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Order of the compensator and the closed-loop eigenvalues in the fractional approach to design

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The order of the stabilizing controller and the closed-loop eigenvalues in LTI control systems designed using the fractional approach are studied. New insight is provided and guidelines are given to attain tighter control over the number of the closed-loop eigenvalues and the compensator order, thus avoiding unnecessarily high order controllers and computationally expensive calculations. The study is carried out using the results of Antsaklis (1986) to derive the closed-loop internal descriptions directly in terms of the proper, stable factorizations of the plant and the controller transfer matrices.

1. Introduction

In the original parametrization of all stabilizing controllers $u = -Cy$ of a plant $y = Pu$ via a stable rational K (Youla *et al.* 1976), the closed-loop eigenvalues are exactly the poles of K; the order ∂C of the resulting C is $\partial K - \partial P$. A difficulty in this approach is, of course, guaranteeing properness for C. In the parametrization of all proper stabilizing controllers C, using proper and stable factorizations, in terms of a proper and stable parameter K' (Desoer *et al.* 1980, Vidyasagar 1985), the closed-loop eigenvalues consist in general, as shown here, of some of the poles of K'; the rest are some of the stable poles of quantities which are usually chosen rather arbitrarily. As a result, if for example stabilization of an unstable plant is attempted using this approach, the number of the closed-loop eigenvalues ($= \partial P + \partial C$) and consequently the order of the stabilizing controller C tend to be larger than anticipated, since it is not clear how to control them tightly.

The fractional approach to the design of linear time-invariant control systems has gained acceptance and a rather large following in recent years. For greater understanding and more efficient designs, it is necessary to study the number and location of the closed-loop eigenvalues and the order of the stabilizing controllers C when the fractional approach is used.

This study is based on recent results presented by Antsaklis (1986), where the relation between proper, stable factorizations and internal descriptions of a plant were established. Here, the internal description of the compensated system is first derived, when $K' = 0$, and the closed-loop eigenvalues are determined directly in terms of the quantities X', Y', N' and D' of the diophantine equation (2). These results are then applied to the case when $K' \neq 0$. The order of the controller C is determined. It is also shown how K' varies for the same P and C depending on the choice of X', Y', etc. This is useful in view of the fact that in H^∞ optimization problems, K' is found first and then the appropriate controller C is determined; and with well-chosen initial factorizations of P and C the solution K' to the optimization problem is simpler.

$$y = Pu; \quad P = N'D'^{-1} \tag{1}$$

where N', D' are proper and stable rational matrices denoted here as $(N', D') \in M(S)$, that is matrices with elements in S, the set of all proper and stable rational functions; y and u are the output and input vectors respectively. Let (N', D') be right coprime (rc) in S; that is, there exists $(X', Y') \in M(S)$ such that the diophantine equation (or Bezout identity)

$$X'D' + Y'N' = I \tag{2}$$

is satisfied. Note that D'^{-1} is also proper, that is D' is biproper.

Consider now the controller

$$u = -Cy; \quad C = \bar{D}_c'^{-1}\bar{N}_c' \tag{3}$$

where C is proper and $\bar{D}_c'$, $\bar{N}_c' \in M(S)$. It is known (Desoer *et al.* 1980), that such a controller internally stabilizes the plant (1) if and only if

$$\bar{D}_c'D' + \bar{N}_c'N' = U' \tag{4}$$

where U', $U'^{-1} \in M(S)$. It can easily be shown that the solutions (X', Y') of the diophantine equation (2) (perhaps with U' in the right-hand side instead of I) which generate proper stabilizing controllers $C = X'^{-1}Y'$ are those for which X' is biproper. If P is strictly proper, then $X' = D'^{-1} - Y'P$ will always be biproper; if P is not strictly proper, however, care should be exercised when solving the diophantine to ensure that X' will be biproper. In the following, P is taken to be proper and the solutions to the diophantine (X', Y') have X' biproper.

Given $P = N'D'^{-1}$ rc, suppose the diophantine (2) is solved to derive a stabilizing controller $C = X'^{-1}Y'$. Although the result described above guarantees internal stability, neither the exact locations nor the number of the closed-loop (cl) eigenvalues or the order of C is directly apparent from (X', Y'). In other words, clear guidelines do not appear to exist as to how to select solutions of the diophantine if there is interest in the order of C and the cl eigenvalues.

To illustrate, consider the following scalar example (Antsaklis 1986).

Example 1

$$P = (s-1)/(s-2)(s+1)$$

and

$$X'D' + Y'N' = \frac{(s-5)(s+3)}{(s+1)(s+2)}\,\frac{(s-2)(s+2)}{(s+1)(s+3)} + \frac{9(s+3)}{(s+2)}\,\frac{(s-1)(s+2)}{(s+1)^2(s+3)} = 1$$

Here $C = X'^{-1}Y' = 9(2+1)/(s-5)$ and the clcp $= (s+1)^3$, as can be verified from $\alpha_p\alpha_c|I + PC|$ where α_p, α_c are the characteristic polynomials of P and C respectively. Note that the cl eigenvalues are in general some of the poles of X', Y', D' and N', which are not unique given P, but not all of them. This difficulty in determining the cl eigenvalues in terms of the quantities the designer selects, becomes greater when all stabilizing controllers are parametrically characterized in terms of K' via (Desoer *et* *al.* 1980, Youla *et al.* 1976, Antsaklis 1979):

$$C = [X' - K'\bar{N}']^{-1}[Y' + K'\bar{D}'] \tag{5}$$

where $K' \in M(S)$ with $(X' - K'\bar{N}')$ biproper; K' can be chosen to satisfy design requirements in addition to internal stability (e.g. H^∞ optimization) if necessary, and

$$U'U'^{-1} = \begin{bmatrix} X' & Y' \\ -\bar{N}' & \bar{D}' \end{bmatrix} \begin{bmatrix} D' & -\bar{Y}' \\ N' & \bar{X}' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{6}$$

As it will be shown below, the cl eigenvalues here are some of the poles of not only X', Y', N', D', but also of $\bar{N}'$, $\bar{D}'$ and K'.

It can easily be seen that the order of the controller C is immediately known from the number of cl eigenvalues via:

$$\partial C = \{\text{no. of cl eigenvalues}\} - \partial P \tag{7}$$

where ∂C is the order of C; this is true as it is assumed that P and C are represented (and implemented) via minimal realizations. So initially, we will concentrate on the cl eigenvalues, but we will also provide expressions for the order of C.

3. Internal descriptions and closed-loop eigenvalues

Let $P = N'D'^{-1}$ (1) and suppose a stabilizing controller $C = X'^{-1}Y'$ is found by solving

$$X'D' + Y'N' = I \tag{2}$$

It is of interest to study the internal descriptions of P, C and of the closed-loop system in terms of their fractional representations.

For this, let

$$\begin{bmatrix} D' \\ N' \end{bmatrix} = \begin{bmatrix} D \\ N \end{bmatrix} \Pi \tag{8}$$

where D, N are rc polynomial matrices ($P = ND^{-1}$) and Π is a rational matrix. Theorem 1 of Antsaklis (1986) states that $(N', D') \in M(S)$ define a rc factorization (1) of P in S if and only if (8) is true where Π, Π^{-1} are stable and $D\Pi$ biproper. Write

$$\begin{bmatrix} D' \\ N' \end{bmatrix} = \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} \hat{D}_1^{-1} = \begin{bmatrix} D \\ N \end{bmatrix} \hat{N}_1 \hat{D}_1^{-1} \tag{9}$$

where $[D_1^{\mathrm{T}} \quad N_1^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{D}_1$ are rc polynomial matrices and $\hat{N}_1$ is a greatest common right divisor of D_1, N_1; then D, N are rc and $\Pi = \hat{N}_1\hat{D}_1^{-1}$. Note that $D_1 z_1 = u$, $y = N_1 z_1$ is a controllable and detectable realization of P, while $Dz = u$, $y = Nz$ is a minimal realization of P.

Similarly

$$[X' \quad Y'] = \Pi_c[\bar{D}_c \quad \bar{N}_c] \tag{10}$$

where $\bar{D}_c$, $\bar{N}_c$ are lc polynomial matrices and Π_c, Π_c^{-1} stable with $\Pi_c\bar{D}_c$ biproper a polynomial matrix. The closed-loop eigenvalues are the zeros of $|\bar{D}_k|$ … the poles of $\Pi\Pi_c$. Note that in general, some of the poles of Π will cancel with zeros of Π_c and some of the zeros of Π with poles of Π_c; the remaining poles in $\Pi\Pi_c$ are the closed-loop eigenvalues.

In Example 1 above, $\Pi = (s+2)/(s+1)^2(s+3)$ and $\Pi_c = (s+3)/(s+1)(s+2)$ in view of (9) and (10); $(\Pi\Pi_c)^{-1} = \bar{D}_k = (s+1)^3$ as expected. Notice that $(s+2)$ and $(s+3)$ cancel out. If D' and X' had been chosen so that their orders were the orders of P and C respectively, then neither Π nor Π_c would have any zeros (Antsaklis 1986). In this case, no cancellations would take place in $\Pi\Pi_c$ and the closed-loop eigenvalues would be all the poles of Π (of $[D'^{\mathrm{T}} \quad N'^{\mathrm{T}}]^{\mathrm{T}}$) and all the poles of Π_c (of $[X' \quad Y']$) as Example 2 illustrates. Note that this is also the case when minimal state space realizations of P are used to determine X', Y', N', D' via state feedback and observer theory (see Doyle 1984, Antsaklis 1986).

Example 2 (Antsaklis 1986)

$P = (s-1)/(s-2)(s+1)$ as in Example 1, and

$$X'D' + Y'N' = \frac{s-5}{s+1}\frac{s-2}{s+1} + 9\frac{s-1}{(s+1)^2} = 1$$

Here $\Pi = 1/(s+1)^2$, $\Pi_c = 1/(s+1)$ and $(\Pi\Pi_c)^{-1} = \bar{D}_k = (s+1)^3$.

The following results can now be formally stated.

Let $P = N'D'^{-1}$ and $C = X'^{-1}Y'$ satisfy (2). Determine Π from (8) and write

$$\Pi[X' \quad Y'] = \bar{D}_k^{-1}[\bar{D}_c \quad \bar{N}_c] \tag{13}$$

an lc polynomial factorization.

Theorem

$Dz = u$, $y = Nz$ and $\bar{D}_c\bar{z}_c = -\bar{N}_c y$, $u = \bar{z}_c$ are minimal internal realizations of P and C respectively. $\bar{D}_k z = 0$, $y = Nz$ is an internal description of the closed-loop system.

Proof

$\Pi[X' \quad Y'] = \Pi\Pi_c[\bar{D}_c \quad \bar{N}_c]$ with $\bar{D}_c, \bar{N}_c$ lc in view of (10). Then (12) gives the desired result since $\bar{D}_k$, $[\bar{D}_c \quad \bar{N}_c]$ are lc. □

Note that the theorem can also be shown directly using Theorem 3 of Antsaklis and Sain (1984).

Corollary 1

The closed-loop eigenvalues are:

(*a*) the poles of $\Pi\Pi_c$, or

(*b*) the poles of $\Pi[X' \quad Y']$ or $\begin{bmatrix} D' \\ N' \end{bmatrix}\Pi_c$; or

(*c*) the poles of $\begin{bmatrix} D' \\ N' \end{bmatrix}[X' \quad Y']$.

Proof

In view of (12) and the Theorem, and by using the fact that premultiplication by rc $[D^T \quad N^T]^T$ does not change the poles. □

Remarks

(i) If instead of (2), $X'D' + Y'N' = U'(U', U'^{-1} \in M(S))$ is solved, all results are valid if instead of $[X' \quad Y']$, $U'^{-1}[X' \quad Y']$ is used; for example, the poles of $\Pi U'^{-1}\Pi_c$ are now the cl eigenvalues.

(ii) It can be shown that

$$\begin{bmatrix} I & C \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} D' \\ N' \end{bmatrix} [X', -Y'] + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \tag{14}$$

In view now of the fact that the poles of the left-hand side are the cl eigenvalues (Antsaklis and Sain 1984, Collier and Desoer 1982), an alternative proof of (*c*) in Corollary 1 is derived.

(iii) In view of (7), $\partial C = \partial(\Pi\Pi_c) - \partial P$.

4. Controller parameterizations

The above results are now used to determine the closed-loop internal descriptions, the eigenvalues and the order of the controller, when a stabilizing C, expressed as a function of the parameter K' is used.

Let $P = N'D'^{-1} = D'^{-1}N'$ be a 'doubly coprime' factorization satisfying (6). In view of (5), all proper stabilizing controllers are given by $C = \bar{D}_c'^{-1}\bar{N}_c$ where

$$[\bar{D}_c' \quad \bar{N}_c'] = [I \quad K']U' \tag{15}$$

with $K' \in M(S)$ and such that $\bar{D}_c'$ is biproper.

In view of the Theorem, if $\Pi[\bar{D}_c' \quad \bar{N}_c'] = \bar{D}_k^{-1}[\bar{D}_c \quad \bar{N}_c]$ an lc polynomial matrix factorization, $\bar{D}_k z = 0$, $y = Nz$ describes the cl system. Using Corollary 1 we have the following result.

Corollary 2

The closed-loop eigenvalues are the poles of:

$$\begin{aligned} \begin{bmatrix} D' \\ N' \end{bmatrix} [\bar{D}_c' \quad \bar{N}_c'] &= \begin{bmatrix} D' \\ N' \end{bmatrix} [I \quad K']U' \\ &= \begin{bmatrix} D' \\ N' \end{bmatrix} [X' \quad Y'] + \begin{bmatrix} D' \\ N' \end{bmatrix} K'[-\bar{N}' \quad \bar{D}'] \end{aligned} \tag{16}$$

or of

$$\begin{bmatrix} I & C \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I & C_0 \\ -P & I \end{bmatrix}^{-1} + \begin{bmatrix} D' \\ N' \end{bmatrix} K'[-\bar{N}' \quad \bar{D}'] \tag{17}$$

$$U'U'^{-1} = \begin{bmatrix} \dfrac{s+3}{s+2} & \dfrac{s+5}{s+2} \\ -\dfrac{1}{s+a} & \dfrac{s-1}{s+a} \end{bmatrix} \begin{bmatrix} \dfrac{s-1}{s+1} & -\dfrac{(s+5)(s+a)}{(s+1)(s+2)} \\ \dfrac{1}{s+1} & \dfrac{(s+3)(s+a)}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with $a > 0$. Using (16), the cl eigenvalues are the poles of $\Pi[I \quad K']U' = \bar{D}_k^{-1}[\bar{D}_c \quad \bar{N}_c]$ with $\Pi = 1/(s+1)$ and $K'(:= n/d)$ proper and stable; then $\bar{D}_k = (s+1)(s+2)(s+a)d$ with $\bar{D}_c = d(s+3)(s+a) - n(s+2)$ and $\bar{N}_c = d(s+5)(s+a) + n(s-1)(s+2)$. Clearly, the cl eigenvalues are the poles of $[D'^{\mathrm{T}} \quad N'^{\mathrm{T}}]$ (at -1), the poles of U' (at -2, $-a$) and the poles of K' (in d). The order of the compensator $C\ (= \bar{D}_c^{-1}\bar{N}_c)$ is $\partial C = 2 + \partial K'$ as can be seen directly from $\bar{D}_c$ or by using (7). There is no doubt that this is an unnecessarily high order for the controller, if stabilization of the first-order plant is the only objective.

The results following Example 3 are correct as long as $\bar{D}_k$, $[\bar{D}_c \quad \bar{N}_c]$ are lc as dictated by the Theorem; and this is generally true if K' is chosen to be proper and stable but otherwise arbitrary. It is of course known that for particular K', the order of C and the number of cl eigenvalues can be reduced. Such K' exists when a stabilizing controller of order less than $\partial(X'^{-1}Y')$ does exist. One way to determine such K' is to use (5) to derive:

$$K' = X'(C - C_0)(I + PC)^{-1}\bar{D}'^{-1} \tag{18}$$

which gives the required value of K' so that starting from $C_0 := X'^{-1}Y'$ to determine another stabilizing controller C. In Example 3, $C = b + 1$, $b > 0$ is a stabilizing controller of order zero, giving $s + b$ as the clcp. $C_0 = (s+5)/(s+3)$ as before. Substituting in (18) the appropriate $K' = [b(s+3) - 2](s+a)/(s+2)(s+b)$; note that here $\Pi[I \quad K']U' = (1/s+b)[s+3, s+5]$. Such K' was found relatively easily here because $C(= b+1)$ was given. In general, if ∂C_0 is high, to find K' so that the resulting stabilizing controller C is of low order, is a non-trivial problem. It can be done by requiring that $\Pi[I \quad K']U' = \bar{D}_k^{-1}[\bar{D}_c \quad \bar{N}_c]$ is of as low order as possible (numerator and denominator have common divisors) and then translating these conditions to restrictions on K'.

The cl eigenvalues depend on the poles of U' (16). In general, the poles of U' are the poles of $[X' \quad Y']$ and the poles of $[-\bar{N}' \quad \bar{D}']$ (6). It is however always possible to choose U' so that the poles of $[X' \quad Y']$ are those of $[-\bar{N}' \quad \bar{D}']$ in which case the number of poles of U' is reduced. In Example 3, this is the case if $a = 2$. This is also the case when the state-space method is used (Doyle 1984, Antsaklis 1986); the poles of U' are the poles of $[X' \quad Y']$ (or of $[-\bar{N}' \quad \bar{D}']$) equal to the zeros of $|sI - (A + HC)|$. These, together with the zeros of $|sI - (A + BF)|$ (poles of Π) and the poles of K', are the cl eigenvalues. H and F are the observer and the state feedback gains respectively. If n is the order of the realization $\{A, B, C, E\}$ of P used, there are $2n + \partial K'$ closed-loop eigenvalues. The order of C in general is $\partial C = n + \partial K'$ in this case. For particular K's, the order of C and the number of cl eigenvalues can of course be reduced as indicated above.

In view of (15)

$$\partial C \leqslant \partial[\bar{D}_c' \quad \bar{N}_c'] = \partial[I \quad K']U' \tag{19}$$

with equality holding when $\bar{D}'_c$ and $\bar{N}'_c$ do not have polynomial common factors in the numerator. If the initial stabilizing controller $C_0 = X'^{-1}Y'$ is of relatively high order, then, if the desired C, which, say, solves an H^∞ optimization problem is of low order, K' has to cancel the excess dynamics in $[I \quad K']U'$ (this was the case in Example 3 when (18) was used); note that H^∞ optimization gives K' which is then used to determine C. Care should be taken in the numerical determination of C from K' in this case, since in practice the cancellations will not be exact.

5. Concluding remarks

In view of the above, it is clear that $\partial K'$, when K' is determined via an optimization technique, say H^∞, can vary widely. It depends of course on P and the appropriate C which solves the control design problem. However, its complexity can be high or low depending on the initial choice for C_0, X', Y', N', D' in the diophantine (2) and $\bar{D}'$, $\bar{N}'$, and it can vary significantly for the same P and C (see also (18)). The suggested procedure in this case is to take $\partial[D'^{\mathrm{T}} \quad N'^{\mathrm{T}}]^{\mathrm{T}}$ and $\partial[\bar{D}' \quad \bar{N}']$ equal to ∂P (i.e. minimum) and try to use a C_0 of the lowest order possible, especially when it is suspected that the desired C is of low order as well. This initial effort will pay handsomely by deriving lower-order controllers when possible and avoiding complicated calculations.

An interesting case, when many of the above drawbacks are avoided, is when P is stable. Here, one usually selects $X = I$, $Y = 0$, $D' = I$ and $N' = P$ in which case

$$C = (I - K'P)^{-1}K' \tag{20}$$

are the stabilizing controllers. The closed-loop eigenvalues are in this case the poles of

$$\begin{bmatrix} I \\ P \end{bmatrix} [I \quad K'] \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}$$

in view of (16). Here the cl eigenvalues and the order of C depend exclusively on K' and P and not on choices which can be made arbitrarily. Notice that here $C_0 = 0$, certainly of low order. It is perhaps worth pointing out that this case of stable plants is the case considered in the original H^∞ paper by Zames (1981).

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