ABSTRACT

A complete treatment of the hidden modes of interconnected systems in control design is presented, which extends and unifies several results that have appeared in the literature in the past decades. The uncontrollable and/or unobservable hidden modes of one and two degrees of freedom control systems are characterized in terms of transfer matrices of the interconnected subsystems and in terms of design parameters. This characterization leads directly to design conditions, which can be used to adequately control the hidden modes; thus, avoiding unnecessarily high order controllers and undesirable behavior. The methods used are based on polynomial matrix internal descriptions, however, all results are expressed so that they can be directly used in control design. Internal stability is guaranteed using a novel stability theorem which adds significant insight to the problem. A method is also presented to characterize the hidden modes of any interconnected system.

I. INTRODUCTION

The hidden modes of a compensated system correspond to the system's eigenvalues which are uncontrollable and/or unobservable from a given input or output, respectively. The hidden modes for single degree of freedom and for particular two degrees of freedom controlled systems have been studied in the literature [1-5, 17-23, 31-33]. In this paper we characterize the hidden modes for the general linear two degrees of freedom controlled system in terms of the frequency domain control design tools: transfer matrices and design parameters. The hidden modes can, of course, be characterized using internal descriptions [4, 11, 17-20], and this is useful mainly in the analysis of control systems. In frequency domain control design methods, where transfer matrices and design parameters are used, these characterizations are not as helpful. This characterization of the hidden modes in terms of the design tools leads directly to design conditions, which can be used to adequately control the hidden modes, in control design, avoiding unnecessarily high order controllers and undesirable behavior. It is recognized that if the interconnected system is internally stable then the hidden modes, if any, will be stable. By undesirable behavior we mean transient responses introduced unintentionally in the design and phenomena such as ringing (see [11] and Example 5.2).

In this paper a complete treatment of the hidden modes of interconnected systems in control design is presented, which extends and unifies several results which have appeared in the literature in the past decades. The methods used are based on polynomial matrix internal descriptions, however, all results are expressed so that they can be directly used in control design. Internal stability is guaranteed using a novel stability theorem which adds significant insight to the problem.

The two degrees of freedom controller, \( C \), provides a unified framework in approaching complicated control problems involving multiple objectives, in a manner which is configuration independent. Several researchers have utilized \( C \) in a time domain state-space formulation (e.g., Bengtsson [7]). Using a transform domain formulation, \( C \) has been incorporated by Pernebo [6] and more recently, by Youla and Bongiorno [6], Doets and Gustafson [9], Vidyasagar [10], Doets and Gündes [12], and Sugie and Yoshikawa [13]. The pole-placement algorithm of Astrom in [11] also uses \( C \) but for scalar plants only. There is much renewed interest in the two degrees of freedom controller that is due to more demanding control problems and to recent advances in understanding and effectively utilizing such control laws. Among the many interconnected systems introduced directly by the interconnections are characterized in terms of transfer matrices and design parameters. The implementation of the controller, \( C \), is usually done by interconnecting available subcontrollers, where each subcontroller is designed to handle a particular task such as stability and regulation. The resulting controller is past causally controllable and observable, and it introduces additional hidden modes; these are also characterized. In addition, given any particular interconnected system we introduce a systematic method to characterize the hidden modes; for this we use an aggregate system representation [4]. The proofs are found in [35].

II. PRELIMINARIES AND HIDDEN MODES OF SINGLE DEGREE OF FREEDOM SYSTEMS

An interconnected system is said to be completely characterized by its proper rational transfer matrix if and only if an internal description of the overall system is controllable from the input and observable from the output. If a transfer matrix does not completely characterize an interconnection of subsystems, then the uncontrollable eigenvalues from the input and the unobservable eigenvalues from the output correspond to the hidden modes of the overall system. Notice that the hidden modes are due exclusively to the interconnections since every subsystem is assumed to be irreducible.

In classical control design of scalar systems it is straightforward to characterize the hidden modes in terms of pole-zero cancellations. In the frequency domain control design of multivariable systems, the hidden modes can also be characterized by considering "pole-zero cancellations." In this case, however, the characterization is not as direct mainly due to the fact that "pole-zero cancellations" are not as well defined in the multivariable case, and also because of the difficulty in associating hidden modes with specific cancellations. Results that refer to particular control configurations have been reported in the literature [1-3, 21]. In [16], these results have been formalized and extended; they are the basis of the results presented here.

Consider the following polynomial matrix description (PMD) of the controlled system:

\[
\begin{align*}
\mathbf{z}(s) &= \mathbf{Q}(s)u(s), \\
\mathbf{y}(s) &= \mathbf{R}(s)z(s) + \mathbf{W}(s)u(s)
\end{align*}
\]

In (2.1), \( \mathbf{Q}(s), \mathbf{R}(s), \mathbf{W}(s) \) are polynomial matrices; the quadruple \( \{\mathbf{Q}(s), \mathbf{R}(s), \mathbf{W}(s)\} \) does not correspond to states of the system represented in (2.1). For the system described in (2.1), the uncontrollable (unobservable) modes from \( u(y) \) correspond to the roots of the determinant of a g.c.l.d. of \( \{\mathbf{Q}(s), \mathbf{R}(s)\} \) (g.c.r.d. of \( \{\mathbf{R}(s), \mathbf{Q}(s)\} \)) [17, 18], where g.c.l.d. denotes greatest common left (right) divisor. To derive the desired characterizations, a better way to express these known conditions is given in Lemma 2.1.

**Lemma 2.1.** The system described by \( \{\mathbf{Q}(s), \mathbf{R}(s), \mathbf{W}(s)\} \) is controllable from \( u \) (observable from \( y \)) if and only if the McMillan degree of the transfer matrix from \( u \) to \( z \) \( (\mathbf{Q}(s)u(s) to y) \) is the same as the degree of \( \mathbf{Q}(s) \).

In the following, appropriate transformations are used to map these products into products of transfer matrices of the interconnected subsystems. Transformations which yield equivalent polynomial matrix descriptions are used. In particular, we apply transformations that maintain system equivalence in the Rosenbrock sense [17].

It is well known that cancellations in products of transfer matrices are not simple extensions of scalar pole-zero cancellations. For example, it is possible to have a cancellation where a polynomials is canceled with a pole of a transfer matrix of another controller that is not necessary.
matrices should be taken as pole cancellations rather than pole-zero cancellations. Notice that a pole of a transfer matrix $T_3(s)$ cancels with a zero of a not necessarily square transfer matrix $T_1(s) \in T_1(s)T_2(s)$ only if rank$(N_3(s))$ is full where $T_1 = D_1^{-1}S_2$ is a left coprime polynomial factorization (i.c.) Moreover, the canceled poles in a product of transfer matrices correspond to hidden modes from the input and/or the output if and only if these poles are eigenvalues of the interconnected system. Additional observations on "multi cancellations" can be found in [16,19,23,30]

The following result will be used in the characterization of hidden modes [16]. Consider

$$T_2 = [T_1, I]T_3$$

(2.3)

$$T_3 = T_3[T_4, I]$$

(2.4)

where $T_1$, $T_2$, $T_3$, and $T_4$ are $p \times m$, $m \times r$, $r \times p$, and $m \times m$, respectively. Note that, $T_1$ can be considered to be the transfer matrix of the cascade connection of system $S_2$ followed by $S_1$, where each system is completely characterized by their transfer matrices $T_2$ and $[T_1, I]$, respectively. It is of interest to characterize the cancellations that result in a reduction in the McMillan degree of the resulting transfer matrix, that is, $\delta T_{12} < \delta T_1 + \delta T_2$ and $\delta T_{34} < \delta T_3 + \delta T_4$, or $T_1$ and $T_3$ are coprime factors.

We now apply the method described above to characterize the hidden modes of the cascade, parallel, and feedback systems shown in Figure 2.1. If the systems, $S_i$, $i=1,2$, are assumed to be completely characterized by their proper transfer matrices $T_i$, then the hidden modes have been characterized in the literature before (for example, see [1-4,17-23,31-33]). Let $S_1$ be completely characterized by $T_1$ and consider the case when $S_2$ is not completely described by $T_2$. Assume that the feedback system is well defined, that is, $[1+T_1T_2]D_2$ and that every input-output map is proper.

![Figure 2.1. Basic Interconnected systems.](image)

A polynomial matrix description for $S_2$ is

$$S_2: \phi_2 = \phi_2 \phi_2$$

(2.5)

where the uncontrollable (unobservable) modes from $u_2$ ($y_2$) correspond to the roots of the determinant of a g.c.d. of $(\phi_2, \phi_2)$ (g.c.r.d. of $(\phi_2, \phi_2)$).

**Lemma 2.3.** CASCADE: The uncontrollable modes from $u_2$ correspond to the uncontrollable modes of $S_2$ and to the poles of $T_1$ that cancel in $T_1T_2$. The unobservable modes eigenvalues of $S_2$ from $y_2$ that cancel in $T_1(T_2P_2^{-1})$.

**PARALLEL:** The uncontrollable modes from $u_2$ correspond to the uncontrollable modes of $S_2$ from $u_2$ and to the controllable eigenvalues of $S_2$ that cancel with poles of $T_1$ in $(\phi_2 \phi_2 \phi_2)D_2$. The uncontrollable modes from $y_2$ correspond to the uncontrollable modes of $S_2$ from $y_2$ and to the observable eigenvalues of $S_2$ from $y_2$ that cancel in $T_1(T_2P_2^{-1})$.

**FEEDBACK:** The uncontrollable modes from $u_1$ correspond to the uncontrollable modes of $S_2$ and to the controllable eigenvalues of $S_2$ from $u_2$ that cancel in $(T_2P_2^{-1})T_1$. The uncontrollable modes from $y_1$ correspond to the observable modes of $S_2$ and to the observable eigenvalues of $S_2$ from $y_2$ that cancel in $T_1T_2P_2^{-1}$.

**Lemma 2.3** demonstrates that when $S_2$ is not completely described by $T_2$, the interconnected systems considered here maintain the hidden modes of $S_2$ only from appropriate inputs and outputs. Furthermore, because of the interconnection additional hidden modes could be introduced.

**III. STABILITY THEOREM. PARAMETERIZATION**

The two degrees of freedom linear controller

$$u = C[y_1; r] + \frac{K}{D_r}[y_1; r]\quad (3.1)$$

where $C = [-C_2, C_1]$ proper, generates the plant input $u$ by independently processing the plant output $y$ and the external reference input $r$ as seen in Figure 3.1.

![Figure 3.1. The controlled system.](image)

$S_p$ is the linear plant described by $y = P u$ with $P$ its proper transfer matrix and $S_c$ is the controller described in (3.1). It is assumed that $[1+P C_p][1+P C_p]^{-1} \neq 0$ and that every input-output map is proper. Under these assumptions, the controlled system is said to be internally stable if the inverse of the denominator polynomial matrix in a polynomial matrix description is stable. If the controlled system is internally stable, we say that $S_c$ is an internally stabilizing controller for $S_p$.

A significant step towards better understanding the role of $C$ in plant compensation was recently accomplished by parametrically characterizing all internally stabilizing two degrees of freedom controllers $C$ as extending the results on parametric characterization of all feedback controllers $C_2$ [24-28,1,2,14-16] which have greatly contributed to control design methods. All internally stabilizing controllers $C$ can be parametrically characterized using two independent stable parameters $K$ and $X$ as

$$C = \left[ x_1 - \frac{K}{N} \right]^{-1} \left[ x_2 + KD \right], \quad X$$

(3.2)

where $N$, $D$, $x_1$, $x_2$ are polynomial matrices, and they are derived from coprime fractional representations of the plant

$$P = ND^{-1} = D^{-1}N$$

(3.3)

and the associated Bezout-Diophantine equation

$$x_1D + x_2N = 1$$

(3.4)

In (3.2), $K$ must be such that $[x_1 - K N] \neq 0$, and for $C$ proper need $D(x_2 + KD)$ proper and $D(x_2 - K N)$ and its inverse proper. In [6], $x_1$ and $x_2$ satisfy the Diophantine equation $x_1D + x_2N = D_b$, with $D_b$ a polynomial matrix ([D_b] Hurwitz). In [9,10], (3.2) involves proper and stable matrices $N$, $D$, $x_1$, $x_2$, $K$, and $X$. The parameter $K$ in (2) is the well known parameter used in the characterization of all stabilizing feedback controllers $C_2$ in [24-27]. The parameter $X$ is actually the response parameter used by Antsaklis and Sain [1] (and Liu and Sung [29]) to parametrically characterize feedback controllers in an error feedback setting. If $D = u = y = N_2$ is an internal polynomial matrix representation of the plant $P$, then it can be shown that

$$x = X$$

(3.5)

that is, $X$ is the transfer matrix between the input $r$ and the partial output $z$ of the plant.

It is advantageous to study internal stability of the system in Figure 3.1 in a novel alternative way [21].
Theorem 3.1 separates the role of \( C \), the feedback part of \( S \), from \( C_f \) in achieving internal stability. Clearly if only feedback action is considered, only (i) is of interest; and if open-loop control is desired, \( C = 0 \), (i) implies that \( P \) must be stable, and \( C = M \) must satisfy (ii). In (ii) the parameter \( M = DX \) appears rather naturally and in (ii) the way is open to use any desired feedback parameterization, not necessarily \( K \) of [5,9-10].

From Theorem 3.1 we can directly characterize the input-output maps attainable from \( r \) with internal stability. In particular, consider the two maps described by \( y = Tr \) and \( u = Mr \) which are characterized in Theorem 3.2.

Theorem 3.2. A pair \((T, M)\) is realizable with internal stability via a two degrees of freedom configuration if and only if \((T, M) = (N, DX)\) with \( X \) stable.

There are many choices in parametrically characterizing all feedback stabilizing controllers \( C_f \) and these are extensively discussed by Antsaklis and Sain in [2]. The stabilizing controller \( C \) can therefore be expressed, in addition to (3.2) as (for example):

\[
C = (I-QP)^{-1}Q, \quad DX = (I-LN)^{-1}L, \quad X
\]

where \( Q = DL \), \( DX = M \) with \( L, X \) stable and \( D(I-QP) = (I-LN)^{-1} \) stable (\([I-QP] \neq 0 \) or \([I-LN] \neq 0 \)).

Parametric characterizations of all internally stabilizing controllers \( C \) proper and nonproper are given in (3.6). For \( C \) proper, \( M \) and \( Q \) are chosen proper and such that \((I-QP)^{-1}Q\) is proper; note that if \( P \) is strictly proper, \( Q \) proper always implies that \((I-QP)^{-1}\) is proper. Notice that \( L \) or \( Q \) in (3.6) must satisfy certain conditions, in addition to being stable, in contrast to \( K \) in (3.2); however, alternative to \( K \) parameterizations, such as in (3.6), are very useful, since they do have certain additional desirable properties (see [2]). The relations between the parameters are

\[
L = x_3 + KD = D^oQ, \quad Q = DK_x + CPC_y = (I+CƤP)^{-1}C_y
\]

where \( Q = DL \), \( DX = M \) with \( L, X \) stable and \( D(I-QP) = (I-LN)^{-1} \) stable (\([I-QP] \neq 0 \) or \([I-LN] \neq 0 \)).

The hidden modes are characterized in terms of the design parameters: \( K, X, \) and \( L \) when \( S \) is completely characterized by their transfer matrices. Using these characterizations we then give conditions in terms of the parameters of interest to avoid the introduction of hidden modes. These conditions can be incorporated in the control system design.

IV. HIDDEN MODES IN TWO DEGREES OF FREEDOM

CONTROLLED SYSTEMS

In this section, the hidden modes of two degrees of freedom controlled systems as depicted in Figure 4.1.1, will be studied. In Section 4.1, the hidden modes from given inputs and outputs will be characterized in terms of transfer matrices. This characterization is done when \( S_p \) and \( S_c \) are completely described by their transfer matrices, and when \( S_p \) is completely described by its transfer matrix, but \( S_c \) is not. In Section 4.2, the hidden modes are characterized in terms of the design parameters: \( K, X, \) and \( L \) when \( S_p \) and \( S_c \) are completely described by their transfer matrices. Using these characterizations we then give conditions in terms of the parameters of interest to avoid the introduction of hidden modes. These conditions can be incorporated in the control system design.

4.1 HIDDEN MODES IN TERMS OF I/O MAPS

Consider

\[
\begin{align*}
\begin{array}{c}
\text{r} \\
u \\
\end{array} \quad \begin{array}{c}
\text{y} \\
\end{array} \\
\begin{array}{c}
\text{C} \\
\text{P} \\
\end{array} \quad \begin{array}{c}
\text{S}_c \\
\text{S}_p \\
\end{array} \\
\end{align*}
\]

Figure 4.1.1. A two degrees of freedom control system. Where the vector of fictitious inputs \( n_o \) is introduced to help with the interpretation of the uncontrollable hidden modes; the other variables were described in Section III.

For \( S_p \) to be characterized by its transfer matrix, that is, \( S_p \) is controllable from \( u^t, r^t \) and observable from \( u \). A polynomial matrix description for \( S_p \) is

\[
S_p = \begin{bmatrix} D_p & S_p \end{bmatrix}, \quad S_p = \begin{bmatrix} D_p & S_p \end{bmatrix}, \quad y = \begin{bmatrix} n \end{bmatrix}
\]

where \( C = [D_p, S_p] \), \( S_p \) is left coprime and \( u^t = n^t \). Two polynomial matrix descriptions for \( S_p \) are

\[
S_p = \begin{bmatrix} D_p & S_p \end{bmatrix}, \quad y = \begin{bmatrix} n \end{bmatrix}
\]

where \( P = [D_p, S_p] \), \( P \) is coprime factorizations. Combining (4.1.1) and (4.1.2) gives a polynomial matrix description for the two degrees of freedom controlled system

\[
D_p = \begin{bmatrix} n & n_p \end{bmatrix}, \quad y = \begin{bmatrix} n \end{bmatrix}, \quad D_p = \begin{bmatrix} n_p \end{bmatrix}
\]

where \( D_p = D_p + S_p \), \( y = \begin{bmatrix} n \end{bmatrix} \). Since \( S_p \) and \( S_p \) are controllable by their transfer matrices the hidden modes are due exclusively to the interconnection.

A preliminary characterization of the hidden modes follows directly from (4.1.4). The uncontrollable modes from \( r \) correspond to the poles of \( D_p \) that cancel in \( D_p + S_p \). This characterization gives insight into the controllability and observability properties of two degrees of freedom systems. For example, notice that the controlled system is observable from \( y^t, u^t \), that is, the uncontrollable modes from \( y \) are observable from \( u \) and vice versa.

Notice that even though \( S_p \) is completely characterized by \( C \), there could be uncontrollable modes from \( r \) or from \( y \). However, the uncontrollable modes from \( r \) are controllable from \( y \) and vice versa. Let \( G_y \) be a g.c.l.d. of \((D_p, S_p)\) and let \( G_y \) be a g.c.l.d. of \((D_p, S_p)\) then

\[
\begin{align*}
\begin{array}{c}
D_p \\
S_p \\
\end{array} = \begin{bmatrix} G_y \end{bmatrix}, \quad \begin{bmatrix} S_p \end{bmatrix} = \begin{bmatrix} G_y \end{bmatrix}, \quad \begin{bmatrix} S_p \end{bmatrix} = \begin{bmatrix} G_y \end{bmatrix}
\end{align*}
\]

where \( G_y \) is a g.c.l.d. \((f) \) and \( G_y \) is a g.c.l.d. \((f) \) coprime polynomial pairs. The roots of \( G_y \) correspond to uncontrollable modes from \( r \) or \( y \). Furthermore, the roots of \( G_y \) are closed-loop eigenvalues, but the roots of \( G_y \) are not (of \( G_y \)). \( \lbrace \begin{bmatrix} D_p \end{bmatrix} = \begin{bmatrix} G_y \end{bmatrix} \rbrace \) and \( \lbrace \begin{bmatrix} D_p \end{bmatrix} = \begin{bmatrix} G_y \end{bmatrix} \rbrace \). This implies that no uncontrollable modes of \( S_p \) from \( r \) will be uncontrollable from \( r \) of the two degrees of freedom controlled system.

Before giving the main result in Theorem 4.1.1 it is useful to characterize the poles of \((I+PC_y)^{-1}\) and of \((I+CP)^{-1}\); the characterization is used to determine when a cancellation of poles of \((I+PC_y)^{-1}\) and of \((I+CP)^{-1}\) can correspond to a hidden mode from an input and/or output.

Lemma 4.1.1. The following relations are true.

(i) \( \lbrace \begin{bmatrix} D_p \end{bmatrix} = \begin{bmatrix} G_y \end{bmatrix} \rbrace \) - closed-loop eigenvalues from \( y \) + [eigenvalues that are uncontrollable from \( y \) and unobservable from \( y \)].

(ii) \( \lbrace \begin{bmatrix} D_p \end{bmatrix} = \begin{bmatrix} G_y \end{bmatrix} \rbrace \) - closed-loop eigenvalues from \( u \) - [roots of \( G_y \) that do not correspond to uncontrollable modes from \( u \)].

The hidden modes are determined by considering cancellations in the products of transfer matrices given in Lemma 4.1.2.

Lemma 4.1.2. The hidden modes are characterized by considering cancellations in the following products of transfer matrices.

Unobservable modes from \( y \): \((I+PC_y)^{-1}[D_p, I] \).

Unobservable modes from \( u \): \((I+CP)^{-1}[D_p, I] \).

Uncontrollable modes from \( r \): \( r \) [closed-loop transfer matrices].

The main result when \( S_p \) and \( S_c \) are completely characterized by their transfer matrices is given next.

Theorem 4.1.1. The hidden modes are characterized as follows.

(i) \( \lbrace \begin{bmatrix} D_p \end{bmatrix} = \begin{bmatrix} G_y \end{bmatrix} \rbrace \) - the unobservable modes from \( y \) (correspond to the poles of \( C \) that cancel in \( PC_y \)).

(ii) \( \lbrace \begin{bmatrix} D_p \end{bmatrix} = \begin{bmatrix} G_y \end{bmatrix} \rbrace \) - the uncontrollable modes from \( r \) correspond to the poles of \( C \) that cancel in \( CP \).
correspond to the poles of $C_y$ that cancel in $C_yP_o$.

Theorem 4.1.1 characterizes the hidden modes introduced by the interconnection of systems in Figure 4.1.1. It is also of interest to characterize the hidden modes when the controller is not completely characterized by its transfer matrix. The conditions for the general case of uncontrollability and unobservability, which requires another internal description of $S_0$, are given in Theorem 4.1.2.

**Theorem 4.1.2.** The uncontrollable modes of $S_0$ from $[u', r']$ will be uncontrollable from $r$, and the uncontrollable modes of $S_0$ from $u$ will be unobservable from $y$.

The conditions in Theorem 4.1.1 are illustrated in Example 4.1.1.

Example 4.1.1. Consider the plant

$$P = \frac{s-1}{(s-2)(s+1)}$$  \hspace{1cm} (4.1.10)

and let the compensator be given by

$$C = \frac{-s(s+1)}{(s+1)(s+2)/3}$$  \hspace{1cm} (4.1.11)

The resulting transfer function is $T = \frac{-s}{s+1}$, which is given by $\phi_T = \frac{s(s+1)}{(s+1)(s+2)/3}$. Comparing $\phi_T$ to the resulting transfer function $T$ gives the hidden modes to be $(-1, -2/3)$. The nature of the hidden modes is directly characterized via Theorem 4.1.1. The three hidden modes are uncontrollable from $r$, one of the closed-loop eigenvalues at $-1$ is unobservable from $u$, and there are no unobservable eigenvalues from $y$.

4.2. Hidden Modes in Terms of Design Parameters

The results in the last section could be used to give conditions to avoid the introduction of hidden modes, but they would not be simple to implement in control design. In this section the hidden modes will be characterized in terms of the parameters utilized in the design of a control system, leading directly to design conditions to avoid unnecessary hidden modes. In particular, the hidden modes will be characterized in terms of $K$, $X$, and $L$, which were used in Section III to parameterize the internally stabilizing controllers.

The results in the following two lemmas are used to characterize the hidden modes in terms of the design parameters. In Lemma 4.2.1, the poles of the parameters of interest are characterized. In Lemma 4.2.2, the uncontrollable modes from $y_0$ are characterized.

First, let $G_p$ be a g.c.d. of $(D_u, N_u)$, where $K = D_u^{-1}N_u$ is 1.c. Then it can be shown that there exist polynomial matrices $D_i$ and $N_i$ such that

$$[D_k, N_k] = [G_k, N_k]$$  \hspace{1cm} (4.2.1)

where $L = D_k^{-1}N_k$ is a coprime factorization.

**Lemma 4.2.1.** The following relations are true.

(i) (poles of $K$) = (closed-loop eigenvalues) - (roots of $[G_k]$)

(ii) (poles of $L$) = (closed-loop eigenvalues) - (uncontrollable eigenvalues from $r$)

(iii) (poles of $X$) = (closed-loop eigenvalues) - (uncontrollable eigenvalues from $r$)

**Lemma 4.2.2.** The uncontrollable modes from $y_0$ correspond to the poles of $[G_k] [N_k]$, the poles of $D_k^{-1}$, and the poles of $P$ in $G_k P_o$. The poles of $G_k^{-1}$ are also given by the poles of $P$ that do not cancel in $[L-NND] L^T$.

The characterization of hidden modes in terms of the design parameters is given next.

**Theorem 4.2.1.** The unobservable modes from $y$ correspond to the poles of $[X, L]$ that cancel in $N[X, L]$. The poles of $[X, L]$ that cancel in $D_k^{-1} N_k$ correspond to unobservable modes from $u$. The unobservable modes from $y$ correspond to the poles of $P$ in $G_y P_o$ and to the poles of $L$ that are not poles of $X$.

The next corollary specializes the conditions in Theorem 4.2.1 to the error feedback configuration.

**Corollary 4.2.1.** In the error feedback configuration, the unobservable modes from $u$ correspond to the poles of $L (L = X)$ which cancel in $N L$. The uncontrollable modes from $r$ correspond to the poles of $P$ that do not cancel in $[I - LN] D^T L^T$, that is, the poles of $G_y P_o$.

The conditions in Corollary 4.2.1 agree with known results for the error feedback configuration in [2].

**Remark 4.2.1.** A characterization of the closed-loop characteristic polynomial in terms of the parameters used in this section is given by

$$\phi_T = \{[G_y] [N_y], [G_y] [D_y], [D_y], \text{ and } \text{ e, R} \}$$  \hspace{1cm} (4.2.2)

where $G_y = G_y G_p$ and $D_y = D_y D_y$. The roots of $[G_y] [N_y]$ correspond to the uncontrollable eigenvalues from $r$, and the uncontrollable eigenvalues from $y$ correspond to the poles of $X$ which cancel in $N X$, and these conditions are interpreted below for single-input, single-output feedback systems. Consider

$$P = \frac{n}{d} = d_g y_g x_g d_l x_l = \frac{d_y}{d_x} y_g d_y d_l$$  \hspace{1cm} (4.2.3)

where $g_y = g_y x_g$, and $d_y = d_y d_y$, and the variables are the same as before except that the factorizations are given in lower case.

The conditions in Theorem 4.2.1 for single-input, single-output systems can be given as follows.

**Corollary 4.2.2.** The uncontrollable modes from $r$ correspond to the zeros of $g_y$ and $d_y$. The unobservable modes from $y$ correspond to the zeros of both $g_y$ and $d_y$. These results are illustrated in the following examples.

Example 4.2.1. The hidden modes of the feedback system considered in Example 4.1.1 are characterized in terms of the design parameters.

**Example 4.2.1.** For the plant and compensator defined in (4.1.7) and (4.1.8), we have the following parameters

$$Q = \frac{8}{s+2/3} L^{-1} (s+1)(s+2/3)^2 X, \frac{8}{s+2/3} (s+2/3)^2 P, \frac{8}{s+2/3} (s+2/3)$$  \hspace{1cm} (4.2.5)

and

$$G_y = \frac{8}{s+2/3} (s+2/3) G_p, G_y = \frac{s+1}{s+2/3} G_y$$  \hspace{1cm} (4.2.6)

and if $C = \frac{s+1}{s+2/3}$, then $d_y = g_y d_y$, and $g_y = g_y d_y$ and $n_y = n_y$. These results are illustrated in the following examples.

In Example 4.2.2.1 the hidden modes of the feedback system considered in Example 4.1.1 are characterized in terms of the design parameters.

**Example 4.2.2.1.** For the plant and compensator defined in (4.1.1) and (4.1.8) we have the following parameters

$$Q = \frac{8}{s+2/3} L^{-1} (s+1)(s+2/3)^2 X, \frac{8}{s+2/3} (s+2/3)^2 P, \frac{8}{s+2/3} (s+2/3)$$  \hspace{1cm} (4.2.7)

and if $C = \frac{s+1}{s+2/3}$, then $d_y = g_y d_y$, and $g_y = g_y d_y$ and $n_y = n_y$. These results are illustrated in the following examples.

From (4.2.9), one of the poles of the plant at $s=\infty$ corresponds to an unobservable mode from $u$. The uncontrollable modes from $r$ correspond to the poles of $G_y$, and of $L$ that are not

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In Example 4.2.2 the conditions given in this section are used to redesign C so that the number of hidden modes is a minimum.

Example 4.2.2. The controller considered in Example 4.1.1 introduces at least three hidden modes. If less hidden modes are desired, a new compensator C need to be designed that attains the desired transfer function $T = (s+\alpha)/(s+\beta)$. Since $T$ is the same and $T = (s+\alpha)/(s+\beta)$ is the same as in (4.6), we need to design $L$.

At this time it should be noted that no internally stabilizing single degree of freedom controller can attain the desired $T$. Therefore, the two degree of freedom design will yield at least one hidden mode. If no hidden modes are desired, the choice for $T$ should be reconsidered.

In order to minimize the number of hidden modes, the poles of $L$ should be the same as the poles of $X$ and no poles of $L$ should be zeros of the plant. Then a possible choice for $L$ is

$$L = \frac{\kappa}{(s+\alpha)(s+\beta)},$$

where $\kappa$ is a constant. The conditions for the compensator to be internally stable and proper are as follows. Only the latter one needs further checking:

$$(1-LN)D^{-1} = \frac{\kappa(s+\alpha)}{(s+\alpha)(s+\beta)} - \frac{\kappa(s-1)}{(s-1)(s+\beta)} (4.11)$$

The transfer function in (4.11) is stable if and only if $(2+\alpha)(2+\beta)<\kappa$, i.e., $\kappa = (\kappa(s-1)/(s+\beta))$. Then, for $\kappa > 0$ need $0 < \beta < \kappa/2$ (4.12) or choose $\alpha > 0$, $\beta > 0$ and $\kappa = (2+\alpha)(2+\beta)$. By choosing $\kappa$ in this way, a controller $C = [-C_0, C_1]$ that attains the desired closed-loop transfer function is

$$C = \begin{bmatrix} -\kappa/(s-2) & (s-2)/(s+1) \\ (s+\alpha)/(s+\beta) & -(s-2)/(s+1) \end{bmatrix}$$

where there is a pole-zero cancellation at $s=-2$ in the expressions for $C_0$ and $C_1$. Suppose $\alpha = 1$, $\beta = 3$, then $\kappa = 15$ and

$$C = \begin{bmatrix} 15(s+1) & s+1 \\ s+1 & s+1 \end{bmatrix}$$

For $C$ in (4.14) there is only one hidden mode due to the interconnection of the controller and the plant; the pole of $P$ at $s=1$ corresponds to an uncontrollable mode from $r$ and to an unobservable mode from $u$.

V. Hidden Modes in Interconnected Systems

The characterization of hidden modes in terms of transfer matrices of a system interconnection can be done starting with the results in Section 4.1. For a complex interconnection of systems it may be simpler to apply Lemma 2.1 to a polynomial matrix description of the interconnected system. A systematic method to do this is explained in this section. For illustration consider the $(R,G,H)$ controlled system in Figure 5.1,

Figure 5.1. An $(R,G,H)$ controlled system.

where the interconnected subsystems are completely described by their transfer matrices $P$, $G$, $H$, and $C$. The $(R,G,H)$ controller is an implementation of a two degrees of freedom compensator, where $C_0 = GH$ and $C_1 = GR$.

The interconnected systems will be represented as in [4, Chapter 4], where a systematic study of internal stability of system interconnections is presented. The aggregate system representation of the $(R,G,H)$ controlled system is given in Figure 5.2,

Figure 5.2. Aggregate system representation.

where $T_1 = \text{block diag} \{P, G, H, R\}; Z = [y_0^T, y_1^T, y_2^T, y_3^T]^T$ and $u = [u_0, u_0, u_0, u_0]$ are the vectors of outputs and inputs of each subsystem, respectively; $y$ is a vector of exogenous inputs entering at the input of each subsystem (for example, the exogenous input of $R$ is $r$, the other ones are fictitious) and

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a constant matrix representing the constant gain interconnections between subsystems. The equations governing the input-output behavior of the aggregate of subsystems in Figure 5.2 are

$$u = y + Fx, \ \ \ \ x = Tw.$$  (5.2)

Assume that $(1-FT_1^{-1})$ exists, then every input-output map is proper and well-defined.

In order to obtain a polynomial matrix description of the $(R,G,H)$ controlled system, consider the following polynomial matrix descriptions of the subsystems completely described by their transfer matrices $P$, $G$, $H$, and $R$:

$$D_{Rg} = N_{Rg} y, \ \ \ \ y_p = z_p, \ \ \ \ D_{Rg} = N_{Rg} y, \ \ \ \ y_p = z_p.$$  (5.3)

Let $D$ be block diagonal $(D, D_0, D_1, D_2)$, $N_1$ be block diagonal $(N_1, N_1, N_2, N_2)$, and $z_1 = [z_1, z_1, z_1, z_1]$. A polynomial matrix description of the $(R,G,H)$ controlled system is:

$$D_{Rg} = N_{Rg} y, \ \ \ \ y_p = z_p, \ \ \ \ D_{Rg} = N_{Rg} y, \ \ \ \ y_p = z_p.$$  (5.4)

where

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In Lemma 5.1, $D_1^{-1}$ is found, making it possible to use Lemma 2.1 directly to characterize the hidden modes of the controlled system.

Lemma 5.1. The inverse of $D_1$ is given by

$$D_1^{-1} = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$

The characterization of the hidden modes is given in Theorem 5.1. The aggregate representation of the interconnected system has helped by simplifying the derivation of a polynomial matrix description, the characteristic polynomial, and the inverse of $D_1$.

Theorem 5.1. The hidden modes are characterized as follows.

(i) The uncontrollable modes from $y$ correspond to the poles of $G$, $H$ and $R$ that cancel in $PG$, $(PG)H$, and $(1+PG)(1+PG)H$, respectively.

(ii) The unobservable modes from $u$ correspond to the poles of $P$, $R$, and $H$ that cancel in $GR$, $(GR)P$, and $(1+GR)(1+GR)P$, respectively.

(iii) The uncontrollable modes from $r$ correspond to the poles of $G$ that cancel in $GR$, $RG$, the poles of $P$ that cancel in $PG$ and in $PM$; the poles of
If that cancel in HPG; and the poles of (1 + HPG)^{-1} that cancel in (1 + HPG)^{-1}GRo.

The characterizations in Theorem 5.1 extend and simplify the results originally presented in [5].

Example 5.1. Consider the following implementation of the controller in (4.2.14):

\[ G = \frac{s+1}{s} , \quad H = 15, \quad R = 1. \]

(5.7)

In view of Theorem 5.1, this choice for {R,G,H} does not introduce any additional hidden modes, so there is still only one, a stable hidden mode can degrade performance. They consider simplify the results initially presented in [5].

Example 5.2. Åström and Wittemark in [11, p. 222] show how a stable hidden mode can degrade performance. They consider the plant

\[ P = \frac{K(s-\alpha)}{(s-\beta)(s-\gamma)} \]

(5.8)

where \( \delta < 0 \) and the two degrees of freedom controller \( C = [C_C, C_D] \). It can be shown that their control law can be implemented via

\[ G = \frac{s+1}{s-\beta}, \quad H = \frac{s+\alpha}{s+\gamma}, \quad R = 1 \]

(5.9)

in the {R,G,H} controller configuration. Note that \( s_0 \) and \( s_1 \) are chosen so that the controlled system is internally stable, and \( \alpha \), \( \beta \), and \( \gamma \) are real constants defined in [11]. The particular implementation of the controller does not affect the following remarks in [11]: A simulation shows that the step response of the system contains an undesirable "ripple" or "ringing" in the control signal u while the output signal is well behaved at the sampling instants. It is pointed out in [11] that the "ringing" is caused by the cancellation of the \( (s-\beta) \) factor. From Theorem 5.1 it is seen that the reason for ringing is that the pole of \( G \) at \( s = \beta \) cancels in PG, corresponds to an unobservable mode from y that is observable from u. Moreover, from Theorem 4.3.1 it is seen that the mode that corresponds to the pole of the controller that cancels with a plant zero will be unobservable from the output, but it will be observable from the plant input in any implementation of the two degrees of freedom controller.

VI. CONCLUSIONS

The results presented here on the hidden modes of interconnected systems in terms of transfer matrices and parameters extend and unify the results in the literature. The emphasis here was in control design. The results and the methodology presented are not limited to the applications shown, but they can be applied to any interconnected system where the study of the hidden modes introduced by the interconnections is of interest. A direct extension of the results would be in the area of decentralized control.

References