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HIDDEN MODES OF INTERCONNECTED SYSTEMS IN CONTROL DESIGN

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ABSTRACT

A complete treatment of the hidden modes of interconnected systems in control design is presented, which extends and unifies several results which have appeared in the literature in the past decades. The uncontrollable and/or unobservable hidden modes of one and two degrees of freedom control systems are characterized in terms of transfer matrices of the interconnected subsystems and in terms of design parameters. This characterization leads directly to design conditions, which can be used to adequately control the hidden modes; thus, avoiding unnecessarily high order controllers and undesirable behavior. The methods used are based on polynomial matrix internal descriptions, however, all results are expressed so that they can be directly used in control design. Internal stability is guaranteed using a novel stability theorem which adds significant insight to the problem. A method is also presented to characterize the hidden modes of any interconnected system.

I. INTRODUCTION

The hidden modes of a compensated system correspond to the compensated system's eigenvalues which are uncontrollable and/or unobservable from a given input or output, respectively. The hidden modes for single degree of freedom and for particular two degrees of freedom controlled systems have been studied in the literature [1–5,17–23,31–33]. In this paper we characterize the hidden modes for the general linear two degrees of freedom controlled system in terms of the frequency domain control design tools: transfer matrices and design parameters. The hidden modes can, of course, be characterized using internal descriptions [4,11,17–20], and this is useful mainly in the analysis of control systems. In frequency domain control design methods, where transfer matrices and design parameters are used, these characterizations are not as helpful. This characterization of the hidden modes in terms of the design tools leads directly to ***design conditions***, which can be used to adequately control the hidden modes in control design; thus, avoiding unnecessarily high order controllers and undesirable behavior. It is recognized that if the interconnected system is internally stable then the hidden modes, if any, will be stable. By undesirable behavior we mean transient responses introduced unintentionally in the design and phenomena such as ringing (see [11] and Example 5.2).

In this paper a complete treatment of the hidden modes of interconnected systems in control design is presented, which extends and unifies several results which have appeared in the literature in the past decades. The methods used are based on polynomial matrix internal descriptions, however, all results are expressed so that they can be directly used in control design. Internal stability is guaranteed using a novel stability theorem which adds significant insight to the problem.

The two degrees of freedom controller, C, provides a unifying framework in approaching complicated control problems involving multiple objectives, in a manner which is configuration independent. Several researchers have utilized C in a time domain state-space formulation (e.g. Bengtsson [7]). Using a transform domain formulation, C has been incorporated by Pernebo [8] and more recently, by Youla and Bongiorno [6], Desoer and Gustafson [9], Vidyasagar [10], Desoer and Gündes [12], and Sugie and Yoshikawa[13]. The pole-placement algorithm of Åström in [11] also uses C but for scalar plants only. There is much renewed interest in the two degrees of freedom controller that is due to more demanding control problems and to recent advances in understanding and effectively utilizing such control laws. Assuming that the plant and controller are controllable and observable, then the hidden modes of the controlled system are introduced exclusively by the interconnections. Under this assumption the hidden modes are completely characterized in terms of the transfer matrices and design parameters. The implementation of the controller, C, is usually done by interconnecting available subcontrollers, where each subcontroller is designed to handle a particular task such as stability and regulation. Therefore, the resulting controller is not necessarily controllable and observable, and it introduces additional hidden modes; these are also characterized. In addition, given any particular interconnected system we introduce a systematic method to characterize the hidden modes; for this we use an aggregate system representation [4]. The proofs are found in [35].

II. PRELIMINARIES AND HIDDEN MODES OF SINGLE DEGREE OF FREEDOM SYSTEMS

An interconnection of irreducible systems is said to be completely characterized by its proper rational transfer matrix if and only if an internal description of the overall system is controllable from the input and observable from the output. If a transfer matrix does not completely characterize an interconnection of subsystems, then the uncontrollable eigenvalues from the input and the unobservable eigenvalues from the output correspond to the ***hidden modes*** of the overall system. Notice that the hidden modes are due exclusively to the interconnections since every subsystem is assumed to be irreducible.

In classical control design of scalar systems it is straightforward to characterize the hidden modes in terms of pole-zero cancellations. In the frequency domain control design of multivariable systems, the hidden modes can also be characterized by considering "pole-zero cancellations." In this case, however, the characterization is not as direct mainly due to the fact that "pole-zero cancellations" are not as well defined in the multivariable case, and also because of the difficulty in associating hidden modes with specific cancellations. Results that refer to particular control configurations have been reported in the literature [1–3,21]. In [16], these results have been formalized and extended; they are the basis of the results presented here.

Consider the following polynomial matrix description (PMD) of the controlled system:

$$\mathcal{P}(s)z(s) = \mathcal{Q}(s)u(s), \quad y(s) = \mathcal{R}(s)z(s)+\mathcal{W}(s)u(s), \tag{2.1}$$

where $\mathcal{P}(s)$, $\mathcal{Q}(s)$, $\mathcal{R}(s)$, $\mathcal{W}(s)$ are polynomial matrices; the quadruple $\{\mathcal{P}(s), \mathcal{Q}(s), \mathcal{R}(s), \mathcal{W}(s)\}$ denotes the system represented in (2.1). For the system described in (2.1), the uncontrollable (unobservable) modes from u (y) correspond to the roots of the determinant of a g.c.l.d. of $(\mathcal{P}(s),\mathcal{Q}(s))$ (g.c.r.d. of $(\mathcal{R}(s),\mathcal{P}(s))$) [17,-18], where g.c.l.(r.)d. denotes greatest common left (right) divisor. To derive the desired characterizations, a better way to express these known conditions is given in Lemma 2.1.

LEMMA 2.1. The system described by $\{\mathcal{P}(s), \mathcal{Q}(s), \mathcal{R}(s), \mathcal{W}(s)\}$ is controllable from u (observable from y) if and only if the McMillan degree of the transfer matrix from u to z ($\mathcal{Q}(s)u(s)$ to y) is the same as the degree of $|\mathcal{P}(s)|$.□

Lemma 2.1 specifies the products of transfer matrices in which a cancellation may result in a hidden mode; the uncontrollable (unobservable) modes are associated with cancellations in $\mathcal{P}(s)^{-1}\mathcal{Q}(s)$ $(\mathcal{R}(s)\mathcal{P}(s)^{-1})$. In the following, appropriate transformations are used to map these products into products of transfer matrices of the interconnected subsystems. Transformations which yield equivalent polynomial matrix descriptions are used. In particular, we apply transformations that maintain system equivalence in the Rosenbrock sense [17].

It is well known that **cancellations** in products of transfer matrices are not simple extensions of scalar pole–zero cancellations. For example, it is possible to have a cancellation where a pole of one transfer matrix does not cancel with a zero of another transfer matrix (the zeros of transfer matrices are the transmission zeros of the system) as in

$$T_1(s)T_2(s) = [1, \ -1]\left[\frac{2s+3}{(s+1)(s+2)}, \ \frac{1}{s+2}\right]^t = \frac{1}{s+1} \tag{2.2}$$

In (2.2), $T_1(s)$ has no zeros and the pole of T_2 at -2 cancels in $T_1(s)T_2(s)$. Let $\delta T_i(s)$ denote the McMillan degree of $T_i(s)$, $i=1,2$. Notice that $\delta T_1(s)=0$ and $\delta T_2(s)=2$, while $\delta T_1T_2(s)=1$. This reduction in the McMillan degree confirms the fact that a pole was canceled in $T_1(s)T_2(s)$; the pole of T_2 at -2. This pole corresponds to a hidden mode from the input and/or the output, for example, when $T_1(s)T_2(s)$ denotes the transfer matrix of the cascade connection of $T_2(s)$ followed by $T_1(s)$.

It is now clear that cancellations in products of transfer

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matrices should be taken as ***pole cancellations*** rather than pole-zero cancellations. Notice that a pole of a transfer matrix $T_2(s)$ cancels with a zero of a not necessarily square transfer matrix $T_1(s)$ in $T_1(s)T_2(s)$ only if $\mathrm{rank}(\tilde{N}_1(s))$ is full where $T_1=\tilde{D}_1^{-1}\tilde{N}_1$ is a left coprime polynomial factorization (l.c.). Moreover, the canceled poles in a product of transfer matrices correspond to hidden modes from the input and/or the output if and only if these poles are eigenvalues of the interconnected system. Additional observations on "multivariable cancellations" can be found in [16,19–23,30].

The following result will be used in the characterization of hidden modes [16]. Consider

$$T_{12} = [T_1^t,\ I_m]^t T_2 \tag{2.3}$$

$$T_{34} = T_3[T_4,\ I_m] \tag{2.4}$$

where T_1, T_2, T_3, and T_4 are p×m, m×r, p×m, and m×r, respectively. Note that, T_{12} can be considered to be the transfer matrix of the cascade connection of system S_2 followed by S_1, where each system is completely characterized by their transfer matrices T_2 and $[T_1^t,\ I_m]^t$, respectively. It is of interest to characterize the cancellations that result in a reduction in the McMillan degree of the resulting transfer matrix, that is, $\delta T_{12} < \delta[T_1^t, I_m]^t + \delta T_2$ and $\delta T_{34} < \delta T_3 + \delta[T_4, I_m]$. This is done in the following lemma.

<u>LEMMA 2.2.</u> The cancellations that result in a reduction in the McMillan degree of the transfer matrices in (2.3) ((2.4)) are given by the poles of T_1 (T_4) which cancel in the product T_1T_2 (T_3T_4).□

In view of Lemma 2.2, $\delta T_{12}=\delta[T_1^t, I_m]^t+\delta T_2$ and $\delta T_{34}=\delta T_3+\delta[T_4, I_m]$ if and only if (D_1,N_2) is l.c. and $(\tilde{N}_3,\tilde{D}_4)$ is right coprime (r.c.), where $T_1=N_1D_1^{-1}$, $T_2=N_2D_2^{-1}$, $T_3=\tilde{D}_3^{-1}\tilde{N}_3$, and $T_4=\tilde{D}_4^{-1}\tilde{N}_4$ are coprime factorizations.

We now apply the method described above to characterize the hidden modes of the cascade, parallel, and feedback systems shown in Figure 2.1. If the systems, S_i, i=1,2, are assumed to be completely described by their proper transfer matrices T_i, then the hidden modes have been characterized in the literature before (for example, see [1–4,17–23,31–33]). Let S_1 be completely characterized by T_1 and consider the case when S_2 is not completely described by T_2. Assume that the feedback system is well defined, that is, $|I+T_1T_2|\neq 0$, and that every input-output map is proper.

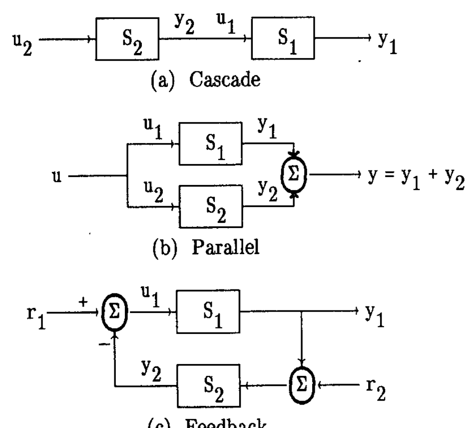

Figure 2.1. Basic interconnected systems.

A polynomial matrix description for S_2 is

$$S_2:\ \mathcal{P}_2 z_2 = \mathcal{Q}_2 u_2,\quad y_2 = \mathcal{R}_2 z_2, \tag{2.5}$$

where the uncontrollable (unobservable) modes from u_2 (y_2) correspond to the roots of the determinant of a g.c.l.d. of $(\mathcal{P}_2,\mathcal{Q}_2)$ (g.c.r.d. of $(\mathcal{R}_2,\mathcal{P}_2)$).

<u>Lemma 2.3.</u>

CASCADE: The uncontrollable modes from u_2 correspond to the uncontrollable modes of S_2 and to the poles of T_1 that cancel in T_1T_2. The unobservable modes eigenvalues of S_2 from y_2 that cancel in $T_1(\mathcal{R}_2\mathcal{P}_2^{-1})$.

PARALLEL: The uncontrollable modes from u correspond to the uncontrollable modes of S_2 from u_2 and to the controllable eigenvalues of S_2 that cancel with poles of T_1 in $(\mathcal{P}_2^{-1}\mathcal{Q}_2)D_1$. The unobservable modes from y correspond to the unobservable modes of S_2 from y_2 and to the observable eigenvalues of S_2 from y_2 which cancel with poles of T_1 in $\tilde{D}_1(\mathcal{R}_2\mathcal{P}_2^{-1})$.

FEEDBACK: The uncontrollable modes from r_1 correspond to the uncontrollable modes of S_2 and to the controllable eigenvalues of S_2 from u_2 that cancel in $(\mathcal{P}_2^{-1}\mathcal{Q}_2)T_1$. The unobservable modes from y_1 correspond to the unobservable modes of S_2 and to the observable eigenvalues of S_2 from y_2 that cancel in $T_1(\mathcal{R}_2\mathcal{P}_2^{-1})$.□

Lemma 2.3 demonstrates that when S_2 is not completely described by T_2, the interconnected systems considered here maintain the hidden modes of S_2 only from appropriate inputs and outputs. Furthermore, because of the interconnection additional hidden modes could be introduced.

III. STABILITY THEOREM. PARAMETERIZATIONS

The two degrees of freedom linear controller

$$u = C[y^t, r^t]^t = [-C_y, C_r][y^t, r^t]^t, \tag{3.1}$$

where $C = [-C_y, C_r]$ proper, generates the plant input u by independently processing the plant output y and the external reference input r as seen in Figure 3.1;

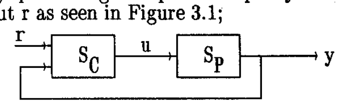

Figure 3.1. The controlled system.

S_P is the linear plant described by y=Pu with P its proper transfer matrix and S_c is the controller described in (3.1). It is assumed that $|I+PC_y|=|I+C_yP|\neq 0$ and that every input-output map is proper. Under these assumptions, the controlled system is said to be internally stable if the inverse of the denominator matrix in a polynomial matrix description is stable. If the controlled system is internally stable, we say that S_c is an internally stabilizing controller for S_P.

A significant step towards better understanding the role of C in plant compensation was recently accomplished by parametrically characterizing all internally stabilizing two degrees of freedom controllers C; thus extending the results on parametric characterization of all feedback controllers C_y [24–28,1,2,14–16] which have greatly contributed to control design methods. All internally stabilizing controllers C can be parametrically characterized using two independent stable parameters K and X as

$$C = (x_1 - K\tilde{N})^{-1}[-(x_2 + K\tilde{D}),\ X], \tag{3.2}$$

where $\tilde{N}$, $\tilde{D}$, x_1, x_2 are polynomial matrices, and they are derived from coprime fractional representations of the plant

$$P = ND^{-1} = \tilde{D}^{-1}\tilde{N}, \tag{3.3}$$

and the associated Bezout–Diophantine equation

$$x_1D + x_2N = I. \tag{3.4}$$

In (3.2), K must be such that $|x_1-K\tilde{N}| \neq 0$, and for C proper need $D(x_2+K\tilde{D})$ proper and $D(x_1-K\tilde{N})$ biproper ($D(x_1-K\tilde{N})$ and its inverse proper). In [6], x_1 and x_2 satisfy the Diophantine equation $x_1\tilde{D}+x_2\tilde{N} = D_0$, with D_0 a polynomial matrix ($|D_0|$ Hurwitz). In [9,10], (3.2) involves proper and stable matrices $\tilde{N}$, $\tilde{D}$, x_1, x_2, K, and X. The parameter K in (2) is the well known parameter used in the characterization of all stabilizing feedback controllers C_y in [24,27]. The parameter X is actually the response parameter used by Antsaklis and Sain [1] (and Liu and Sung [29]) to parametrically characterize feedback controllers in an error feedback setting. If Dz = u, y = Nz is an internal polynomial matrix representation of the plant P, then it can be shown that

$$z = Xr, \tag{3.5}$$

that is, X is the transfer matrix between the input r and the partial state z of the plant.

It is advantageous to study internal stability of the system in Figure 3.1 in a novel alternative way [21].

<u>THEOREM 3.1.</u> The compensated system is internally stable if and only if

(i) $u = -C_yy$ internally stabilizes the system y = Pu, and

(ii) C_r is such that $M := (I + C_yP)^{-1}C_r$ satisfies $D^{-1}M = X$, a stable rational, where C_y satisfies (i) and $P = ND^{-1}$ a right coprime polynomial factorization.□

Theorem 3.1 separates the role of C_y, the feedback part of C, from C_r in achieving internal stability. Clearly if only feedback action is considered, only (i) is of interest; and if open loop control is desired, $C_y=0$, (i) implies that P must be stable, and $C_r=M$ must satisfy (ii). In (ii) the parameter M (=DX) appears rather naturally and in (i) the way is open to use any desired feedback parameterization, not necessarily K of [6,9–10].

From Theorem 3.1 we can directly characterize the input-output maps attainable from r with internal stability. In particular, consider the two maps described by y=Tr and u=Mr which are characterized in Theorem 3.2.

<u>THEOREM 3.2</u>. A pair (T,M) is realizable with internal stability via a two degrees of freedom configuration if and only if (T,M) = (NX,DX) with X stable.□

There are many choices in parametrically characterizing all feedback stabilizing controllers C_y and these are extensively discussed by Antsaklis and Sain in [2]. The stabilizing controllers C can therefore be expressed, in addition to (3.2) as (for example):

$$C = (I-QP)^{-1}[-Q,\ DX] = ((I-LN)D^{-1})^{-1}[-L,\ X], \tag{3.6}$$

where $Q = DL$, $DX = M$ with L, X stable and $D^{-1}(I-QP) = (I-LN)D^{-1}$ stable ($|I-QP| \neq 0$ or $|I-LN| \neq 0$). Parametric characterizations of all internally stabilizing controllers C, proper and nonproper are given in (3.6). For C proper, M and Q are chosen proper and such that (I–QP) is biproper; note that if P is strictly proper, Q proper always implies that $(I-QP)^{-1}$ is proper. Notice that L or Q in (3.6) must satisfy certain conditions, in addition to being stable, in contrast to K in (3.2); however, alternative to K parameterizations, such as in (3.6), are very useful, since they do have certain additional desirable properties (see [2]).

The relations between the parameters are

$$L = x_2+K\tilde{D} = D^{-1}Q, \quad Q = DL = C_y(I+PC_y)^{-1}=(I+C_yP)^{-1}C_y$$

$$X = (x_1-K\tilde{N})C_r = D^{-1}M, \quad M = DX = (I+C_yP)^{-1}C_r. \tag{3.7}$$

These relations will be useful in Section 4.2 where the hidden modes of two degrees of freedom systems are characterized in terms of these parameters.

IV. HIDDEN MODES IN TWO DEGREES OF FREEDOM CONTROLLED SYSTEMS

In this section, the hidden modes of two degrees of freedom controlled systems as depicted in Figure 4.1.1, will be studied. In Section 4.1, the hidden modes from given inputs and outputs will be characterized in terms of transfer matrices. This characterization is done when S_p and S_c are completely described by their transfer matrices, and when S_p is completely described by its transfer matrix, but S_c is not. In Section 4.2, the hidden modes are characterized in terms of the design parameters: K, X, and L when S_p and S_c are completely described by their transfer matrices. Using these characterizations we then give conditions in terms of the parameters of interest to avoid the introduction of hidden modes. These conditions can be incorporated in the control system design.

4.1 <u>Hidden Modes in Terms of I/O Maps</u>

Consider

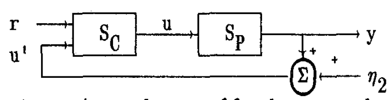

Figure 4.1.1. A two degrees of freedom control system.

where the vector of fictitious inputs η_2 is introduced to help with the interpretation of the uncontrollable hidden modes; the other variables were described in Section III.

First, consider S_c to be completely described by its transfer matrix, that is, S_c is controllable from $[u'^t, r^t]^t$ and observable from u. A polynomial matrix description for S_c is

$$S_c: \qquad \tilde{D}_cz_c = -\tilde{N}_yy+\tilde{N}_rr-\tilde{N}_y\eta_2,\ u=z_c \tag{4.1.1}$$

where $C = \tilde{D}_c^{-1}[-\tilde{N}_y,\ \tilde{N}_r]$ is left coprime and $u' = \eta_2+y$. Two possible polynomial matrix descriptions for S_p are

$$S_P: \qquad Dz = u,\ \ y = Nz \tag{4.1.2}$$

$$S_P: \qquad \tilde{D}\tilde{z} = \tilde{N}u,\ \ y = \tilde{z} \tag{4.1.3}$$

where $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$ are coprime factorizations. Combining (4.1.1) and (4.1.2) gives a polynomial matrix description for the two degrees of freedom controlled system

$$D_0z = \tilde{N}_rr-\tilde{N}_y\eta_2,\ [y^t,\ \ u^t]^t = [N^t,\ \ D^t]^tz \tag{4.1.4}$$

where $D_0 = \tilde{D}_cD+\tilde{N}_yN$. Since S_P and S_c are assumed to be completely characterized by their transfer matrices the hidden modes are due exclusively to the interconnection.

A preliminary characterization of the hidden modes follows directly from (4.1.4): The uncontrollable modes from r (η_2) correspond to the poles of D_0^{-1} that cancel in $D_0^{-1}\tilde{N}_r$ ($D_0^{-1}\tilde{N}_y$). The unobservable modes from y (u) correspond to the poles of D_0^{-1} that cancel in ND_0^{-1} (DD_0^{-1}). This characterization gives insight into the controllability and observability properties of two degrees of freedom systems. For example, notice that the controlled system is observable from $[y^t, u^t]^t$, that is, the unobservable modes from y are observable from u and vice versa.

Notice that even though S_c is completely characterized by C, there could be uncontrollable modes from r or from y. However, the uncontrollable modes from r of S_c are controllable from y and vice versa. Let G_y be a g.c.l.d. of $(\tilde{D}_c,\tilde{N}_y)$ and let G_r be a g.c.l.d. of $(\tilde{D}_c,\tilde{N}_r)$ then

$$[\tilde{D}_c,\ \tilde{N}_y] = G_y[\tilde{D}_{cy},\ \tilde{N}_{cy}] \tag{4.1.5}$$

$$[\tilde{D}_c,\ \tilde{N}_r] = G_r[\tilde{D}_{cr},\ \tilde{N}_{cr}] \tag{4.1.6}$$

where $(\tilde{D}_{cy},\tilde{N}_{cy})$ and $(\tilde{D}_{cr},\tilde{N}_{cr})$ are coprime polynomial pairs. The roots of $|G_r|$ ($|G_y|$) correspond to uncontrollable modes from r (y) of S_c. Furthermore, the roots of $|G_y|$ are closed-loop eigenvalues, but the roots of G_r are not ($|D_0|=\alpha|G_y|\ |\tilde{D}_{cy}D+\tilde{N}_{cy}N|$, $\alpha\in\mathbb{R}$). This implies that no uncontrollable modes of S_c from r will be uncontrollable from r of the two degrees of freedom controlled system.

Before giving the main result in Theorem 4.1.1 it is useful to characterize the poles of $(I+PC_y)^{-1}$ and of $(I+C_yP)^{-1}$; the characterization is used to determine when a cancellation of poles of $(I+PC_y)^{-1}$ and of $(I+C_yP)^{-1}$ can correspond to a hidden mode from an input and/or output.

<u>LEMMA 4.1.1.</u> The following relations are true.

(i) {poles of $(I+PC_y)^{-1}$} = {{closed-loop eigenvals.} – {uncontrollable eigenvals. from η_2} – {unobservable eigenvals. from y} + {eigenvals. that are uncontrollable from η_2 and unobservable from y}}.

(ii) {poles of $(I+C_yP)^{-1}$} ⊂ {{closed-loop eigenvals.) – unobservable eigenvals. from u} – {roots of $|G_y|$ that do not correspond to unobservable modes from u}}□

The hidden modes are determined by considering cancellations in the products of transfer matrices given in Lemma 4.1.2.

<u>LEMMA 4.1.2.</u> The hidden modes are characterized by considering cancellations in the following products of transfer matrices.

$$\text{Unobservable modes from y:} \quad (I+PC_y)^{-1}P[\tilde{D}_c^{-1},\ I]. \tag{4.1.7}$$

$$\text{Unobservable modes from u:} \quad (I+C_yP)^{-1}[\tilde{D}_c^{-1},\ I]. \tag{4.1.8}$$

$$\text{Uncontrollable modes from r:} \quad [I,\ P^t]^t(I+C_yP)^{-1}\tilde{D}_c^{-1}\tilde{N}_r.\square \tag{4.1.9}$$

The main result when S_P and S_c are completely characterized by their transfer matrices is given next.

<u>THEOREM 4.1.1.</u> The hidden modes are characterized as follows.

(i) The unobservable modes from y (u) correspond to the poles of C (P) that cancel in PC ($\tilde{N}_yP$).

(ii) The uncontrollable modes from r correspond to the poles of $(I+C_yP)^{-1}$ that cancel in $(I+C_yP)^{-1}C_r$, and to the poles of P that cancel in both PM and C_yP.□

The next two corollaries specialize the conditions in Theorem 4.1.1 for two single degree of freedom configurations. First, we consider the error feedback configuration where $u = -C_yy+C_yr-C_y\eta_2$ ($C_r = C_y$), and then the feedback configuration where $u = -C_yy+r-C_y\eta_2$ ($C_r = I$).

<u>COROLLARY 4.1.1.</u> In the error feedback configuration, the unobservable modes from y (u) correspond to poles of C_y (P) that cancel in PC_y (C_yP). The uncontrollable modes from r correspond to the poles of P that cancel in PC_y.□

<u>COROLLARY 4.1.2.</u> In the second feedback configuration the unobservable modes from y (u) correspond to the poles of C_y (P) that cancel in PC_y (C_yP). The uncontrollable modes from r

correspond to the poles of C_y that cancel in C_yP.□

Theorem 4.1.1 characterizes the hidden modes introduced by the interconnection of systems in Figure 4.1.1. It is also of interest to characterize the hidden modes when the controller is not completely characterized by its transfer matrix. The conditions for the general case of uncontrollability and unobservability, which requires another internal description of S_c, are given in Theorem 4.1.2.

<u>THEOREM 4.1.2.</u> The uncontrollable modes of S_c from $[u'^t, r^t]^t$ will be uncontrollable from r, and the unobservable modes of S_c from u will be unobservable from y.□

The conditions in Theorem 4.1.1 are illustrated in Example 4.1.1.

<u>Example 4.1.1.</u> Consider the plant

$$P = \frac{s-1}{(s-2)(s+1)} \tag{4.1.10}$$

and let the compensator be given by

$$C = \left[\frac{-8(s+1)}{(s-13/3)}, \frac{(s+1)^2(s+2/3)}{(s-13/3)(s+\alpha)(s+\beta)}\right]. \tag{4.1.11}$$

The resulting transfer function is $T = (s-1)/((s+\alpha)(s+\beta))$. The closed-loop characteristic polynomial is given by $\psi_2 = (s+\alpha)(s+\beta)(s+1)^2(s+2/3)$. Comparing ψ_2 to the resulting transfer function T gives the hidden modes to be $\{-1,-1,-2/3\}$. The nature of the hidden modes is directly characterized via Theorem 4.1.1. The three hidden modes are uncontrollable from r, one of the closed-loop eigenvalues at -1 is unobservable from u, and there are no unobservable eigenvalues from y.□

4.2. <u>Hidden Modes in Terms of Design Parameters</u>

The results in the last section could be used to give conditions to avoid the introduction of hidden modes, but they would not be simple to implement in control design. In this section the hidden modes will be characterized in terms of the parameters utilized in the design of a control system, leading directly to design conditions to avoid unnecessary hidden modes. In particular, the hidden modes will be characterized in terms of K, X, and L, which were used in Section III to parameterize the internally stabilizing controllers.

The results in the following two lemmas are used to characterize the hidden modes in terms of the design parameters. In Lemma 4.2.1, the poles of the parameters of interest are characterized. In Lemma 4.2.2, the uncontrollable modes from η_2 are characterized.

First, let G_P be a g.c.l.d. of $(\tilde{D}_k, \tilde{N}_{cy})$, where $K = \tilde{D}_k^{-1}\tilde{N}_k$ is l.c. Then it can be shown that there exist polynomial matrices $\tilde{D}_l$ and $\tilde{N}_l$ such that

$$[\tilde{D}_k, \tilde{N}_{cy}] = G_P[\tilde{D}_l, \tilde{N}_l], \tag{4.2.1}$$

where $L = \tilde{D}_l^{-1}\tilde{N}_l$ is a coprime factorization.

<u>LEMMA 4.2.1.</u> The following relations are true.

(i) {poles of K} = {{closed-loop eigenvals.} − {roots of $|G_y|$}}

(ii) {poles of L} = {{closed-loop eigenvals.} − {uncontrollable eigenvals. from η_2}}

(iii) {poles of X} = {{closed-loop eigenvals.} − {uncontrollable eigenvals. from r}}□

<u>LEMMA 4.2.2.</u> The uncontrollable modes from η_2 correspond to the roots of $|G_y|\ |G_P|$, where the poles of G_P^{-1} are poles of P that cancel in PC_y. The poles of G_P^{-1} are also given by the poles of P that do not cancel in $(I-LN)D^{-1}$.□

The characterization of hidden modes in terms of the design parameters is given next.

<u>THEOREM 4.2.1.</u> The unobservable modes from y correspond to the poles of [X, L] that cancel in N[X, L]. The poles of [X, L] that cancel in D[X, L] and the poles of P (in G_P^{-1}) that cancel in $D_0^{-1}[\tilde{N}_y, \tilde{N}_r]$ correspond to unobservable modes from u. The uncontrollable modes from r correspond to the poles of P in G_P^{-1} and to the poles of L that are not poles of X.□

The next corollary specializes the conditions in Theorem 4.2.1 to the error feedback configuration.

<u>COROLLARY 4.2.1.</u> In the error feedback configuration, the unobservable modes from u correspond to the poles of L (L=X) which cancel in NL. The uncontrollable modes from r correspond to the poles of P that do not cancel in $(I-LN)D^{-1}$, that is, the poles of G_P^{-1}.□

The conditions in Corollary 4.2.1 agree with known results for the error feedback configuration in [2].

<u>Remark 4.2.1.</u> A characterization of the closed-loop characteristic polynomial in terms of the parameters used in this section is given by

$$\psi_2 = k|G_y|\ |G_{P1}|\ |G_{P2}|\ |\tilde{D}_{l1}|\ |\tilde{D}_{l2}|,\ k \in \mathbb{R} \tag{4.2.2}$$

where $G_P = G_{P1}G_{P2}$ and $\tilde{D}_l = \tilde{D}_{l1}\tilde{D}_{l2}$. The roots of $|G_y|\ |G_{P1}|\ |\tilde{D}_{l1}|$ correspond to the controllable eigenvalues from r; hence, they are poles of X. The roots of $|G_{P2}|\ |\tilde{D}_{l2}|$ correspond to the uncontrollable eigenvalues from r. The controllable eigenvalues from r that are unobservable from y correspond to the poles of X which cancel in NX, and these eigenvalues can correspond only to some of the roots of $|G_y|\ |\tilde{D}_{l1}|$. The uncontrollable eigenvalues from r that are unobservable from y correspond to the roots of $|\tilde{D}_{l2}|$ since none of the plant poles can correspond to unobservable eigenvalues from y.

The final result of this section gives the design conditions that can be used to avoid unnecessary hidden modes. Notice that these conditions could be used the other way around when it is desirable to introduce a cancellation that does not affect internal stability. These conditions follow directly from Theorem 4.2.1.

<u>DESIGN CONDITIONS FOR NO HIDDEN MODES:</u> *To avoid unobservable modes from y do not choose poles of [X, L] that cancel in N[X, L]. To avoid uncontrollable modes from r make all the poles of L and the poles of P in G_P^{-1} poles of X. To avoid unobservable modes from u don't choose poles of [X, L] as poles of P.*

These conditions are interpreted below for single-input, single-output feedback systems. Consider

$$P = \frac{n}{d} = \frac{n}{d_l g_{P1} g_{P2}},\ L = \frac{n_l}{d_l},\ X = \frac{n_x}{d_x} = \frac{n_x}{g_y g_{P1} d_{l1}}, \tag{4.2.3}$$

where $g_P = g_{P1}g_{P2}$ and $d_l = d_{l1}d_{l2}$ and the variables are the same as before except that the factorizations are given in lower case. The conditions in Theorem 4.2.1 for single-input, single-output systems can be given as follows.

<u>COROLLARY 4.2.2.</u> The uncontrollable modes from r correspond to the zeros of g_{P2} and d_{l2}. The unobservable modes from y correspond to the zeros of both g_y and d_l that are zeros of the plant. The unobservable modes from u correspond to all the zeros of g_{P1}, and to zeros of g_y and d_l that are poles of the plant.□

<u>In order to avoid hidden modes from r and y, choose</u> $d_{l2} = 1$ <u>and</u> $g_{P2} = 1$, <u>and choose</u> g_y <u>and</u> d_l <u>to have no zeros equal to zeros of the plant, respectively. To avoid hidden modes from u, choose</u> $g_P = 1$, <u>and</u> g_y <u>and</u> d_l <u>to have no zeros equal to the poles of the plant.</u> In terms of these desired values for the parameters, the expressions for C_y and C_r are given by

$$C_y = \frac{g_P n_l}{d_{cy}} = \frac{g_{P1} n_l}{d_{cy}},\quad C_r = \frac{d_l g_P}{d_{cy}} \frac{n_x}{g_y g_{P1} d_{l1}} = \frac{n_x}{g_y d_{cy}}, \tag{4.2.4}$$

and if $C = d_c^{-1}[n_y, n_r]$, then $d_c = g_y d_{cy}$, $n_y = g_y g_{P1} n_l$, and $n_r = n_x$. These results are illustrated in the following examples.

In Example 4.2.1 the hidden modes of the feedback system considered in Example 4.1.1 are characterized in terms of the design parameters.

<u>Example 4.2.1.</u> For the plant and compensator defined in (4.1.7) and (4.1.8) we have the following parameters

$$Q = \frac{8(s-2)}{s+2/3},\ L = \frac{8}{(s+1)(s+2/3)},\ X = \frac{1}{(s+\alpha)(s+\beta)} \tag{4.2.5}$$

$$G_y = (s+\alpha)(s+\beta),\quad G_P = s+1 = G_{P2}, \tag{4.2.6}$$

$$G_{P1} = 1,\ \tilde{D}_l = \tilde{D}_{l2},\ \text{and}\ \tilde{D}_{l1} = 1. \tag{4.2.7}$$

The unobservable modes from y correspond to the cancellations in:

$$(s-1)\left[\frac{1}{(s+\alpha)(s+\beta)}, \frac{8}{(s+1)(s+2/3)}\right]. \tag{4.2.8}$$

Since there are no cancellations in (4.2.8), there are no unobservable modes from y. The unobservable modes from u correspond to cancellations in:

$$(s-2)(s+1)\left[\frac{1}{(s+\alpha)(s+\beta)}, \frac{8}{(s+1)(s+2/3)}\right]. \tag{4.2.9}$$

From (4.2.9), one of the poles of the plant at s=−1 corresponds to an unobservable mode from u. The uncontrollable modes from r correspond to the poles of G_P^{-1} and of L that are not poles of X. So the uncontrollable modes from r correspond to $\{-1, -1, -2/3\}$.

In Example 4.2.2 the conditions given in this section are used to redesign C so that the number of hidden modes is a minimum.

Example 4.2.2. The controller considered in Example 4.1.1 introduces at least three hidden modes. If less hidden modes are desired, a new compensator C needs to be designed that attains the desired transfer function $T = (s-1)/((s+\alpha)(s+\beta))$. Since T is the same and $T = NX = (s-1)X$, X is the same as in (4.2.6), we need to design L.

At this time it should be noted that no internally stabilizing single degree of freedom controller can attain the desired T. Therefore, the two degree of freedom design will yield at least one hidden mode. If no hidden modes are desired, the choice for T should be reconsidered.

In order to minimize the number of hidden modes, the poles of L should be the same as the poles of X and no poles of L should be zeros of the plant. Then a possible choice for L is

$$L = \frac{\kappa}{(s+\alpha)(s+\beta)}, \tag{4.2.10}$$

where κ is a constant. The conditions for the compensator to be internally stabilizing and proper are X, L, and $(1-LN)D^{-1}$ must be stable (see Section III). Only the latter one needs further checking:

$$(1-LN)D^{-1} = \frac{(s+\alpha)(s+\beta) - (s-1)\kappa}{(s-2)(s+1)(s+\alpha)(s+\beta)}. \tag{4.2.11}$$

The transfer function in (4.2.11) is stable if and only if $(2+\alpha)(2+\beta)=\kappa$, i.e., $\alpha=(\kappa-4-2\beta)/(2+\beta)$. Then, for $\alpha>0$ need

$$0 < \beta < \frac{\kappa-4}{2} \tag{4.2.12}$$

or choose $\alpha > 0$, $\beta > 0$ and $\kappa = (2+\alpha)(2+\beta)$. By choosing κ in this way, a controller $C=[-C_y, C_r]$ that attains the desired closed-loop transfer function is

$$C = \left[\frac{-\kappa(s-2)(s+1)}{(s+\alpha)(s+\beta)-(s-1)\kappa}, \frac{(s-2)(s+1)}{(s+\alpha)(s+\beta)-(s-1)\kappa}\right], \tag{4.2.13}$$

where there is a pole-zero cancellation at $s=-2$ in the expressions for C_y and C_r. Suppose $\alpha=1$, $\beta=3$, then $\kappa=15$ and

$$C = \left[-\frac{15(s+1)}{s-9}, \frac{s+1}{s-9}\right]. \tag{4.2.14}$$

For C in (4.2.14) there is only one hidden mode due to the interconnection of the controller and the plant; the pole of P at $s=-1$ corresponds to an uncontrollable mode from r and to an unobservable mode from u.□

V. HIDDEN MODES IN INTERCONNECTED SYSTEMS

The characterization of hidden modes in terms of transfer matrices of a system interconnection can be done starting with the results in Section 4.1. For a complex interconnection of systems it may be simpler to apply Lemma 2.1 to a polynomial matrix description of the interconnected system. A systematic method to do this is explained in this section. For illustration consider the {R;G,H} controlled system in Figure 5.1,

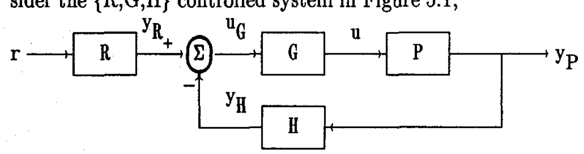

Figure 5.1. An {R;G,H} controlled system.

where the interconnected subsystems are completely described by their transfer matrices P, R, H, and G. The {R;G,H} controller is an implementation of a two degrees of freedom compensator, where $C_y = GH$ and $C_r = GR$.

The interconnected systems will be represented as in [4, Chapter 4], where a systematic study of internal stability of system interconnections is presented. The aggregate system representation of the {R;G,H}) controlled system is given in Figure 5.2,

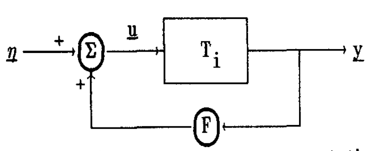

Figure 5.2. Aggregate system representation.

where T_i = block diag {P, G, H, R}; $\underline{y} = [y_P^t, y_g^t, y_h^t, y_r^t]^t$ and $\underline{u} = [u_P, u_g, u_h, u_r]$ are the vectors of outputs and inputs of each subsystem, respectively; $\underline{\eta}$ is a vector of exogenous inputs entering at the input of each subsystem (for example, the exogenous input of R is r, the other ones are fictitious); and

$$F = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & -I & I \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.1}$$

is a constant matrix representing the constant gain interconnections between subsystems. The equations governing the input-output behavior of the aggregate of subsystems in Figure 5.2 are

$$\underline{u} = \underline{\eta} + F\underline{y}, \quad \underline{y} = T_i\underline{u}. \tag{5.2}$$

Assume that $(I-FT_i)^{-1}$ exists, then every input-output map is proper and well-defined..

In order to obtain a polynomial matrix description of the {R;G,H} controlled system, consider the following polynomial matrix descriptions of the subsystems completely described by their transfer matrices P, G, H, and R:

$$\tilde{D}z_P = \tilde{N}u_g,\ y_P=z_P,\ \tilde{D}_g z_g = \tilde{N}_g u_g,\ y_g=z_g,$$
$$\tilde{D}_h z_h = \tilde{N}_h u_h,\ y_h=z_h,\ \tilde{D}_r z_r = \tilde{N}_r u_r,\ y_r=z_r. \tag{5.3}$$

Let $\underline{D}$ = block diag$\{\tilde{D}, \tilde{D}_g, \tilde{D}_h, \tilde{D}_r\}$, $\tilde{N}_i$ = block diag $\{\tilde{N}, \tilde{N}_g, \tilde{N}_h, \tilde{N}_r\}$, and $z_i = [z_P^t, z_g^t, z_h^t, z_r^t]^t$. A polynomial matrix description of the {R;G,H} controlled system is:

$$\tilde{D}_i z_i = \tilde{N}_i\underline{\eta}, \underline{y} = z_i, \underline{u} = Fz_i + \eta, \tag{5.4}$$

where

$$\tilde{D}_i = \underline{D}-\tilde{N}_iF = \begin{bmatrix} \tilde{D} & -\tilde{N} & 0 & 0 \\ 0 & \tilde{D}_G & \tilde{N}_G & -\tilde{N}_G \\ -\tilde{N}_H & 0 & \tilde{D}_H & 0 \\ 0 & 0 & 0 & \tilde{D}_R \end{bmatrix}. \tag{5.5}$$

In Lemma 5.1, $\tilde{D}_i^{-1}$ is found, making it possible to use Lemma 2.1 directly to characterize the hidden modes of the controlled system.

LEMMA 5.1. The inverse of $\tilde{D}_i$ is given by

$$\begin{bmatrix} (I+PGH)^{-1}\tilde{D}^{-1} & (I+PGH)^{-1}P\tilde{D}_G^{-1} & -(I+PGH)^{-1}PG\tilde{D}_H^{-1} & (I+PGH)^{-1}PG\tilde{D}_R^{-1} \\ -GH(I+PGH)^{-1}\tilde{D}^{-1} & (I+GHP)^{-1}\tilde{D}_G^{-1} & -(I+GHP)^{-1}G\tilde{D}_H^{-1} & (I+GHP)^{-1}G\tilde{D}_R^{-1} \\ H(I+PGH)^{-1}\tilde{D}^{-1} & H(I+PGH)^{-1}P\tilde{D}_G^{-1} & (I+HPG)^{-1}\tilde{D}_H^{-1} & H(I+PGH)^{-1}PG\tilde{D}_R^{-1} \\ 0 & 0 & 0 & \tilde{D}_R^{-1} \end{bmatrix} \tag{5.6}$$

The characterization of the hidden modes is given in Theorem 5.1. The aggregate representation of the interconnected system has helped by simplifying the derivation of a polynomial matrix description, the characteristic polynomial, and the inverse of $\tilde{D}_i$.

THEOREM 5.1. The hidden modes are characterized as follows.

(i) The unobservable modes from y correspond to the poles of G, H and R that cancel in PG, (PG)H, and $(I+PGH)^{-1}(PG)R$, respectively.

(ii) The unobservable modes from u correspond to the poles of H, P, and R that cancel in GH, (GH)P, and $(I+GHP)^{-1}GR$, respectively.

(iii) The uncontrollable modes from r correspond to the poles of G that cancel in G[HP, R]; the poles of P that cancel in HPG and in PM; the poles of

H that cancel in HPG; and the poles of $(I+GHP)^{-1}$ that cancel in $(I+GHP)^{-1}GR$.□

The characterizations in Theorem 5.1 extend and simplify the results originally presented in [5].

Example 5.1. Consider the following implementation of the controller in (4.2.14):

$$G = \frac{s+1}{s-9}, \quad H = 15, \text{ and } R = 1. \tag{5.7}$$

In view of Theorem 5.1, this choice for {R;G,H} does not introduce any additional hidden modes, so there is still only one, corresponding to the pole of P at $s = -1$.□

Example 5.2. Åström and Wittenmark in [11, p. 232] show how a stable hidden mode can degrade performance. They consider the plant

$$P = \frac{K(z-b)}{(z-1)(z-a)} \tag{5.8}$$

where $b<0$ and the two degrees of freedom controller $C=[-C_y, C_r]$. It can be shown that their control law can be implemented via

$$G = \frac{t_0 z}{(z-b)}, \; H=\frac{(s_0 z+s_1)}{t_0 z}, \text{ and } R=1 \tag{5.9}$$

in the {R;G,H} controller configuration. Note that s_0 and s_1 are chosen so that the controlled system is internally stable, and t_0, K, and a are real constants defined in [11]. The particular implementation of the controller does not affect the following remarks in [11]: A simulation shows that the step response of the system contains an undesirable "ripple" or "ringing" in the control signal u while the output signal is well behaved at the sampling instants. It is pointed out in [11] that the "ringing" is caused by the cancellation of the $(z-b)$ factor. From Theorem 5.1 it is seen that the reason for ringing is that the pole of G at $z=b$, which cancels in PG, corresponds to an unobservable mode from y that is observable from u. Moreover, from Theorem 4.1.1 it is seen that the mode that corresponds to the pole of the controller that cancels with a plant zero will be unobservable from the output, but it will be observable from the plant input in any implementation of the two degrees of freedom controller.

VI. CONCLUSIONS

The results presented here on the hidden modes of interconnected systems in terms of transfer matrices and parameters extend and unify the results in the literature. The emphasis here was in control design. The results and the methodology presented are not limited to the applications shown, but they can be applied to any interconnected system where the study of the hidden modes introduced by the interconnections is of interest. A direct extension of the results would be in the area of decentralized control.

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