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Optimal control of hybrid switched systems: A brief survey

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Abstract This paper surveys recent results in the field of optimal control of hybrid and switched systems. We first summarize results that use different problem formulations and then explore the underlying relations among them. Specifically, based on the type of switching, we focus on two important classes of problems: internally forced switching (IFS) problems and externally forced switching (EFS) problems. For IFS problems, we focus on optimal control techniques for piecewise affine systems. For EFS problems, methodologies of two-stage optimization, embedding transformation and switching LQR design are investigated. Detailed optimization methods found in the literature are discussed.

Keywords Hybrid systems · Switched systems · Optimal control · Survey

1 Introduction

Hybrid systems are heterogeneous dynamical systems which involve both continuous models and discrete event models Antsaklis and Nerode (1998). Continuous models consist of time-driven continuous variable dynamics, which are usually described by differential or difference equations; discrete event models contain event-driven discrete dynamics, often described by finite-state machines or Petri nets. As such, hybrid systems theory combines ideas originating in the computer science and software engineering disciplines on one hand, and systems theory and control engineering on the other. This mixed character explains the terminology "hybrid systems". Hybrid systems have been identified as important in a wide variety of applications: in the control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, among many other fields. More detailed historical review of hybrid systems can be found in Antsaklis (2000), Antsaklis

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and Koutsoukos (2002), and Antsaklis et al. (1998), and more recent reviews are given in Antsaklis and Koutsoukos (2005) and Goebel et al. (2009). Published books on hybrid systems include Matveev and Savkin (2000), van der Schaft (2000), Sebastain Engell and Schnieder (2002), Savkin and Evans (2002), Wassim M Haddad and Nersesov (2006), Lunze and Lamnabhi-Lagarrigue (2009), and Tabuada (2009).

Switched systems are a particular class of hybrid systems consisting of several subsystems and a switching law specifying the active subsystems at each time instant. Although switched system models are relatively simple and straightforward, this system class exhibits several typical behaviors of hybrid dynamical systems. For the recent results on stability and stabilization of switched systems, one can refer to the survey Lin and Antsaklis (2009) and the references therein.

In addition to stability and stabilization issues, the problem of determining optimal control laws for hybrid systems and in particular for switched systems, has been extensively investigated in recent years and it has attracted researchers from various fields in science and engineering. There are both theoretical and computational results in the open literature. The available theoretical results usually extend the classical maximum principle or the dynamic programming approach to switched systems Riedinger et al. (1999), Piccoli (1999), and Sussmann (2000). The computational results take advantage of various optimization techniques and high-speed computers to develop efficient numerical methods for the optimal control of switched systems Xu and Antsaklis (2003, 2004a), Bengea and DeCarlo (2005), Baotic et al. (2006), Seatzu et al. (2006a), Axelsson et al. (2008), Ding et al. (2009b), Zhang et al. (2009b), and Görges et al. (2011).

This paper surveys recent progress in computational methods for optimal control problems of switched systems. Such problems are difficult to solve due to switching of subsystem dynamics. The past decade has seen some breakthroughs in theory as well as the development of efficient computational methods. However there are no theoretical or computational results applicable to general optimal control problems for all types of switched systems. The existing literature results are often based on different models and differ in problem formulation and approaches. Here, we focus on summarizing recent results that have been used in different problem formulations and then exploring the underlying relations among them. Note that some problem formulations focus on guaranteeing the decrease of the cost function using various computational techniques and thus the (local or global) optimality of the solution is not of primary concern.

The present paper is organized as follows. In Section 2, a brief overview of theoretical results on optimal control of hybrid systems is presented and the general optimal control problem formulation of switched systems is given. Section 3 reviews existing optimal control methodologies for switched systems with internally forced switching (IFS). Optimal control problems of piecewise affine systems are discussed. Section 4 focuses on the results on optimal control of switched systems with externally forced switching (EFS). The two-stage optimization, embedding transformation and switching LQR design are discussed. A brief summary in Section 5 concludes the paper.

2 Optimal control problems in hybrid and switched dynamical systems

2.1 Hybrid systems

Several approaches to determine control laws for hybrid systems have been reported in the control and computer science literature. Numerous results on necessary conditions for optimality have appeared for a variety of models of hybrid systems Branicky and Mitter (1995), Branicky et al. (1998), Piccoli (1999), Riedinger et al. (1999), Sussmann (2000), and Rantzer and Johansson (2000), Hedlund and Rantze (1999b, 2002), Shaikh and Caines (2002, 2003), Caines and Shaikh (2005, 2006). However, most of the results consider general problem settings and it is not always possible to develop tractable algorithms to numerically compute the optimal solution.

For continuous-time hybrid systems, Branicky and Mitter (1995) compared several algorithms for optimal control, while Branicky et al. (1998) discussed general necessary conditions for the existence of optimal control laws for hybrid systems by using dynamic programming. They established a general hybrid framework for the optimal control problem, proved the existence of optimal (relaxed or chattering) controls and near-optimal (precise or nonchattering) controls, and derived generalized quasi-variational inequalities (GQVI's) that the associated value function is expected to satisfy.

Necessary optimality conditions for trajectories of hybrid systems were derived using the maximum principle by Sussmann (2000) and Piccoli (1999), who considered a fixed sequence of finite length. Several versions of hybrid maximum principles were proposed. A similar approach was used by Riedinger et al. (1999), who considered both autonomous and controlled switchings with linear quadratic cost functionals.

Hedlund and Rantzer (1999b, 2002) used convex dynamic programming (CDP) to approximate hybrid optimal control laws and to compute lower and upper bounds of the optimal costs. The case of piecewise-affine systems was discussed by Rantzer and Johanson (2000). For determining the optimal feedback control law these techniques require the discretization of the state space in order to solve the corresponding Hamilton-Jacobi-Bellman equations.

Shaikh and Caines (2002) considered a finite-time hybrid optimal control problem and gave necessary optimality conditions for a fixed sequence of modes using the maximum principle. In Shaikh and Caines (2003) these results were extended to non-fixed sequences by using a suboptimal result based on the Hamming distance permutations of an initial given sequence. Finally, in Caines and Shaikh (2005, 2006), a feedback law for a finite time LQR problem was derived by integrating the computation of the optimality zones into the hybrid maximum principle algorithms class.

A special class of hybrid systems motivated by the structure of manufacturing systems was considered by Cassandras et al. (1998, 2001), Cho et al. (2000), Gokbayrak and Cassandras (2000b, a), Zhang and Cassandras (2001). The hybrid system model considered extends event-driven models to include time-driven dynamics where the event-driven dynamics follow a queueing theory model. Due to the non-differentiability in event-driven processes, the problem to find the optimal control input is a non-smooth optimization problem. The results showed the necessary condition for an optimal control input and properties of the optimal sample paths, which can be used to identity the non-smooth part in the solution.

2.2 Switched systems

In order to find ways to numerically compute the optimal control in hybrid systems, many researchers have been focusing on switched systems. A switched system may be obtained from a hybrid system by simplifying the details of the discrete behavior to switching patterns from a certain class, which usually represent discontinuity in vector fields. For simplicity, it is often assumed that all subsystems are in the same state space, e.g. $\mathbb{R}^n$.

The mathematical representation of switched systems is based on the state-space model.

Definition 1 A switched system is described by a collection of indexed differential (or difference) equations

$$\dot{x}(t) = f_{i(t)}(x(t), u(t)),\ x(0) = x_0 \quad (1)$$

$$y(t) = g_{i(t)}(x(t), u(t)). \quad (2)$$

where the input is $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ is the continuous state vector and $i(t) \in \{1, 2, \cdots, M\} \triangleq Q$.

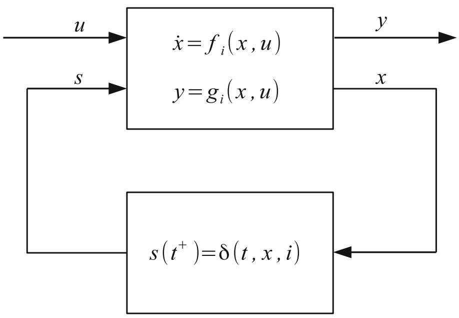
The discrete event dynamics are modeled by a switching signal, which is usually described as a timed sequence, $s = ((\tau_0, i_0), (\tau_1, i_1), \cdots, (\tau_N, i_N))$ where $0 \leq N < \infty$, $t_0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_N = t_f$, and $i_k \in Q$ for $0 \leq k \leq N$. t_0 and t_f are the given intial time and final time, respectively. It can be seen that a switching signal s consists of two components: a switching sequence (of indices) $\sigma = \{i_k\}_{k=0}^N$ and a sequence of switching instants $\tau = \{\tau_k\}_{k=0}^N$.

According to the nature of switching signals, switched systems may be classified into two classes: switched systems with *internally forced switching* (IFS) or *externally forced switching* (EFS). As shown in Fig. 1, the switching signal s of IFS is based on the information of the state x and the current mode i. In general, the switching law can be a function of t, x, u and i. The switching signal s of EFS is an exogenous input to the system, as is the continuous input u, as shown in Fig. 2. Hence one has freedom to choose a specific switching signal or a class of switching signals of interests to study the behavior of systems.

Remark 1 For a switched system with EFS, the exogenous control input is a pair (s, u). For a switched system with IFS, the exogenous control input is u. The switching sequence s is generated implicitly based on the evolution of x and u.

Remark 2 A variety of switched system models can be derived from Eqs. 1–2. Based on the different dynamics of subsystems (characterized by indexed differential or difference equations), we have *continuous-time (discrete-time)* switched systems, if the subsystems are continuous (discrete) time systems, or *linear (nonlinear)* switched systems, if subsystems are linear (nonlinear) systems. If the continuous control input u is absent from the model, we call it an *autonomous* switched system. The continuous state x usually does not exhibit jumps at switching instants (i.e. $x(\tau_i^+) \neq x(\tau_i^-)$). However, we note that some methods reported here can be extended to problems with jumps.

Fig. 1 Switched system with *internally forced switching(IFS)*



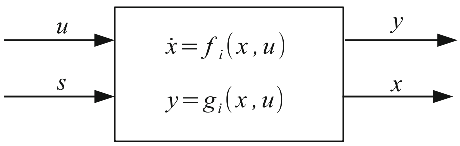


Fig. 2 Switched systems with *externally forced switching* (EFS)

2.3 Two optimal control problems of switched systems

Although in general optimal control problems could be formulated for switched systems with both EFS and IFS, results would be difficult to obtain. Therefore we focus on two important classes of problems which can be solved individually, namely, optimal control problem with EFS only (EFS Problems), and problems for systems with IFS only (IFS problem). Most of the literature in this paper addresses one of these problems.

Problem 1 (**Internally Forced Switching Problem**) Xu and Antsaklis (2003) Consider a switched system with IFS. Find an admissible control $u(t)$ such that x departs from a given initial state $x(t_0) = x_0$ at the given initial time t_0 and meets he terminal manifold defined by $\psi(x(t_f)) = 0$ where ψ is a vector function and

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))dt + \sum_{1\leq k\leq K} \gamma(x(t_k), i_{k-1}, i_k) \tag{3}$$

is minimized (here K is the number of switchings).

Problem 2 (**Externally Forced Switching Problem**) Xu and Antsaklis (2003) Consider a switched system with EFS. Find an admissible control pair (s, u) (u is piecewise continuous) such that x departs from a given initial state $x(t_0) = x_0$ at the given initial time t_0 and meets the terminal manifold defined by $\psi(x(t_f)) = 0$ where ψ is a vector function and

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))dt + \sum_{1\leq k\leq K} \gamma(x(t_k), i_{k-1}, i_k) \tag{4}$$

is minimized (here K is the number of switchings).

Remark 3 Problem (1) and (2) are formulated as general optimal control problems with terminal cost ψ, running cost $\int_{t_0}^{t_f} Ldt$, and switching cost function γ. Subject to applicability and solvability, different problems may adopt specific forms for the terminal, running and switching costs or drop certain cost terms.

Remark 4 The main difference between the two problems is whether switching is an exogenous input or autonomously generated. For EFS problem, one needs to optimize both the continuous control input u and the switching signal s, which are strongly coupled in the optimization process. On the other hand, the difficulty in IFS problem is that the switching depends on the specific initial state x_0 and the control u (e.g. switching surfaces characterized by functions of states and input) and cannot be explicitly determined unless a specific control input is given.

3 Optimal control for switched systems with IFS

IFS problems concentrate on finding a continuous control input u to minimize the cost function when switching is IFS. The difficulty in this problem is that the switching instants may depend on the continuous control input u in a very complicated way. Among the models of switched systems, piecewise affine system models are suitable for IFS problems since the switchings are implicitly determined by the partition of the state and input spaces.

A piecewise affine (PWA) system is defined by partitioning the state space into polyhedral regions, and associating with each region a different linear state-update equation

$$x(t+1) = A_i x(t) + B_i u(t) + f_i \tag{5}$$

$$\text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : H_i x + J_i u \leq K_i \right\} \tag{6}$$

where $x \in \mathbb{R}^{n_c} \times \{0, 1\}^{n_l}$, $u \in \mathbb{R}^{m_c} \times \{0, 1\}^{m_l}$, $\{\mathcal{X}_i\}_{i=0}^{s-1}$ is a polyhedral partition of the sets of state+input space $\mathbb{R}^{n+m}$, $n \triangleq n_c + n_l$, $m \triangleq m_c + m_l$. f_i is a suitable constant vector. PWA systems can model a large number of physical processes, such as systems with static nonlinearities, and can approximate nonlinear dynamics via multiple linearizations at different operating points Sontag (1981) and Sontag (1996). Details and recent results on PWA can be found in Bemporad et al. (2000c), Imura and van der Schaft (2000), Mignone et al. (2000), Kerrigan and Mayne (2002), and Roll et al. (2004).

When hard input and state constraints

$$Ex(t) + Lu(t) \leq M \tag{7}$$

are introduced, the constrained PWA system (CPWA) can be described by Eqs. 8–9

$$x\,(t+1) = A_i x(t) + B_i u(t) + f_i \tag{8}$$

$$\text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{X}}_i \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : \tilde{H}_i x + \tilde{J}_i u \leq \tilde{K}_i \right\} \tag{9}$$

where $\left\{\tilde{\mathcal{X}}_i\right\}_{i=0}^{s-1}$ is the new polyhedral partition of the sets of state+input space $\mathbb{R}^{n+m}$ by intersecting the polyhedrons $\mathcal{X}_i$ in Eq. 6 with the polyhedron described by Eq. 7.

After defining the cost function

$$J\left(U_0^{T-1}, x(0)\right) \triangleq \sum_{k=0}^{T-1} (\|Qx(k)\|_p + \|Ru(k)\|_p) + \|Px(T)\|_p \tag{10}$$

the finite-time optimal control problem (FTCOC) is formulated as in Eqs. 11–12

$$J^*(x(0)) \triangleq \min_{\{U_0^{T-1}\}} J(U_0^{T-1}, x(0)) \tag{11}$$

$$s.t. \begin{cases} x(t+1) = A_i x(t) + B_i u(t) + f_i \\ \qquad \text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{X}}_i \end{cases} \tag{12}$$

where the column vector $U_0^{T-1} \triangleq \left[u^T(0), \ldots, u(T-1)^T\right]^T \in R^{mT}$, is the optimization vector and T is the time horizon. In Eq. 10, $\|Qx\|_p = x^T Qx$ for $p = 2$ and $\|Qx\|_p = \|Qx\|_{1,\infty}$ for $p = 1, \infty$, where $R = R^T > 0$, $Q = Q^T$, $P = P^T > 0$ if $p = 2$ and Q, R, P non-singular if $p = \infty$ or $p = 1$.

The main results on FTCOC can be found in Baotic et al. (2003a, b, 2006), Baric et al. (2005, 2008), Bemporad et al. (2000b, 2002a, 2000a), Christophersen et al. (2005), Fotiou et al. (2006a, 2006b), Borrelli et al. (2003, 2005). It has been proved that the closed form of the state-feedback solution to the finite time optimal control problem based on quadratic or linear norm performance criteria, is a time-varying piecewise affine feedback control law. Moreover, two computational methods are provided to numerically find the optimal solution.

One way is describing the PWA system by a set of inequalities with integer variables as the system switches between the different dynamics. An appropriate modeling framework for such class of systems is the mixed logical dynamic framework. Mixed logical dynamical (MLD) systems are computationally oriented representations of hybrid systems that consist of a collection of linear difference equations involving both real and Boolean (i.e. 1 or 0) variables, subject to a set of linear inequalities. Details on MLD can be found in Bemporad and Morari (1999a) and Ferrari-Trecate et al. (2002).

After transforming the PWA system to its equivalent MLD system, Eqs. 10–12 can be rewritten as:

$$\min_{\{U_0^{T-1}\}} J\left(U_0^{T-1}, x(0)\right) \triangleq \sum_{k=0}^{T-1}(\|Qx(k)\|_p + \|Ru(k)\|_p) + \|Px(T)\|_p \tag{13}$$

$$\text{s.t.} \begin{cases} x(k+1) = Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(t), \\ E_1\delta(k) + E_2z(k) \leq E_3u(t) + E_4x(t) + E_5 \end{cases} \tag{14}$$

where x is a vector of continuous and binary states and u is the input. δ and z represent auxiliary binary and continuous variables, respectively, which are introduced when transforming logic relations into MLD system Bemporad and Morari (1999b). A, B_1, B_2, B_3, $E_1, \ldots, E_5$ are matrices of suitable dimensions.

The optimal control problem in Eqs. 13–14 can be formulated as a Mixed Integer Quadratic Program (MIQP) when the squared Euclidean norm $p = 2$ is used (Bemporad and Morari 1999a), or as a Mixed Integer Linear Program (MILP), when $p = \infty$ or $p = 1$ (Bemporad et al. 2000b). In addition, multiparametric programming can be used to efficiently compute the explicit form of the optimal state-feedback control law (Bemporad et al. 2000b, a).

Another method of combining a dynamic programming strategy with a multi-parametric program solver was proposed in Borrelli et al. (2003, 2005). The optimization is of the following form

$$J_j^*(x(j)) \triangleq \min_{u(j)} \|Qx(j)\|_p + \|Ru(j)\|_p + J_{j+1}^*(f_{PWA}(x(j), u(j))) \tag{15}$$

$$\text{s.t. } f_{PWA}(x(j), u(j)) \in \mathcal{X}^{j+1} \tag{16}$$

for $j = T-1, \ldots, 0$, with boundary conditions

$$X^T = X^f \tag{17}$$

and

$$J_T^*(x(T)) = \|Px(T)\|_p \tag{18}$$

where

$$X^j = \left\{x \in \mathbb{R}^n | \exists u,\ f_{PWA}(x, u) \in \mathcal{X}^{j+1}\right\} \tag{19}$$

is the set of all initial states for which the problem Eqs. 15–16 is feasible.

Depending on the norm used in the cost function Eq. 15, the algorithm based on dynamic programming recursion and multiparametric quadratic solvers was used when $p = 2$ (Borrelli et al. 2003, 2005). Similarly, when $p = 1$ or $p = \infty$, the algorithm based on dynamic programming recursion and multiparametric linear program solvers was shown in Baotic et al. (2003a, 2006), Borrelli et al. (2005). Compared with the algorithm based on MIP, the dynamic programming algorithm is more efficient and less complex due to fewer underlying inequality constraints. Also, the dynamic programming algorithm can be used to approximate infinite time horizon solutions through finite time horizon solutions. Recent work Baric et al. (2005, 2008) showed how to exploit the underlying geometric structure of the optimization problem with a linear performance index in order to significantly improve the efficiency of the off-line computations. Fotiou et al. (2006a, b) studied the constrained finite-time optimal control problem of discrete-time nonlinear systems using algebraic geometry methods.

4 Optimal control for switched systems with EFS

In order to solve the EFS problem in general, one needs to optimize jointly over both the continuous input and the switching signal. Recall that a switching signal consists of both switching instants and ordered indices of systems. Even when the continuous control input is absent, finding an optimal switching signal can be challenging. Therefore many researchers decided to start with *autonomous switched systems* (i.e. $u = 0$) so that they only need to focus on deriving the optimal switching signal. Various approaches were proposed to find the optimal switching instants and sequences. Although different approaches have their specific switched systems models and cost functions, most of them can be classified into two kinds of methodologies: ***two-stage optimization*** and ***embedding transformation***. For *non-autonomous switched systems* the existing approaches, which are typically extensions or improvements of the methods for autonomous systems, also fall into the two categories.

In this section, two main methodologies (two-stage optimization and embedding transformation) are discussed respectively. For each methodology, the corresponding literature results are introduced from non-autonomous nonlinear switched systems to autonomous linear switched systems. At the end, a special class of EFS problems, ***switching LQR problem*** is presented separately, due to the particular structure in problem formulations and algorithms.

4.1 Two-stage optimization

A two-stage optimization algorithm consists of two separate levels of optimization. The idea was first introduced by Xu and Antsaklis (2000b). At the lower level, the algorithm assumes a fixed switching sequence and minimizes the cost functional with respect to the switching times and continuous control input. At the upper level, it updates the switching sequence to minimize the cost function. The whole optimization process is based on alternating between these two procedures.

Two-stage optimization procedures:

Stage 1 *Given a switching sequence σ, calculate the optimal switching instants τ and the optimal control input u (for non-autonomous switched systems).*

Stage 2 *Calculate a new sequence $\tilde{\sigma}$, which is the result of updating the original sequence σ. Repeat Stage 1 using $\tilde{\sigma}$.*

It can be seen that Stage 1 and Stage 2 are (relatively) decoupled problems and therefore can be solved separately. Since Stage 2 can be formulated as another search problem, many results in the literature assumed the switching sequence to be prefixed and focus on Stage 1 optimization as a starting point.

For *non-autonomous switched nonlinear systems*, a computational method was proposed in Xu and Antsaklis (2000b, 2002c, 2003, 2004a) to solve the optimization problem in Stage 1. They formulated the optimization problem in terms of minimizing a cost functional of the switching instants. Then the gradient-based nonlinear programming algorithms were used to numerically calculate the switching times. The conceptual algorithm for Stage 1 optimization is as follows.

Stage 1 *(a) Fix the switching sequence. Let the minimum value of J with respect to u be a function of the switching instants, i.e., $J = J(\tau)$ where $\tau = (\tau_1, \ldots, \tau_N)^T$. Find the form of $J(\tau)$*

Stage 1 *(b) Minimize J with respect to τ. i.e.*

$$\min_{\tau} J(\tau)$$

subject to $\tau \in \mathcal{T}$, where $\mathcal{T} \triangleq \{\tau = (\tau_1, \ldots, \tau_N)^T | t_0 \le \tau_1 \le \cdots \le \tau_N \le t_f\}$.

In Stage 1(a), we need to find an optimal continuous input u and the corresponding minimum J. Note that although different subsystems are active in different time intervals, the problem is conventional since these intervals are fixed. Therefore, most of the available numerical methods for unconstrained conventional optimal control problems with fixed end-time can be used. Stage 1(b) is a constrained nonlinear optimization problem, which can be solved using various gradient-based computational methods, such as gradient projection (using $\frac{\partial J}{\partial \tau}$) and constrained Newton's method (using $\frac{\partial J}{\partial \tau}$ and $\frac{\partial^2 J}{\partial^2 \tau}$). By combining Stage 1(a) and Stage 1(b), the following iterative algorithm provides a framework for the optimization methodologies.

1. *Set the iteration index $j = 0$. Choose an initial τ^j.*
2. *By solving an optimal control problem, find $J(\tau^j)$.*
3. *Find $\frac{\partial J}{\partial \tau}$ (and $\frac{\partial^2 J}{\partial^2 \tau}$ if second-order method is to be used).*
4. *Use some feasible direction method to update τ^j to be $\tau^{j+1} = \tau^j + \alpha^j d\tau^j$ (here $d\tau^j$ is formed by using the gradient information of J, the step-size α^j can be chosen using some step-size rule, e.g., Armijo's rule). Set the iteration index $j = j + 1$.*
5. *Repeat Steps (2), (3), (4) and (5), until a pre-specified termination condition is satisfied (e.g., the norm of the projection of $\frac{\partial J}{\partial \tau}$ on any feasible direction is smaller than a given small number $\epsilon > 0$).*

Note that Step 2 hinders the usage of the algorithm because the values of the derivatives $\frac{\partial J}{\partial \tau}$ and $\frac{\partial^2 J}{\partial^2 \tau}$ are not readily available. In Xu and Antsaklis (2000a, 2002c), a method based on direct differentiations of the value function was proposed to approximate the values of the derivatives. In Xu and Antsaklis (2004b), based on an equivalent problem formulation, accurate values of the derivatives were obtained by solving a two point boundary value differential algebraic equation (DAE).

For *autonomous switched nonlinear systems*, the controlled parameters to optimize the cost function are the switching instants only. Egerstedt et al. (2003, 2006b) derived a simpler formula for the gradient $\frac{\partial J}{\partial \tau}$ than the one in Xu and Antsaklis (2002b). Then the formula was used in a steepest descent algorithm with Armijo's stepsizes to find the optimal switching instants. It is noted that Xu and Antsaklis (2002b) considered the autonomous switched nonlinear systems, which is a special class of non-autonomous switched systems studied by Xu and Antsaklis (2002c, 2000a).

Various practical issues were considered later, as extensions to the results in Egerstedt et al. (2003, 2006b). State estimation and on-line computation were addressed in Azuma et al. (2006) and Egerstedt et al. (2006a) and Ding et al. (2009a, b), Wardi et al. (2007, 2010). Azuma et al. (2006) and Egerstedt et al. (2006a) considered the problem when the state of the system is only partially known through the outputs. A method was presented to guarantee that the current switch-time estimates remain optimal as the state estimates evolve in a computationally feasible manner. Inspired by this work, more results (Ding et al. 2009a, b; Wardi et al. 2007, 2010) appeared on the on-line optimization of switched systems. The need for real-time on-line algorithms typically arises when complete information about the system is not available but the algorithm can acquire partial information about it in real time. Notice that the results in on-line computation often assume that the second derivative $\frac{\partial^2 J}{\partial^2 \tau}$ is known.

Recently, Johnson and Murphey (2009, 2011) used a different approach to find the gradient $\frac{\partial J}{\partial \tau}$, which relies on fundamental principles in calculus instead of constrained Lagrange multiplier techniques Egerstedt et al. (2003, 2006b), Xu and Antsaklis (2002c, 2000a). Moreover, the approach was also extended to an explicit derivation of the second derivative $\frac{\partial^2 J}{\partial^2 \tau}$. By using both first and second order derivatives in the gradient-based method, examples showed the significant reduction in iterations needed to converge, compared to the method using first derivative only in Egerstedt et al. (2003). Caldwell and Murphey (2010) compared the convergence between the first-order method (steepest descent) and second-order method (Newton's method) in the optimization of switching instants. They emphasized the importance of the second-order method due to its quadratic convergence, considering the concern of convergence rate for on-line implementation.

Giua et al. (2001a, b) considered *switched affine systems with state jumps*, defined in Eqs. 20 and 21

$$\dot{x}(t) = A_{i(t)}x(t) + f_{i(t)}, \tag{20}$$

$$x(t^+) = M_{i,j}x(t^-). \tag{21}$$

Equation 21 models a jump condition: whenever at time t a switch from mode $i(t^-) = i$ to mode $i(t^+) = j \neq i$ occurs, the state jumps from $x(t^-)$ to $x(t^+) = M_{i,j}x(t^-)$, where $M_{i,j} \in \mathbb{R}^{n\times n}$ are constant matrices. The cost function is

$$J = \int_0^\infty x^T(t)Q_{i(t)}x(t)dt + \sum_{k=1}^{N} \gamma_{i_{k-1},i_k} \tag{22}$$

where Q_i are positive semi-definite matrices and $\gamma_{i,j}$ satisfies $\gamma_{i,j} \geq 0$ for $i \neq j$ and $\gamma_{i,i} = 0$ for $\forall i$. The cost (Eq. 22) consists of two components: a quadratic cost that depends on the time evolution (the integral) and a cost that depends on the switches (the sum). Giua et al. (2001a, b) assumed a fixed switching sequence and thus the unknown switching instants are the optimization parameters. It was shown that the optimal control takes the form of a state feedback. A region on the state was numerically calculated so that an optimal switch should occur if and only if the present state is in this region. It is worth noting that the switching times are implicitly determined by the regions, instead of being calculated explicitly as in Xu and Antsaklis (2000b, 2002c, 2003, 2004a) and Egerstedt et al. (2003, 2006b). This method was used as "slave" procedure in Bemporad et al. (2002b, c, d) and later was generalized as a switching table procedure (STP) in Bemporad et al. (2002d) and Seatzu et al. (2006a).

For *autonomous switched time-varying linear systems,* Caldwell and Murphey (2012b, 2011) considered the switching time optimization problem subject to quadratic cost (also potentially time-varying). By making full use of the linearity of the switched system, the problem was formulated so that a single set of differential equations could be solved prior to optimization. Once the equations are solved and the solutions are stored, the problem may be reformulated as a problem that does not rely on solving any differential equations during calculation of the descent direction. Compared with the approaches (Egerstedt et al. 2003; Xu and Antsaklis 2002b, a) requiring a constant number of differential equations at each step of steepest descent, the algorithm in Caldwell and Murphey (2012b, 2011) trades additional memory demands for the computational advantage of all the integrations being independent of the switching sequence and switching instants.

As we mentioned earlier, Stage 2 can be formulated as a discrete search problem. However, the search for all possible switching sequences could grow exponentially after each iteration. In order to further reduce the value of cost function, a *single mode insertion* technique can be used to avoid the "combinatorial explosion". It updates the switching sequence by inserting two switching points at some time with a subsystem index between them. The idea was originally proposed in Axelsson et al. (2005a, 2008).

For *non-autonomous switched nonlinear systems*, a bi-level hierarchical algorithm was proposed in Gonzalez et al. (2010b, a). At the lower level, the algorithm assumes a fixed modal sequence and determines the optimal mode duration and optimal continuous input (Stage 1). It is pointed out that this stage of optimization can be solved by transforming the problem into a classical optimal control problem over the switching instants and continuous control input Gonzalez et al. (2010a). At the higher level, the algorithm employs the mode insertion method to find either a lower cost or less infeasible switching sequence (Stage 2). It is noted that Gonzalez et al. (2010b, a) considered a "constrained" optimal control problem. By "constrained", it is meant that the state x is constrained to a set described by

$$x(t) \in \left\{x \in \mathbb{R}^n | h_j(x) \leq 0,\ j = 1, ..., N\right\}. \tag{23}$$

for all time. The cost function is defined in Eq. 22 but without the switching cost term. An improved algorithm was presented recently (Gonzalez et al. 2010a), in which new features are implemented to overcome certain shortcomings of the original algorithm.

For *autonomous switched nonlinear systems*, Egerstedt et al. (2006b) presented a simple case of extending the gradient-based nonlinear programming algorithms in Egerstedt et al. (2003) to obtain derivative information about inserting new modes into a given switching sequence. A complete problem formulation and algorithm were shown in Axelsson et al. (2005a, 2008). The resulting algorithm first minimizes the cost functional with respect to the switching instants (Stage 1) and then updates the switching sequence by adding new modes to it at each iteration (Stage 2). Since the algorithm is defined on a sequence of spaces with increasing dimensions, a notion of local optimality and a suitable concept of convergence were defined in Axelsson et al. (2005b) to prove that the proposed algorithm converges in that sense.

Although the single model insertion provides a way of varying switching sequences in practice, Wardi and Egerstedt (2012) pointed out that the two-stage optimization involving the single mode insertion technique may be inefficient due to a possible infinite-loop procedure at each step of the algorithm. Therefore different algorithms were proposed to solve the two-stage optimization problem by using other optimization techniques for Stage 1 and 2.

Feng et al. (2009) considered the optimal control problem of *discrete-time non-autonomous switched nonlinear systems*. The algorithm for Stage 2 searches for the optimal switching sequence through the discrete filled function method. For *autonomous switched nonlinear systems*, Wardi and Egerstedt (2012) developed an alternative that changes any finite (but possibly unbounded) number of modes at each iteration, and its computational workload appears to be independent of the number of modes that are being changed. If the *state jumps* appear in autonomous switched nonlinear systems, Loxton et al. (2009) showed that an approximate solution for this optimal control problem can be computed by solving a sequence of conventional dynamic optimization problem. This approach can reduce the excessive switching between subsystems by merging two or more switching times into one.

For *autonomous switched linear systems*, a master-slave procedure (MSP) is used in Bemporad et al. (2002c, b), Giua et al. (2001a, b). The "slave" procedure, originated from Giua et al. (2001a, b), is based on the construction of the switching regions which explicitly determines the optimal switching instants, assuming the switching sequence is known (Stage 1). The "master" procedure is based on mixed-integer quadratic programming (MIQP) which aims to find an optimal switching sequence assuming the switching instants are known (Stage 2). Later, a switching table procedure (STP) was presented in Bemporad et al. (2002d) and Seatzu et al. (2006a) as a generalization of the slave procedure. STP is based on dynamic programming and allows one to avoid the explosion of the computational burden as the number of switching sequences increases. It relies on the construction of switching tables which specify when the switching should occur and what the next mode should be. Moreover, Bemporad et al. (2003), Bemporad et al. (2004a, b), Seatzu et al. (2006b) showed that STP can be applied to other hybrid system frameworks.

Recently, a projection-based method was reported in Caldwell and Murphey (2012c, a). The optimization problem was first formulated as an infinite dimensional optimal control problem where the switching control design variables are constrained to be integers. Then the projection-based techniques were introduced to handle the integer constraint. A necessary condition for optimality of switched systems, based on this approach, was derived in Caldwell and Murphey (2012c). Caldwell and Murphey (2012a) was concerned with the line search step of such projection-based numerical optimization procedure.

4.2 Embedding transformation

The basic idea of embedding transformation is that one can transform switched systems to a larger family of systems where the number of switchings, switching sequence, and switching instants are embedded as decision variables. Then the various conventional optimization techniques can be utilized. By adopting such problem transformation, there is no need to make any assumptions about the number of switches nor about the switching sequence at the beginning of the optimization.

Bengea and DeCarlo (2005) and Wei et al. (2007) considered *non-autonomous switched nonlinear systems* through embedding transformation. The switched system is first embedded into a larger family of continuous systems, as in Eq. 24

$$x(t) = \sum_{i=1}^{N} v_i(t) f_i(x(t), u_i(t)) \tag{24}$$

where $v_i \in [0, 1]$ and $\sum_{i=1}^{N} v_i(t) = 1$. Then the embedded system can be solved using conventional optimal control techniques. As a modification of the theorem in Berkovitz (1974), the theoretical results in Bengea and DeCarlo (2005) showed that the set of trajectories of the switching system is dense in the set of trajectories of the embedded system. Furthermore, the results also imply that if one solves the embedded optimal control problem and obtains a solution, either the solution is the solution of the original problem, or suboptimal solutions can be constructed. Wei et al. (2007) further explored the possible numerical nonlinear programming techniques under this framework. It was shown that sequential quadratic programming (SQP) can be utilized to reduce the computational complexity; this is an issue that cannot be tackled by mixed integer programming (MIP). The effectiveness of the proposed approach was demonstrated through several examples.

The application of embedding transformation to *autonomous switched nonlinear systems* was considered in Mojica et al. (2008) and Mojica-Nava et al. (2013). The original switched system was transformed into a an equivalent polynomial representation, as Eq. 25.

$$\dot{x} = \sum_{k=0}^{q} f_k(x) l_k(v) \tag{25}$$

where $l_k(v) = \prod_{i=0, i\neq k}^{q} \frac{(v-i)}{(k-i)}$ and $v \in \Omega = \left\{ v \in \mathbb{R} | \prod_{k=0}^{q} (v-k) \right\}$. Also, the original cost function

$$J = \varphi(x(t_f)) + \int_{t_0}^{t_f} L_{\sigma(t)}(x) dt \tag{26}$$

was transformed into the equivalent form

$$J = \varphi(x(t_f)) + \int_{t_0}^{t_f} \mathcal{L}(x, v) dt \tag{27}$$

where $\mathcal{L}(x, v) = \sum_{k=0}^{q} L_k(x) l_k(v)$. Note that the equivalent optimal control problem uses v as a control variable, which mimics the behavior of the original switched system. Since the equivalent problem has a constraint which is non-convex with a disjoint feasible set, traditional optimization solvers have a disadvantaged performance. Then the problem was further relaxed and convexified using the method of moments so that the existing numerical methods in convex programming can be used to solve the problem. The necessary and sufficient condition for optimality was also obtained. A more detailed and complete optimization algorithm based on semi-definite programming can be found in Mojica-Nava et al. (2013).

For *autonomous switched linear* systems, Das and Mukherjee (2008) used a different embedding transformation, as shown in Eq. (28).

$$\dot{x} = A_1 x u_1 + A_2 x (1-u_1) u_2 + \ldots + A_{N-1} x \prod_{i=1}^{N-2} (1-u_i) u_{N-1} + A_N x \prod_{i=1}^{N-1} (1-u_i) \tag{28}$$

where A_i is the state matrix for i-th linear subsystem and $u_i \in \{0, 1\}$ for $i \in \{1, 2, \ldots N\}$. It showed that the optimal solution can be determined by solving the two-point boundary value problem after applying Pontryagin's Minimum Principle.

4.3 Switching LQR problem

Görges et al. (2011) and Zhang and Hu (2008b, a), Zhang et al. (2009a, b) studied the optimal control and scheduling problem of a special class of switched systems, *discrete-time switched linear systems*. The model considered is

$$x(k+1) = A_{i(k)}x(k) + B_{i(k)}u(k). \tag{29}$$

Switching between subsystems is described by the switching index $j(k)$ which is subject to control. Further, the cost function is in the quadratic form

$$J = \sum_{k=0}^{N-1} \left(x^T(k)Q_{i(k)}x(i) + u^T(k)R_{i(k)}u(i) \right) + x^T(N)Q_f x(N) \tag{30}$$

where Q_f and Q_i are symmetric positive semi-definite and R_i is symmetric positive definite. The variables to be optimized are the sequence of control inputs $\{u(k)\}_{k=1}^N$ and the switching sequence $\{i(k)\}_{k=1}^N$.

It is noted that Görges et al. (2011) and Zhang and Hu (2008b) showed that the explicit feedback form of the optimal control input can be obtained by solving a set of difference Riccati Equations (DRE). The resulting optimal control input $u(k)$ is in piecewise linear state feedback form and the switching sequence and times are obtained through dynamic programming. Moreover, Zhang and Hu (2008b) and Görges et al. (2011) also realized the problem that the size of the positive semi-definite matrix set will grow exponentially as the time evolves, which calls for an efficient computational algorithm in practice.

To reduce the solution space, Görges et al. (2011) considered a sub-optimal cost function so that the optimal control problem can be approached by relaxed dynamic programming (Rantzer 2006; Lincoln and Rantzer 2006). A different approach in Zhang and Hu (2008a) and Zhang et al. (2009a) finds the minimum equivalent subset of the positive semi-definite matrix set by removing the redundant matrices. It should be noted that the optimality of the problem is not jeopardized in Zhang and Hu (2008a) and Zhang et al. (2009a). Zhang et al. (2009a) also showed that a similar algorithm can be extended to the case when sub-optimal performance is acceptable.

5 Conclusion

This survey gives a brief overview of the results on optimal control of switched systems, with emphasis on computational results recently developed. We first presented the general formulation of optimal control problems and then introduced two important problem classes based on the nature of switching. For each class of problems, the main methodologies under different assumptions and types of switched systems were summarized. For internally forced switching problems, the computational results on piecewise affine systems were explored. For externally forced switching problems, optimization techniques such as two-stage optimization, embedding transformation and switching LQR design were discussed. Note that the research on optimal control of switched and hybrid systems is still an active area, the survey is not attempting to include all the results in the literature but primarily aims to show the essential ideas and trends in this field. We apologize for any omissions due to space limitation or because of not being aware of them.

It is worth to note that several software packages are available to compute the optimal control solutions of switched and hybrid systems. Multi-Parametric Toolbox (MPT) (Kvasnica et al.) is a free MATLAB toolbox for design, analysis and deployment of optimal controllers for constrained linear, nonlinear and hybrid systems. Efficiency of the code is guaranteed by the extensive library of algorithms from the field of computational geometry and multi-parametric optimization. YALMIP (Lofberg 2004) features an intuitive and flexible modeling language for solving optimization problems. The main emphasis is on convex conic programming (linear, quadratic, second order cone and semi-definite programming), but it also supports integer programming and non-convex problems. Moreover, YALMIP can be used together with the MPT toolbox to setup and solve multiparametric optimization problems. CDP (Convex Dynamic Programming) Tool is another MATLAB toolbox developed to solve hybrid optimal control problems. The user manual can be found in Hedlund and Rantzer (1999a).

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