)$ and $H(x,u)$ contains higher order terms of $f(x,u)$ and $h(x,u)$, respectively. It can be verified that the terms contained in $F(x,u)$ and $H(x,u)$ are at least linear in either x or u, for instance $x^2, xu, u^2, x^3, x^2u, u^2x, u^3, \cdots$. The following result is immediate.

Proposition 1: Denote σ_u (resp. σ_x) as the smallest order for u (resp. x) contained in a polynomial of x and u. Either $\sigma_x \geq 2$ or $\sigma_u \geq 2$ for $x^T F(x,u)$ and $u^T H(x,u)$.

Theorem 8: Consider a nonlinear system given by (22) and its linearized system given by (24). Suppose the linearized system is controllable and observable. If the linearization (24) is strict passive and $D+D^T>0$, then the nonlinear system (22) is locally SSIP.

Proof: From Lemma 5, we know the linearized model is SSIP with a storage function $V(x) = \frac{1}{2}x^TPx$, such that for some $\upsilon_1 > 0$ and $\upsilon_2 > 0$, $u^Ty - \dot{V} \geq \upsilon_1 x^Tx + \upsilon_2 u^Tu$. Apply $V(x)$ as locally defined storage function for the nonlinear system (22), we obtain

$$\begin{aligned}
&u^Th(x,u) - \frac{\partial V}{\partial x}f(x,u) \\
=&u^T(Cx+Du) + u^TH(x,u) \\
&- x^TP(Ax+Bu) - x^TPF(x,u) \\
\geq&\upsilon_1 x^Tx + \upsilon_2 u^Tu - u^TH(x,u) - x^TPF(x,u).
\end{aligned}$$

From Proposition 1, $x^TPF(x,u) + u^TH(x,u)$ contains higher order terms either in x ($\sigma_x \geq 2$) or in u ($\sigma_u \geq 2$). Denote the terms with $\sigma_x \geq 2, \sigma_u \leq 1$ by $\Gamma_1(x,u)$ and the terms with $\sigma_u \geq 2$ by $\Gamma_2(x,u)$. Thus,

$$\begin{aligned}
&u^Th(x,u) - \frac{\partial V}{\partial x}f(x,u) \\
\geq&\upsilon_1 x^Tx - \Gamma_1(x,u) + \upsilon_2 u^Tu - \Gamma_2(x,u).
\end{aligned}$$

According to Taylor's theorem, there exist a ball around $x = 0$ and a ball around $u = 0$ and constants $c_i \geq 0$ for $i = 1, 2, 3, 4$ (and at least one $c_i > 0$), for which

$$\begin{aligned}
\|\Gamma_1(x,u)\| \leq \|x\|^2(c_1\|u\| + c_2\|x\|), \\
\|\Gamma_2(x,u)\| \leq \|u\|^2(c_3\|u\| + c_4\|x\|).
\end{aligned}$$

Thus, the following relation holds when (x,u) close to $(0,0)$,

$$\begin{aligned}
&u^Th(x,u) - \frac{\partial V}{\partial x}f(x,u) \\
\geq&\|x\|^2(\upsilon_1 - c_1\|u\| - c_2\|x\|) + \|u\|^2(\upsilon_2 - c_3\|u\| - c_4\|x\|) \\
\geq&\frac{1}{2}\upsilon_1\|x\|^2 + \frac{1}{2}\upsilon_2\|u\|^2.
\end{aligned}$$

Therefore, the nonlinear system (22) is locally SSIP. ∎

In discrete-time domain, we have the following result based on Lemma 6 and similar arguments for (22).

Theorem 9: Consider a nonlinear system given by (23) and its linearized system given by (24). Suppose the linearized system is controllable and observable. If the linearization (24) is strict passive with a storage function $V(x) = \frac{1}{2}x^TPx$ and $D + D^T - B^TPB > 0$, then the nonlinear system (23) is locally SSIP.

Remark 9: These results are not surprising since the nonlinear system affine in control Σ_c (resp. Σ_d) contains the lower-order terms for the general nonlinear system (22) (resp. (23)). In this sense, Σ_c (resp. Σ_d) can be viewed as an *approximation* for a general nonlinear system (22) (resp. (23)) when talking about *local* properties for the system. Clearly, the the results for strict dissipative linearizations can also be extended to general nonlinear systems as well.

VI. Final Remarks

In this paper, we study a class of nonlinear system whose local passivity is implied from its linearization. For such systems, the storage function of its linearized system works as a locally defined storage function of the nonlinear system. This storage function can be used to show passivity of the nonlinear system within a neighborhood of the equilibrium. Our main results show that for a linearized system which is simultaneously strict passive and strict input passive (SSIP), the nonlinear system will hold the same property locally. Algebraic conditions are established based on positive-real lemmas under which a linear system is shown to be SSIP. The property of SSIP is analogous to asymptotic stability in stability analysis using an indirect Lyaponuv approach. We also investigate linearization of QSR-dissipative systems (more general than passivity) and relate passivity indices for a nonlinear system and its linearization.

A storage function for a linear system can be obtained by solving algebraic equations and further used as a locally defined storage function for the nonlinear system. In this way, we can guarantee local passivity of a complicated nonlinear system for which the storage function may not be easily found. This is one benefit of this work. On the other hand, the neighborhood for local passivity may be sufficiently small indicated from the linearized model. Moreover, some intricacies of the nonlinear system may be neglected by linearization. However, this information may play a dominate role, such as the coupling in power systems. In this case, other approximation techniques may be useful such as model reduction for nonlinear systems that preserves passivity [26].

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