Passivity Analysis of a System and its Approximation

M. Xia, P. J. Antsaklis and V. Gupta

***Abstract*— We consider the following problem: given two mathematical system models, one of which could represent accurately a physical system and the other an approximation of the system, what passivity properties of the system can be inferred from studying only the approximate model. Our results show that an excess of passivity (whether in the form of input strictly passive, output strictly passive or very strictly passive) in the approximate model guarantees a certain passivity index for the system, provided that the norm of the error between the two models is sufficiently small in a suitably defined sense. Further, we consider QSR dissipative systems and show that QSR dissipativity has a similar robustness property, even though the supply rates for the system and its approximation may be different.**

I. Introduction

Energy dissipation is a fundamental concept in dynamical systems. Passivity and dissipativity characterize the "energy" consumption of a dynamical system and form a powerful tool in many real applications. Passivity is closely related to stability and exhibits a compositional property for parallel and feedback interconnections [1], [2], [3]. Passivity-based control is especially useful in the analysis of complex coupled systems.

In this paper, we are particularly interested in the passivity of a system as inferred from studying an approximate model of its dynamical behavior. In physical systems, precise knowledge of the mathematical model is difficult to obtain. Moreover, even if such a model were obtainable, the classical tradeoff between model accuracy and tractability may lead to the use of a simpler model [4]. A variety of approximation methods can be used, for analysis, simulation or control design of the 'real' systems [5]. While it is known that under some conditions, linearization [1], [6] and model reduction [7], [8] can be performed so as to preserve passivity, the question of whether passivity of a system can be guaranteed if a model 'close' to it is passive still remains open.

The main contribution of this paper is the establishment of relationships between *passivity levels* of two mathematical system models, one of which could represent accurately a physical system and the other could represent an approximation. Of course, the two mathematical models can represent two different approximations of the same physical system as well. The approximate model is assumed to have an *excess* of passivity, defined as passivity levels (similar to passivity indices [3]) that characterize how passive it is (how much of the energy introduced into the system is dissipated). If the error between the system and its approximation is "small" in a suitably defined sense, we show that the passivity levels for the system can be guaranteed. Since there is a rich theory of using passivity levels (or indices) to design controls [3], [9], our results imply that it is possible to use the (hopefully more tractable) approximate model for control design. An alternative interpretation of the results is as robustness properties with respect to model uncertainties [10], [11]. Further, if the approximate model does not have an excess of passivity, we assume that it is QSR dissipative and similar robustness properties can be derived. We apply our results to a particular approximation method: model reduction of a higher-order system to obtain a lower-order model which can be used to facilitate control designs or speed up simulations [5], [7]. Specifically, we consider linear systems and their positive-real truncations [8] and derive variations in the passivity levels for the full-order and reduced-order systems. The works such as [5], [7], [8] focus on how to preserve passivity. As opposed to these works, we show how passivity levels vary as a function of the order.

The rest of the paper is organized as follows. Section II provides background material on passivity and model reduction preserving passivity. Section III presents the problem statement. The main results are given in Section IV. Numerical examples are provided in Section V. Concluding remarks are given in Section VI.

Notation. The signal space under consideration is either the standard $\mathcal{L}_2$ space or the extended $\mathcal{L}_2$ space. The exact space will be clear from the context. The Euclidean space of dimension m is denoted by $\mathbb{R}^m$. Denote the truncation of $u(t)$ up to time T $(0 \leq T < \infty)$ by $u_T(t)$. The inner product of truncated signals $u_T(t), y_T(t)$ is denoted by $\langle u, y\rangle_T$, where $\langle u, y\rangle_T \triangleq \int_0^T u^T(t)y(t)\mathrm{d}t$ and $u^T(t)$ denotes the transpose of $u(t)$. The $\mathcal{L}_2$-induced norm of the signal $u_T(t)$ is denoted by $\|u_T(t)\|_{\mathcal{L}_2}$, where $\|u_T(t)\|^2_{\mathcal{L}_2} \triangleq \int_0^T u^T(t)u(t)\mathrm{d}t$. The H_∞ norm of a transfer function $G(s)$ is denoted by $\|G\|_{H_\infty}$. For a matrix $A \in \mathbb{R}^{n\times n}$, the minimum eigenvalue of A is denoted by $\underline{\lambda}(A)$ and the maximum eigenvalue by $\overline{\lambda}(A)$. Re$[A]$ is the real part of a complex matrix A. $A \geq 0$ denotes that A is positive semi-definite and $A > 0$ implies that A is positive definite. The n-dimensional identity matrix is denoted by $I_{n\times n}$ or simply I by omitting the dimensions if clear from the context. The notation $\max\{a, b\}$ denotes the larger value of $a, b \in \mathbb{R}$ and $\min\{a, b\}$ denotes the smaller value of $a, b \in \mathbb{R}$.

II. Preliminaries

Definition 1 ([1], [12]): Consider a system Σ with input u and output y where $u(t), y(t) \in \mathbb{R}^m$. It is said to be

The authors are with the department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA. (e-mail: mxia, antsaklis.1, vgupta2 @nd.edu). Research supported by the National Science Foundation under Grant No. CNS-1035655.

- *passive*, if there exists a constant $\beta \leq 0$ such that
$$\langle u, y\rangle_T \geq \beta.$$
- *input strictly passive* (ISP), if there exist constants $\nu > 0$ and $\beta \leq 0$ such that
$$\langle u, y\rangle_T \geq \beta + \nu\langle u, u\rangle_T. \tag{1}$$
- *output strictly passive* (OSP), if there exist constants $\rho > 0$ and $\beta \leq 0$ such that
$$\langle u, y\rangle_T \geq \beta + \rho\langle y, y\rangle_T. \tag{2}$$
- *very strictly passive* (VSP), if there exist constants $\rho > 0$, $\nu > 0$ and $\beta \leq 0$ such that
$$\langle u, y\rangle_T \geq \beta + \rho\langle y, y\rangle_T + \nu\langle u, u\rangle_T. \tag{3}$$
- *finite-gain $\mathcal{L}_2$ stable*, if there exist constants $\kappa > 0$ and $\beta \leq 0$ such that
$$\langle y, y\rangle_T \leq -\beta + \kappa^2\langle u, u\rangle_T. \tag{4}$$

In all cases, the inequality should hold for $\forall u(t)$, $\forall T \geq 0$ and the corresponding $y(t)$.

The constant β is related to the initial condition of the system Σ and plays an important role in the stability analysis of Σ [12]. The inner product $\langle u, y\rangle_T$ may be interpreted as the externally supplied energy to Σ during the interval $[0, T]$ [1], [13]. The above definitions can be viewed as special cases of QSR-dissipative systems [2], [14], defined as systems for which there exist $Q = Q^T, R = R^T$ and S, such that $\forall u(t)$, $\forall T \geq 0$ and the corresponding $y(t)$,
$$r(u, y) \triangleq \langle y, Qy\rangle_T + 2\langle y, Su\rangle_T + \langle u, Ru\rangle_T \geq 0. \tag{5}$$
The function $r(u, y)$ is called the *supply rate* for Σ.

If a system Σ is ISP for $\nu > 0$, it is also ISP for $\nu - \epsilon$, where $0 \leq \epsilon < \nu$. Analogously, if Σ is OSP for $\rho > 0$, it is also OSP for $\rho - \epsilon$, where $0 \leq \epsilon < \rho$ [3]. Finally, if Σ is VSP for (ρ, ν), i.e. (3) holds, it is also VSP for $(\rho-\epsilon, \nu-\epsilon)$, where $0 \leq \epsilon < \min\{\rho, \nu\}$ (see [15] for a complete proof). A positive value of ρ or ν can thus be interpreted as an *excess* of passivity and these two values (called *passivity levels*) characterize 'how passive' Σ is. If ρ or ν is negative, we say Σ has a *shortage* of passivity. This intuition is captured by the concept of passivity indices [3].

Definition 2: For a system Σ with input u and output y where $u(t), y(t) \in \mathbb{R}^m$,

- the *input feedforward passivity index* (IFP) is the largest $\nu > 0$ such that (1) holds for $\forall u$ and $\forall T \geq 0$,
- the *output feedback passivity index* (OFP) is the largest $\rho > 0$ such that (2) holds for $\forall u$ and $\forall T \geq 0$.

The two indices are denoted by IFP(ν) and OFP(ρ), respectively.

Note the fact that a system has IFP(ν) and OFP(ρ) does not necessarily imply that the system is VSP for (ρ, ν). In other words, (3) may not hold for the passivity indices ρ and ν. A necessary condition for ρ and ν to be VSP is given by $\rho\nu \leq \frac{1}{4}, \rho > 0, \nu > 0$ (see [15] for a complete proof). As a result, for VSP, it may not make sense to define the largest $\rho > 0$ and the largest $\nu > 0$ (simultaneously) such that (3) holds for $\forall u$ and $\forall T \geq 0$. We thus use the notion of passivity levels. Consider the system Σ,

- any $\tilde{\nu} \in (0, \nu]$ is a *passivity level* of Σ if Σ has IFP(ν);
- any $\tilde{\rho} \in (0, \rho]$ is a *passivity level* of Σ if Σ has OFP(ρ);
- any $(\tilde{\rho}, \tilde{\nu})$ are *passivity levels* of Σ if Σ is VSP for (ρ, ν) such that $0 < \tilde{\rho} \leq \rho, 0 < \tilde{\nu} \leq \nu$.

Consider a linear time-invariant system with transfer function $G(s)$, a *minimal* state-space realization is given by
$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{6}$$
where $\{A, B\}$ is controllable and $\{A, C\}$ is observable. The following result is useful to test whether system (6) is passive.

Lemma 1 ([13]): System (6) is passive if and only if there exist matrices $P = P^T > 0$, L and W, such that
$$\begin{aligned} PA + A^TP &= -L^TL, \\ PB &= C^T - L^TW, \\ W^TW &= D + D^T. \end{aligned} \tag{7}$$

A special case of system (6) is of relaxation type, i.e.
$$A = A^T, A \leq 0, B^T = C, D \geq 0. \tag{8}$$
Relaxed systems play an important role in applications and examples of such systems include integrated circuits and mechanical systems where inertial effects may be neglected, see e.g. [8], [14].

The algorithm for model reduction considered in this paper preserves passivity, called *positive-real truncated balancing realization* (PR-TBR for short), as presented in [8]. The observability grammian W_o and the controllability grammian W_c of system (6) can be used as a basis for PR-TBR procedure when (8) is satisfied. The square roots of the eigenvalues of the product W_cW_o are called Hankel singular values and can be used to establish upper bounds on the difference between the transfer function of the full-order system G_1 and its reduced order approximation G_2. If we denote σ_i as the ith Hankel singular value (where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ and n is the order of G_1), then
$$\|G_1 - G_2\|_{H_\infty} \leq 2\sum_{i=r+1}^{n} \sigma_i, \tag{9}$$
where r is the order of the reduced order model G_2 [5], [7].

Remark 1: There exist various methods for model reduction (of linear or nonlinear systems), but we do not concentrate on that problem. Linear models of relaxation type and model reduction preserving passivity are used merely as illustrating examples of our main results.

III. Problem Statement

Consider two system models Σ_1 and Σ_2 as shown in Fig. 1. One can view Σ_i as the system we are interested in and Σ_j as an *approximation* of Σ_i, where $i, j \in \{1, 2\}$ and $j \neq i$. A commonly used measure for judging how well Σ_j approximates Σ_i is to compare the outputs for the same

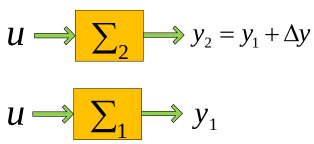

Fig. 1. Illustration of two systems: Σ_1 with input u and output y_1 and Σ_2 with input u and output $y_2 = y_1 + \Delta y$, where u, y_1, y_2 and Δy are of the same dimensions.

excitation function u [5]. We denote the difference in the outputs by Δy. Note that in general Δy will depend on the exact function u. The error may be due to modeling, linearization or model reduction or a host of other reasons. For a 'good' approximation, we require that the "worst" case Δy over all control inputs u be small. More formally, we say that Σ_j is a *good* approximation of Σ_i if there exists a positive constant $\gamma > 0$ such that

$$\langle \Delta y, \Delta y\rangle_T \leq \gamma^2 \langle u, u\rangle_T, \quad \forall u \text{ and } \forall T \geq 0. \tag{10}$$

The value of γ obviously reflects how good the approximation is. In the following analysis, without loss of generality, we view Σ_2 as an approximation of Σ_1.

Remark 2: Note that (10) actually requires the 'error system' with input u and output Δy to be $\mathcal{L}_2$ stable. For stable linear systems with zero initial conditions, Σ_1 (resp. Σ_2) is characterized by the transfer function G_1 (resp. G_2). Defining $\Delta G = G_1 - G_2$, if $\|\Delta G\|_{H_\infty} \leq \gamma$, (10) is satisfied. Thus, γ is an upper bound on the H_∞ norm of the difference in the transfer functions G_1 and G_2. In particular, if G_2 is a reduced order model of G_1 that obtained through the PR-TBR procedure, from (9), we obtain that the 'error' constraint γ in (10) can be calculated as $2\sum_{i=r+1}^{n}\sigma_i$.

We are now ready to state the problem of interest. Assume that Σ_2 has an *excess* of passivity, namely Σ_2 has IFP(ν) or OFP(ρ) or is VSP for (ρ, ν). What passivity property for Σ_1 can be inferred from that of Σ_2? For the case when Σ_2 does not have an excess of passivity, we assume it to be (Q_2, S_2, R_2)-dissipative and consider the problem of obtaining conditions under which Σ_1 is (Q_1, S_1, R_1)-dissipative as well. The problem is summarized as follows.

Problem 1: Suppose that an approximate model Σ_2

1) has IFP(ν); or
2) has OFP(ρ); or
3) is VSP for (ρ, ν); or
4) is (Q_2, S_2, R_2)-dissipative.

What corresponding passivity or QSR-dissipativity properties can be derived for the system Σ_1 based on (10)?

IV. Main Results

We begin by considering the case when the approximate model is ISP and then move on to the cases when the approximation is OSP, VSP or QSR-dissipative, subsequently.

A. Input Strictly Passive Systems

We have the following result that guarantees a certain passivity level given the error constraint γ and IFP of the approximate model.

Theorem 1: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (10) is satisfied for some $\gamma > 0$. If Σ_2 has IFP(ν) and $\gamma < \nu$, then Σ_1 is ISP for $\tilde{\nu} = \nu - \gamma$.

Proof: From Cauchy-Schwarz inequality and the assumption (10), we obtain

$$|\langle u, \Delta y\rangle_T| \leq \sqrt{\langle u, u\rangle_T}\sqrt{\langle \Delta y, \Delta y\rangle_T} \leq \gamma\langle u, u\rangle_T, \tag{11}$$

For the system Σ_2 with input u and output y_2, we have

$$\begin{aligned}
&\langle u, y_2\rangle_T - \nu\langle u, u\rangle_T \\
=&\langle u, y_1\rangle_T - \nu\langle u, u\rangle_T + \langle u, \Delta y\rangle_T \\
\leq&\langle u, y_1\rangle_T - \nu\langle u, u\rangle_T + |\langle u, \Delta y\rangle_T| \\
\leq&\langle u, y_1\rangle_T - (\nu - \gamma)\langle u, u\rangle_T.
\end{aligned}$$

Now, by assumption, Σ_2 is ISP for $\nu > 0$, then

$$\langle u, y_2\rangle_T - \nu\langle u, u\rangle_T \geq \beta.$$

Therefore, defining $\tilde{\nu} = \nu - \gamma > 0$, we obtain $\langle u, y_1\rangle_T - \tilde{\nu}\langle u, u\rangle_T \geq \beta$. This implies Σ_1 is ISP for $\tilde{\nu} > 0$. ■

Note $\tilde{\nu}$ does *not* represent the IFP of Σ_1 (Σ_1 may have IFP larger than $\tilde{\nu}$). If we are merely interested in determining whether Σ_1 is passive (rather than characterizing the passivity level of Σ_1), we can allow γ to be equal to ν.

Corollary 1: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (10) is satisfied for some $\gamma > 0$. If Σ_2 has IFP(ν) and $\gamma \leq \nu$, then, Σ_1 is passive.

Proof: From (11) and $\gamma \leq \nu$, we obtain

$$|\langle u, \Delta y\rangle_T| \leq \gamma\langle u, u\rangle_T \leq \nu\langle u, u\rangle_T.$$

The following relation holds for Σ_1

$$\begin{aligned}
\langle u, y_1\rangle_T &= \langle u, y_2\rangle_T - \langle u, \Delta y\rangle_T \\
&\geq \langle u, y_2\rangle_T - |\langle u, \Delta y\rangle_T| \\
&\geq \langle u, y_2\rangle_T - \nu\langle u, u\rangle_T \geq \beta.
\end{aligned}$$

Therefore, $\langle u, y_1\rangle_T \geq \beta$, i.e. Σ_1 is passive. ■

B. Output Strictly Passive Systems

For OSP systems, we assume along the lines of [3] that the inverse of Σ_2 is causal and $\mathcal{L}_2$ stable. For linear system (6), a necessary condition to satisfy this assumption is that $G(s)$ has relative degree zero and is minimum phase, i.e. all the zeros of $G(s)$ have negative real parts. In this case, the OFP for $G(s)$ is shown to be equivalent to the IFP of the inverse of $G(s)$, see e.g. [3].

Assumption 1: Consider Σ_2 with input u and output y_2. Assume the inverse of Σ_2 is causal and stable, i.e. there exist $\eta > 0$, such that $\forall y_2$, $\forall T \geq 0$

$$\langle u, u\rangle_T \leq \eta^2\langle y_2, y_2\rangle_T. \tag{12}$$

With this assumption, we have the following result.

Theorem 2: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (10) holds for some $\gamma > 0$ and (12) holds for some $\eta > 0$. If Σ_2 has OFP(ρ) and $\gamma < \rho$, then Σ_1 is OSP for $\tilde{\rho} = \rho - \gamma$ if

$$\frac{1}{\eta^2} - \left(1 + 2(\rho - \gamma)\frac{1}{\rho} + (\rho - \gamma)\gamma\right) \geq 0. \tag{13}$$

Proof: For all $\rho > 0$, we have the following relation $u^T y_2 \le \frac{1}{2\rho} u^T u + \frac{\rho}{2} y_2^T y_2$ and thus

$$u^T y_2 - \rho y_2^T y_2 \le \frac{1}{2\rho} u^T u - \frac{\rho}{2} y_2^T y_2.$$

Σ_2 is assumed to be OSP for $\rho > 0$, thus

$$\frac{1}{2\rho}\langle u,u\rangle_T - \frac{\rho}{2}\langle y_2,y_2\rangle_T \ge \langle u,y_2\rangle_T - \rho\langle y_2,y_2\rangle_T \ge \beta,$$

and therefore $\langle y_2,y_2\rangle_T \le \frac{1}{\rho^2}\langle u,u\rangle_T - \frac{2\beta}{\rho}$. From Cauchy-Schwarz inequality, (10) and the fact $\beta \le 0$, we obtain

$$\begin{aligned} |\langle y_2,\Delta y\rangle_T| &\le \sqrt{\langle \Delta y,\Delta y\rangle_T}\sqrt{\langle y_2,y_2\rangle_T} \qquad (14)\\ &\le \frac{\gamma}{\rho}\sqrt{\langle u,u\rangle_T}\sqrt{\langle u,u\rangle_T - 2\beta\rho} \\ &\le \frac{\gamma}{\rho}\left(\langle u,u\rangle_T - 2\beta\rho\right) = \frac{\gamma}{\rho}\langle u,u\rangle_T - 2\beta\gamma. \end{aligned}$$

Together with (11), if we define $a \triangleq \rho - \gamma > 0$, we obtain

$$\begin{aligned} \Phi \triangleq & \gamma\langle y_2,y_2\rangle_T - \langle u,\Delta y\rangle_T + 2a\langle \Delta y,y_2\rangle_T - a\langle \Delta y,\Delta y\rangle_T \\ \ge & \gamma\langle y_2,y_2\rangle_T - |\langle u,\Delta y\rangle_T| - 2a|\langle \Delta y,y_2\rangle_T| - a\gamma^2\langle u,u\rangle_T \\ \ge & \gamma\langle y_2,y_2\rangle_T - \left(\gamma + 2a\frac{\gamma}{\rho} + a\gamma^2\right)\langle u,u\rangle_T + 4a\beta\gamma. \end{aligned}$$

If (13) is satisfied, from assumption (12), we obtain

$$\begin{aligned} &\gamma\langle y_2,y_2\rangle_T - \left(\gamma + 2a\frac{\gamma}{\rho} + a\gamma^2\right)\langle u,u\rangle_T \\ &\ge \left[\frac{1}{\eta^2} - \left(1 + 2a\frac{1}{\rho} + a\gamma\right)\right]\gamma\eta^2\langle y_2,y_2\rangle_T \ge 0. \end{aligned}$$

Thus, $\Phi \ge 4a\beta\gamma$. For Σ_1 with $y_1 = y_2 - \Delta y$,

$$\begin{aligned} &\langle u,y_1\rangle_T - (\rho-\gamma)\langle y_1,y_1\rangle_T \\ =&\langle u,y_2\rangle_T - \rho\langle y_2,y_2\rangle_T + \Phi \ge \beta + 4a\beta\gamma \triangleq \bar{\beta}, \end{aligned}$$

for all functions u, all $T \ge 0$ and $\bar{\beta} \le 0$. Therefore, for $\gamma < \rho$, Σ_1 is OSP for $\tilde{\rho} = \rho - \gamma$. ■

Note that Σ_1 may have OFP larger than $\tilde{\rho}$. If we are merely interested in passivity of Σ_1, we have the following result.

Corollary 2: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (10) holds for some $\gamma > 0$ and (12) holds for some $\eta > 0$. If Σ_2 has OFP(ρ) and $\gamma\eta^2 \le \rho$, then, Σ_1 is passive.

Proof: From (11) and the assumption (12), we obtain

$$|\langle u,\Delta y\rangle_T| \le \gamma\langle u,u\rangle_T \le \gamma\eta^2\langle y_2,y_2\rangle_T.$$

Thus, the following relation holds if $\gamma\eta^2 \le \rho$,

$$\begin{aligned} \langle u,y_1\rangle_T &= \langle u,y_2\rangle_T - \langle u,\Delta y\rangle_T \\ &\ge \langle u,y_2\rangle_T - \rho\langle y_2,y_2\rangle_T - |\langle u,\Delta y\rangle_T| + \rho\langle y_2,y_2\rangle_T \\ &\ge \beta + (\rho - \gamma\eta^2)\langle y_2,y_2\rangle_T \ge \beta. \end{aligned}$$

Therefore, $\langle u,y_1\rangle_T \ge \beta$, i.e. Σ_1 is passive. ■

C. Very Strictly Passive Systems

We have the following result.

Theorem 3: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (10) holds for some $\gamma > 0$. Suppose Σ_2 is VSP for (ρ,ν), where $\rho > \gamma, \nu > \gamma$. Then, Σ_1 is VSP for $(\rho-\gamma,\nu-\gamma)$ if

$$\nu^2 - \frac{2(\rho-\gamma)}{\rho} - (\rho-\gamma)\gamma \ge 0. \qquad (15)$$

Proof: We use the relation $u^T y_2 - \nu u^T u \le \frac{1}{2\nu} y_2^T y_2 - \frac{\nu}{2} u^T u$. Σ_2 is assumed to be ISP for $\nu > 0$, thus

$$\frac{1}{2\nu}\langle y_2,y_2\rangle_T - \frac{\nu}{2}\langle u,u\rangle_T \ge \langle u,y_2\rangle_T - \nu\langle u,u\rangle_T \ge \beta,$$

and therefore $\langle y_2,y_2\rangle_T \ge \nu^2\langle u,u\rangle_T + 2\beta\nu$. Also, Σ_2 is OSP for $\rho > 0$, thus (14) is satisfied. Together with (10) and (11), if we define $a = \rho - \gamma > 0, \psi = 2a\langle y_2,\Delta y\rangle_T - \langle u,\Delta y\rangle_T - a\langle \Delta y,\Delta y\rangle_T$, we obtain

$$\begin{aligned} |\psi| &\le |\langle u,\Delta y\rangle_T| + 2a|\langle y_2,\Delta y\rangle_T| + a\langle \Delta y,\Delta y\rangle_T \\ &\le \left(\gamma + 2a\frac{\gamma}{\rho} + a\gamma^2\right)\langle u,u\rangle_T - 4a\beta\gamma. \end{aligned}$$

Thus, the following relation holds

$$\begin{aligned} &\gamma\langle u,u\rangle_T + \gamma\langle y_2,y_2\rangle_T + \psi \\ \ge&\gamma(1+\nu^2)\langle u,u\rangle_T + 2\beta\nu\gamma - |\psi| \\ \ge&\left[\gamma(1+\nu^2) - (\gamma + 2a\frac{\gamma}{\rho} + a\gamma^2)\right]\langle u,u\rangle_T + 2\beta\nu\gamma + 4a\beta\gamma \\ =&\gamma\left(\nu^2 - \frac{2a}{\rho} - a\gamma\right)\langle u,u\rangle_T + 2\beta\nu\gamma + 4a\beta\gamma. \end{aligned}$$

We assume that $\nu^2 - \frac{2a}{\rho} - a\gamma \ge 0$ from (15), thus

$$\gamma\langle u,u\rangle_T + \gamma\langle y_2,y_2\rangle_T + \psi \ge 2\beta\nu\gamma + 4a\beta\gamma.$$

For Σ_1 with input u and output $y_1 = y_2 - \Delta y$, we have

$$\begin{aligned} &\langle u,y_1\rangle_T - (\nu-\gamma)\langle u,u\rangle_T - (\rho-\gamma)\langle y_1,y_1\rangle_T \\ =&\langle u,y_2\rangle_T - \nu\langle u,u\rangle_T - \rho\langle y_2,y_2\rangle_T \\ &+ \gamma\langle u,u\rangle_T + \gamma\langle y_2,y_2\rangle_T + \psi \\ \ge&\langle u,y_2\rangle_T - \nu\langle u,u\rangle_T - \rho\langle y_2,y_2\rangle_T + 2\beta\nu\gamma + 4a\beta\gamma. \end{aligned}$$

Σ_2 is assumed to be VSP for (ρ,ν) and therefore

$$\langle u,y_2\rangle_T - \nu\langle u,u\rangle_T - \rho\langle y_2,y_2\rangle_T \ge \beta.$$

Defining $\bar{\beta} = \beta + 2\beta\nu\gamma + 4a\beta\gamma \le 0$, we have

$$\langle u,y_1\rangle_T - (\nu-\gamma)\langle u,u\rangle_T - (\rho-\gamma)\langle y_1,y_1\rangle_T \ge \bar{\beta}.$$

Thus, for $\gamma < \rho, \gamma < \nu$, Σ_1 is VSP for $(\rho-\gamma,\nu-\gamma)$. ■

Σ_1 is VSP for (ρ,ν) implies that ρ is a passivity level for OSP and ν is a passivity level for ISP. The OFP of Σ_1 is larger than ρ and the IFP is larger than ν in general.

Corollary 3: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (10) holds for some $\gamma > 0$. If Σ_2 is VSP for (ρ,ν) and $\rho\nu^2 + \nu - \gamma \ge 0$, then, Σ_1 is passive.

Proof: Σ_2 is ISP for ν, it has been shown that $\langle y_2,y_2\rangle_T \ge \nu^2\langle u,u\rangle_T + 2\beta\nu$. From (11), we obtain

$$\begin{aligned} \chi \triangleq& -|\langle u,\Delta y\rangle_T| + \rho\langle y_2,y_2\rangle_T + \nu\langle u,u\rangle_T \\ &\ge (\rho\nu^2 + \nu - \gamma)\langle u,u\rangle_T + 2\beta\rho\nu. \end{aligned}$$

Thus, if $\rho\nu^2 + \nu - \gamma \geq 0$, we obtain $\chi \geq 2\beta\rho\nu$. Σ_2 is VSP for (ρ, ν), thus $\langle u, y_2\rangle_T - \rho\langle y_2, y_2\rangle_T - \nu\langle u, u\rangle_T \geq \beta$. For Σ_1 with input u and output y_1, we have

$$\begin{aligned}\langle u, y_1\rangle_T &= \langle u, y_2\rangle_T - \langle u, \Delta y\rangle_T \\ &\geq \langle u, y_2\rangle_T - \rho\langle y_2, y_2\rangle_T - \nu\langle u, u\rangle_T + \chi \\ &\geq \beta + 2\beta\rho\nu \triangleq \bar{\beta}.\end{aligned}$$

Thus, $\langle u, y_1\rangle_T \geq \bar{\beta}$ and $\bar{\beta} \leq 0$, i.e. Σ_1 is passive. ■

Remark 3: It can be verified that the above results hold when Σ_1 and Σ_2 exchange places. In other words, it does not really matter whether we view Σ_1 as an approximation of Σ_2 or Σ_2 as an approximation of Σ_1. In practice, however, a simple model is usually used as an approximation of a complex system, e.g. linearized model vs. nonlinear model and lower-order model vs. higher-order model.

Remark 4: Theorem 1-3 relate passivity levels between Σ_1 and Σ_2 for ISP, OSP and VSP systems. It is worth stressing that these results are applicable to *any* approximation methods and *any* system structure in general. In particular, if we consider linear systems and PR-TBR as a particular approximation approach, then the results in Theorem 1-3 provide a tool to trade off the 'error' constraint γ in (10) as a function of *variations in the passivity levels* for the full-order system Σ_1 (or Σ_2) and the reduced-order system Σ_2 (or Σ_1) (see [15] for more details).

D. Extension to QSR-dissipative Systems

In this section, we extend the results to QSR-dissipative systems, for which the system may be not passive or have a shortage of passivity.

Theorem 4: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (10) holds for some $\gamma > 0$. Let Σ_2 be (Q_2, S_2, R_2)-dissipative and assume $S_1 - S_2 = 0, Q_1 - Q_2 > 0, R_1 - R_2 > 0$. If there exists a constant $\xi > 0$ such that

$$\begin{aligned}\underline{\lambda}(R_1 - R_2) - \frac{\gamma^2}{\xi} - 2\lambda_1\gamma - b \geq 0,\\ \underline{\lambda}(Q_1 - Q_2) - \xi\lambda_2 \geq 0,\end{aligned} \tag{16}$$

where $b = 2\max\{0, \overline{\lambda}(-Q_1)\gamma^2\}$, and

$$\lambda_1 \triangleq \sqrt{\overline{\lambda}(S_1^T S_1)} \geq 0, \lambda_2 \triangleq \overline{\lambda}(Q_1^T Q_1) \geq 0,$$

then Σ_1 is (Q_1, S_1, R_1)-dissipative.

Proof: From Cauchy-Schwarz inequality, we obtain

$$|\langle S_1 u, \Delta y\rangle_T| \leq \sqrt{\overline{\lambda}(S_1^T S_1)}\gamma\langle u, u\rangle_T = \lambda_1\gamma\langle u, u\rangle_T.$$

Also, for some $\xi > 0$, the following relation holds

$$2\langle Q_1 y_2, \Delta y\rangle_T \leq \frac{\gamma^2}{\xi}\langle u, u\rangle_T + \xi\lambda_2\langle y_2, y_2\rangle_T.$$

Define the supply rate for Σ_i as $r_i(u, y_i) \triangleq \langle y_i, Q_i y_i\rangle_T + 2\langle y_i, S_i u\rangle_T + \langle u, R_i u\rangle_T$, then

$$\begin{aligned}r_1 =& r_2 + \langle y_2, (Q_1 - Q_2)y_2\rangle_T + \langle u, (R_1 - R_2)u\rangle_T \\ &- 2\langle y_2, Q_1\Delta y\rangle_T + \langle \Delta y, Q_1 \Delta y\rangle_T - 2\langle \Delta y, S_1 u\rangle_T \\ \geq& r_2 + \left(\underline{\lambda}(Q_1 - Q_2) - \xi\lambda_2\right)\langle y_2, y_2\rangle_T + \langle \Delta y, Q_1\Delta y\rangle_T \\ &+ \left(\underline{\lambda}(R_1 - R_2) - \frac{\gamma^2}{\xi} - 2\lambda_1\gamma\right)\langle u, u\rangle_T.\end{aligned}$$

Since Σ_2 is (Q_2, S_2, R_2)-dissipative, $r_2 \geq 0$. Two cases are possible. If $Q_1 > 0$, we have $b = 0, \langle \Delta y, Q_1 \Delta y\rangle_T \geq 0$. Thus, from (16), we obtain $r_1 \geq r_2 \geq 0$. If $Q_1 \leq 0$, we have $b = \overline{\lambda}(-Q_1)\gamma^2$ and from (10),

$$\langle \Delta y, Q_1 \Delta y\rangle_T \geq -\overline{\lambda}(-Q_1)\gamma^2\langle u, u\rangle_T.$$

If (16) holds, we obtain $r_1 \geq r_2 \geq 0$. In summary, $r_1 \geq 0$ if (16) is satisfied and thus Σ_1 is (Q_1, S_1, R_1)-dissipative. ■

Remark 5: Similar arguments can be developed when $S_1 - S_2 = 0, Q_1 - Q_2 > 0, R_1 - R_2 > 0$ does not hold. However, the analysis is more involved.

V. Numerical Examples

In this section, we provide a few numerical examples to illustrate the theoretical results developed in this paper. In general, Σ_1 represents the system we are interested in and and Σ_2 is an approximation of Σ_1. We assume zero initial conditions. For linear systems, the behavior of system Σ_i is determined by the corresponding transfer function G_i. The approximation method we consider here is the PR-TBR procedure for model reduction (see e.g. [8]) and the system models are of relaxation type.

Example 1: The original system G_1 given by (17) is of order 8. The Hankel singular values are given by Λ in (18) and ordered as $\sigma_1 \geq \cdots \geq \sigma_8$. Its second-order approximation is given by

$$G_2 = \frac{0.5s^2 + 21.96s + 47.85}{s^2 + 5.8s + 4.456},$$

and the IFP(ν) for G_2 (defined in [3]) can be computed as

$$\nu = \min_{w \in \mathbb{R}} \mathrm{Re}[G_2(jw)] = 0.5.$$

The error in the transfer functions is given as [8]

$$\|G_1 - G_2\|_{H_\infty} \leq 2\sum_{k=3}^{8}\sigma_k = 0.0803.$$

Thus, $\gamma = 0.0803 < 0.5$. According to Theorem 1, G_1 is ISP for $\tilde{\rho} = \rho - \gamma = 0.5 - 0.0803 = 0.4197$. This is true because the IFP for G_1 is actually 0.5, larger than $\tilde{\rho} = 0.4197$.

Example 2: The original system G_1 is given by

$$\frac{1.8s^5 + 53.56s^4 + 590.8s^3 + 3034s^2 + 7279s + 6543}{s^5 + 23s^4 + 203.1s^3 + 861.7s^2 + 1759s + 1382}.$$

Its first-order approximation is given by

$$G_2 = \frac{1.8s + 19.37}{s + 4.132},$$

$$G_1 = \frac{0.5s^8 + 28.6s^7 + 352.2s^6 + 1887s^5 + 5299s^4 + 8295s^3 + 7190s^2 + 3173s + 542.9}{s^8 + 18.5s^7 + 133.5s^6 + 496.1s^5 + 1047s^4 + 1290s^3 + 911.1s^2 + 337.5s + 50.18} \tag{17}$$

$$\Lambda = \text{diag}\{4.6357, 0.4834, 0.0375, 0.0023, 3.5\times 10^{-4}, 1.9\times 10^{-5}, 0, 0\}. \tag{18}$$

$$A = \begin{pmatrix} -5 & 0.1 & 1.2 & 0 & 0 & 1 \\ 0.1 & -3 & 0 & -0.3 & 0 & -1 \\ 1.2 & 0 & -6 & -2 & 0.5 & -2 \\ 0 & -0.3 & -2 & -4 & 0.4 & 0.5 \\ 0 & 0 & 0.5 & 0.4 & -4 & -0.8 \\ 1 & -1 & -2 & 0.5 & -0.8 & -8 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 2 \\ 0.8 \end{pmatrix}, C = B^T, D = 2. \tag{19}$$

whose inverse is causal and stable. We can compute

$$\eta = \|(G_2)^{-1}\|_{H_\infty} = 0.5556, \rho = 0.213.$$

The error in the transfer function G_1 and G_2 is given by $\gamma = 0.0461$. Thus, $\gamma < \rho$ and (13) holds because

$$\frac{1}{\eta^2} - \left(1 + 2(\rho-\gamma)\frac{1}{\rho} + (\rho-\gamma)\gamma\right) = 0.6695 > 0.$$

From Theorem 2, we can conclude that G_1 is OSP for $\tilde{\rho} = \rho - \gamma = 0.213 - 0.0461 = 0.1669$. This is true because the OFP for G_1 is given by 0.211, larger than $\tilde{\rho} = 0.1669$.

Example 3: The original system G_1 is given by (19). Its second-order approximation is given by

$$G_2 = \frac{2s^2 + 42.06s + 183.8}{s^2 + 11.22s + 26.79},$$

which is VSP for (ρ, ν), where $\nu = 1.2, \rho = 0.01$. This can be verified through $\Pi \le 0$ [16], where Π is given by

$$\begin{bmatrix} A^TP + PA + \rho C^TC & PB - (1/2C^T - \rho C^TD) \\ B^TP - (1/2C - \rho D^TC) & \nu I + \rho D^TD - D \end{bmatrix},$$

with A, B, C, D as a minimal realization of G_2 and $P = I$.

The error in G_1 and G_2 is given by $\gamma = 0.0042$. For our choice of ρ, ν, we obtain

$$\nu^2 - \frac{2(\rho-\gamma)}{\rho} - (\rho-\gamma)\gamma = 0.2869 > 0,$$

therefore (15) is satisfied. According to Theorem 3, the original system G_1 is VSP for $(\tilde{\rho}, \tilde{\nu})$, where

$$\tilde{\nu} = \nu - \gamma = 1.1958, \tilde{\rho} = \rho - \gamma = 0.0058.$$

This can also be verified through $\Pi \le 0$ by setting $P = I$ and substituting $\tilde{\rho}, \tilde{\nu}$ for ρ, ν, respectively.

Remark 6: Note that a higher-order reduced model will result in smaller difference in the transfer functions or the passivity levels. To verify Theorem 4, a simple example is presented in [15] (not in this paper due to space limitations).

VI. Concluding Remarks

In this paper, we established conditions under which the passivity properties of a system can be obtained by analyzing its approximation. The approximate model is assumed to be input/output/very strictly passive and the general result states that if the error between the system and its approximation is small, the original system has a guaranteed passivity level as well. The analysis is extended to a general case when the approximation is QSR dissipative (not necessarily passive). The results may be interpreted as robustness properties of passivity with respect to model uncertainties. Our results can be used to derive variations in the passivity levels of a linear system and its reduced-order approximation.

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