Z. Gao and P. J. Antsaklis, "On Stable Solutions of the One and Two Sided Model Matching Problems,"Control Systems Technical Note #59, Dept. of Electrical and Computer Engineering, University of Notre Dame, April 1988; Proc. of the 26th Allerton Conference on Communication, Control and Computing, pp. 191-198, Monticelo, IL, Sept. 28-30, 1988. Also in IEEE Trans. Automatic Control, Vol. 34, No. 9, pp. 978-982, September 1989.

Technical Notes and Correspondence

On Stable Solutions of the One- and Two-Sided Model Matching Problems

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Abstract—**An algorithm is introduced to determine proper and stable solutions to the model matching problem. It utilizes the theory of inverses and state-space algorithms, and it guarantees a proper and stable solution to the problem when one exists. This approach is also used to determine proper and stable solutions of the two-sided matching problem. Examples are included.**

I. Introduction

The model matching problem (MMP) is defined as follows. Given proper rational matrices $P(p \times m)$ and $T(p \times q)$, find a proper rational matrix M such that the equation

$$PM = T \tag{1}$$

holds. This problem is also being referred to as the exact model matching problem.

The model matching problem has drawn the attention of many researchers both because of its importance in control and its attractive mathematical formulation. Note that (1) has also been studied in the literature over polynomials, rationals, and rings using different mathematical tools and methods.

The MMP was formulated and proposed by Wolovich [1] in the early 1970's. Its significance in control problems is well known. Here the main interest is in the case where P is proper and T is proper and stable, and a proper and stable M is to be determined. We call this problem stable MMP (SMMP). Although the conditions for existence of proper and stable solutions are known [10], [15], [16], it appears that there is no satisfactory approach to determine the solution when one exists. It should be mentioned that the algorithms which have been reported in the literature [6], [7] exhibit unsatisfactory performance. Early attempts used linear state feedback realizations of M to solve a restricted version of the problem. In particular, it was recognized early that a state feedback controller can be presented as a mathematically equivalent open-loop control law M of the form $PM = T$. This led to attempts to solve a restricted form of the SMMP (T constrained) solvable via some state feedback "equivalent" M [1], [2]; M in this case was proper and stable and it corresponded to a stabilizing stable feedback control law. The MMP was later formulated as a kernel problem [3], [4], [12] in which the stability of solutions was ignored; in [5] the properness of solutions was not considered. In this note an algorithm is presented to determine proper and stable solutions of the SMMP.

Assume that a proper and stable solution M to (1) does exist. The interest here is common in control cases where rank $[P] = p \leq m$. In this case there exists a right inverse P_{ri} of P, so that $PP_{ri} = I_{p\times p}$. Clearly

$$M = P_{ri}T \tag{2}$$

satisfies (1), and it is shown below that M can always be chosen to be proper and stable. The advantage of this approach is that the extensive results on inverses, which include algorithms in state space, can be used here with few modifications; furthermore, after P_{ri} has been found, solutions M for different T's can be easily calculated. Note that in many problems T is not really fixed but it can vary within certain limits [10]. In many cases the specifications of the problem allow us to choose T so that a proper and stable solution M to (1) does exist. A method for choosing such T is given in [10]; see also Section C in the Appendix. Here it is assumed that P and T are such that a proper and stable solution M to SMMP does exist, and this is the starting point in our approach.

The approach to solve SMMP is subsequently used to determine proper and stable solutions to the two-sided matching problem (TSMP). The two-sided matching problem (TSMP) [8], [9] arises in the multivariable control synthesis problem of designing a controller which makes the outputs of a physical system respond in a desirable manner to reference inputs and disturbances. In particular, consider the closed-loop system of Fig. 1 where S is the system to be controlled.

Let

$$\begin{bmatrix} y_c(s) \\ y_m(s) \end{bmatrix} = \begin{bmatrix} G(s) & H(s) \\ M(s) & N(s) \end{bmatrix} \begin{bmatrix} u_c(s) \\ u_r(s) \end{bmatrix} \tag{3}$$

where u_r is the vector of reference and disturbance inputs, u_c is the vector of control inputs, y_m is the available measurement output vector, y_c is the vector of outputs to be controlled, and G, H, M, and N are rational matrices. Let the controller be

$$u_c = Fy_m$$

so that the closed-loop system of Fig. 1 is stable and a desired transfer function H_d is obtained between u_r and y_c. Through a simple calculation

$$y_c = \{GF[I-MF]^{-1}N + H\}u_r = H_d u_r.$$

Now the problem can be formulated as follows.

Given G, H, M, N, and H_d, find a proper controller F so that:

i) the closed-loop system is stable; and ii) $GF[I-MF]^{-1}N + H = H_d$. (4)

This is the so-called general servomechanism problem (GSP) [8]. Define

$$X := F[I-MF]^{-1}$$

so that

$$F = [I+XM]^{-1}X.$$

X must be stable for internal stability. Substitute F in (4) to obtain

$$GXN + H = H_d$$

or

$$GXN = H_d - H. \tag{5}$$

We call this type of problem the two-sided matching problem (TSMP). In general, the TSMP is defined as follows.

Given the rational matrices A, B, and C, find X such that

$$AXB = C. \tag{6}$$

We are interested in proper and stable solutions X where A, B are proper and C is proper and stable. The approach used for the SMMP can be directly applied to determine proper and stable solutions of the TSMP. Based on the method presented below, a solution X is found of the form

$$X = A_{ri}CB_{li} \tag{7}$$

Manuscript received May 4, 1988; revised September 7, 1988.
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IEEE Log Number 8929486.

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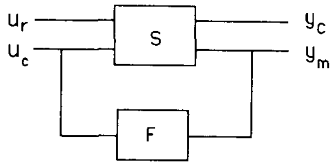

Fig. 1.

where A_{ri} and B_{li} are the corresponding right and left inverses of A and B.

In Section II, the existence of a proper and stable solution of the SMMP is discussed. In Section III, an algorithm to determine such solutions and an example are given. In Section IV, it is explained how to apply the algorithm to solve the TSMP. An example in [8] is used to compare our results to the previous ones. Finally, in Section V, some concluding remarks are included.

II. Preliminaries

Consider (1) and the SMMP problem. For a proper and stable solution M to exist, P and T have to satisfy certain conditions. We give the following existence theorem without proof.

Theorem 1 [10]: Given P proper, T proper and stable with rank $[P]$ = rank $[T] = p$, there exist proper and stable solutions M if and only if T has as its zeros all the RHP finite zeros and all the zeros at infinity of P together with their associated structure (described below).

Note that similar results have appeared elsewhere in the literature, e.g., [15], [16]. The theorem and its significance is discussed below.

Let $P = ND^{-1}$ and $T = N_T D_T^{-1}$ be right coprime polynomial fraction descriptions of P and T, and write

$$N = N_b \underline{N} \tag{8}$$

where the $p \times p$ matrix N_b contains all the RHP zeros of P in the sense that the roots of $|N_b|$ ($\neq 0$) are exactly the RHP zeros of P. This can always be achieved by using, for example, the Smith form of N. Note that since P has full row rank, N_b is a left divisor of a greatest left divisor (gld) of the rows of N, the determinant of which has roots the finite (transmission) zeros of P. According to Theorem 1, for a stable solution to exist, N_T has to have the form

$$N_T = N_b \underline{N}_T. \tag{9}$$

That is, the RHP zeros of P together with their structure (zero directions), which appear in N_b, appear unchanged in T. To illustrate, let z be a RHP zero of P, in N_b, and let the vector a satisfy $aN_b(z) = 0$; that is, a gives the direction associated with z. Then in view of (9), $aN_T(z) = 0$; that is, z together with its direction a [(z, a) imposes certain restrictions on the structure of N_T, via $aN_T(z) = 0$] will appear in T. For the zeros at infinity, the same argument is true. This can be proved in the same way by substituting $s = 1/w$ for s and studying the zeros at $w = 0$.

Note that Theorem 1 is also valid when rank $[T] < p$ with slight modifications. In this case the solvability condition is also relation (9), which, however, does not necessarily imply that the RHP zeros of P in N_b are also zeros of T.

III. Main Results

The solutions derived here are of the form $M = P_{ri}T$ (2). It is first shown that solutions M can always be chosen to be proper and stable when such solutions exist. Then an algorithm is given to derive such M.

Lemma 1: Let $PP_{ri} = I_{p\times p}$; then P_{ri} can always be written as

$$P_{ri} = N_i(N_b \underline{D}_i)^{-1} \tag{10}$$

where N_i and $(N_b\underline{D}_i)$ are right coprime polynomial matrices with N_b defined in (8) and $\underline{D}_i$ a polynomial matrix.

Proof: Let $P = ND^{-1}$ and $P_{ri} = N_i D_i^{-1}$ be right coprime polynomial matrix fraction descriptions of P and P_{ri}; then

$$ND^{-1}N_i D_i^{-1} = I$$

which implies that

$$ND^{-1}N_i = D_i. \tag{11}$$

Clearly, the left-hand side is a polynomial matrix and all the poles of P in D cancel with N_i since N and D are right coprime. Substitute (8) for (11)

$$N_b(\underline{N}D^{-1}N_i) = D_i.$$

Therefore, D_i can be written as

$$D_i = N_b \underline{D}_i \tag{12}$$

where $\underline{D}_i = \underline{N}D^{-1}N_i$ is a polynomial matrix. Q.E.D.

Remark: Lemma 1 shows that all the RHP zeros of P, in N_b, will appear as poles of any right inverse of P. This is, of course, a well-known result. The lemma, however, shows more than this. In particular, the contribution here is that the lemma shows that not only the RHP zero locations of P will be poles of P_{ri}, but also the associated RHP zero structure (zero directions) of P will appear in the pole structure of P_{ri}.

Theorem 2: If the conditions of Theorem 1 are satisfied and P_{ri} does not introduce any new RHP poles other than RHP zeros of P, then M in (2) is proper and stable.

Proof: In view of Theorem 1 and (9), $P = N_b\underline{P}$ and $T = N_b\underline{T}$. Since N_b is nonsingular, $\underline{P}$ still have full row rank. From (2) and Lemma 1,

$$M = P_{ri}T = (\underline{P}_{ri}N_b^{-1})(N_b\underline{T}) = \underline{P}_{ri}\underline{T}.$$

Since $\underline{T}$ is stable and $\underline{P}_{ri}$ contains only the stable poles of P_{ri}, M is stable. The properness of M can be proven by substituting $s = 1/w$. The zeros of $P(w)$ and $T(w)$ at $w = 0$ are the zeros of infinity of $P(s)$ and $T(s)$ and by using a similar approach, it is shown that M has no poles at $w = 0$. Q.E.D.

The algorithm introduced below to derive such proper and stable M utilizes results from the literature on inverses. See [11] for a survey. Here we are only interested in those algorithms which can arbitrarily assign the nonfixed poles of the inverse system. Antsaklis [13] gives a simple stable inverse algorithm but it works only for plants P where $\lim_{s\to\infty} P$ is of full rank. Patel [14] extended the results to also apply to strictly proper plants. In the algorithm introduced below, we combine the merits of both inverse algorithms to solve SMMP (see Sections A and B in the Appendix).

Given P with rank $[P] = p$, a right inverse P_{ri} is first determined. The only RHP poles of P_{ri} are the RHP zeros of P. In Step 5 a solution M of SMMP is determined as $M = P_{ri}T$. It is proper and stable, in view of the above result.

The SMMP Algorithm:

Step 1: Find an irreducible state-space realization of P as $\{A, B, C, E\}$, where A, B, C, and E are $n \times n$, $n \times m$, $p \times n$, and $p \times m$ real matrices, respectively.

Step 2: If rank $[E] = p$, set $A' = A$, $B' = B$, $C_q' = C$, $E_q' = E$, and $\Phi(s) = I$, and go to Step 4.

If rank $[E] < p$, find C_q, E_q, and $\Phi(s)$ such that rank $[E_q] = p$ and $P = \Phi^{-1}(s)\ [C_q(sI - A)^{-1}B + E_q]$ where $\Phi(s)$ is a $p \times p$ nonsingular polynomial matrix.

Step 3: Find $\{A', B', C_q', E_q'\}$, an irreducible representation of $\{A, B, C_q, E_q\}$.

Step 4: Find a proper right inverse of the system $\{A', B', C_q', E_q'\}$ as $\{A_I, B_I, C_I, E_I\}$. Calculate the transfer matrix $P_I = C_I(sI - A_I)^{-1}B_I + E_I$.

Step 5: Calculate $M = P_{ri}T = P_I\Phi(s)T$ which is a solution of the SMMP.

Remark 1: For Step 1, many algorithms are available, e.g., the Hankel method [12], or the coprime fraction method [3], [12].

Remark 2: In Step 2, a method in [14] can be used to find C_q, E_q, and $\Phi(s)$. It is also outlined in the Appendix.

Remark 3: In Step 4, we can use the algorithm in (see the Appendix). Assume P is of order n and it has k zeros, then per nth-order right inverse can always be found with k zeros of P as oles and remaining $(n - k)$ poles arbitrarily assignable.

Remark 4: Note that this algorithm can be directly used in the more general case of Ω-stabilization, considered in [15], although here we concentrate only on the standard stabilization problems (no poles in the RHP). To do so the location of the zeros included in N_b must be changed from the RHP to Ω^c, the complement of Ω, which is a (good) region of the complex plane symmetric with respect to the real axis.

Example 1: Let

$$P=\begin{bmatrix} \frac{s-1}{s(s+1)} & \frac{s-1}{s(s+2)} \end{bmatrix}=[0,\ s-1]\begin{bmatrix} -(s+1) & s(s+1) \\ s+2 & 0 \end{bmatrix}^{-1}$$

be a right coprime fraction.

Step 1: An irreducible realization of P is

$$A=\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix},\ B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix},\ C=[0\ \ -1\ \ 1],\ E=[0\ \ 0].$$

Step 2: Since rank $[E]=0<p$, we apply the method in [14] and find

$$C_q=[-1\ \ 0\ \ -2] \quad E_q=[1\ \ 1] \text{ and } \Phi(s)=s.$$

Step 3: An irreducible representation of the system $\{A, B, C_q, E_q\}$ is

$$A'=\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad B'=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C_q'=[-2\ \ -3] \quad E_q'=[1\ \ 1].$$

Step 4: Using the algorithm in [13], we find

$$A_l=\begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix} \quad B_l=\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_l=\begin{bmatrix} -2 & 0 \\ 3 & 3 \end{bmatrix} \quad E_l=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$P_l=C_l(sI-A_l)^{-1}B_l+E_l=\begin{bmatrix} \frac{(s+1)}{(s+2)} \\ \frac{3}{(s-1)} \end{bmatrix} \quad P_{ri}=P_l\Phi(s)=\begin{bmatrix} \frac{s(s+1)}{(s+2)} \\ \frac{3s}{(s-1)} \end{bmatrix}.$$

Step 5: If T is chosen as

$$T=\begin{bmatrix} \frac{(s-1)}{(s+1)(s+3)} & \frac{(s-1)}{(s+1)(s+4)} \end{bmatrix}$$

then a proper and stable M to $PM=T$ is

$$M=P_{ri}T=\begin{bmatrix} \frac{s(s-1)}{(s+2)(s+3)} & \frac{s(s-1)}{(s+2)(s+4)} \\ \frac{3s}{(s+1)(s+3)} & \frac{3s}{(s+1)(s+4)} \end{bmatrix}.$$

Notice that P_{ri} is not proper.

IV. Application to the Two-Sided Matching Problem

The basic idea is to use the stable inverse algorithm, that was used for SMMP, to solve the two-sided matching problem (TSMP). We believe that this approach has certain advantages over the earlier results by Ohm *et al.* [8] and by Ozguler *et al.* [9]. First, it not only gives the conditions on the existence of solution but also gives a computational method to find the solution. Second, it provides some insight into the problem in that the existence of proper and stable solution X depends on the locations and directions of the RHP and infinite zeros of the transfer matrices A, B, and C. This leads to a practical approach to pick C which is related to the closed-loop transfer matrix in view of (5). To illustrate the approach, consider (5), and suppose that z is a zero of G (or N), and the row vector a is determined from $aG(z)=0$ (or $aN(z)=0$); then according to the theory in [10] (see Section C in the Appendix), the zero z together with its structure will appear in (H_d-H) if and only if

$$a(H_d(z)-H(z))=0$$

or

$$aH_d(z)=aH(z). \tag{13}$$

It appears that this method gives clearer constraints on H_d and is more computationally efficient than the results in [8] and [9].

Consider

$$AXB=C. \tag{6}$$

Theorem 3: Suppose C is proper and stable, A and B are proper and have full row rank and full column rank, respectively. Then the TSMP in (6) has a proper and stable solution if C has all the RHP zeros and infinite zeros of A and B together with their associated structures (in the sense of Theorem 1).

Proof: The stability of solution X is proved below; the properness can be proved by using the same variable substitution, $s=1/w$, as in SMMP.

Consider solutions of the form

$$X=A_{ri}CB_{li}. \tag{14}$$

Let $A=N_{b1}\underline{N}_A D_A^{-1}=N_{b1}\underline{A}$ and $B=D_B^{-1}\underline{N}_B N_{b2}=\underline{B}N_{b2}$, where N_{b1} and N_{b2} are similarly defined as N_b in (8); then if C satisfies the condition of the theorem it can be factorized as

$$C=N_{b1}\underline{C}N_{b2} \tag{15}$$

with $\underline{C}$ an appropriate stable transfer matrix. By Lemma 1, a solution will be

$$X=(\underline{A}_{ri}N_{b1}^{-1})(N_{b1}\underline{C}N_{b2})(N_{b2}^{-1}\underline{B}_{li})=\underline{A}_{ri}\underline{C}\underline{B}_{li}. \tag{16}$$

Since the inverse algorithm does not introduce any RHP zeros, $\underline{A}_{ri}$ and $\underline{B}_{li}$ will be stable. Therefore, X will also be stable. Q.E.D.

Note that Theorem 3 is valid regardless of the row or column rank of C with slight modifications. When C does not have full row or column rank, the solvability condition is also (15), where, however, the zeros contained in N_{b1} and N_{b2} are not necessarily the zeros of C.

Example 2: A simple single-input single-output example from [8] is used to compare our results to the known results in literature, although the method suggested above shows better performance on multiinput multioutput systems. Given the system in Fig. 2, characterize all admissible H_d.

The system matrix is

$$\begin{bmatrix} G(s) & H(s) \\ M(s) & N(s) \end{bmatrix}=\begin{bmatrix} \frac{-(s-1)}{s(s-2)} & 1 \\ \frac{-(s-1)^2}{s(s+1)(s-2)} & \frac{(s-1)}{(s+1)} \end{bmatrix}.$$

Let H_d be the desired transfer function from u_r to y_c. We want to characterize all the H_d such that there exist proper and stable solutions X of the equation

$$GXN=H_d-H.$$

Here $A=G$, $B=N$, $C=H_d-H$, and $N_{b1}=N_{b2}=(s-1)$. According to Theorem 3, for a proper and stable solution X to exist, C must have the form

$$C=(s-1)^2\underline{C}$$

where $\underline{C}$ must be proper and stable with at least one infinite zero. Considering in this example $H=1$ and H_d is a scalar transfer function, $C=(n(s)-s(s))/d(s)$, where $H_d=n(s)/d(s)$, and it is straightforward to find the following constraints on $H_d(s)$:

1) $d(s)$ must be a Hurwitz polynomial;
2) $n(s)$ and $d(s)$ must satisfy

$$n(s)-d(s)=(s-1)^2k(s)$$

for some polynomial $k(s)$; and

3) the relative degree of $(n(s)-d(s))$ and $d(s)$ must be greater than zero.

From these constraints we can see that $d(s)$ is almost an arbitrary Hurwitz polynomial of degree higher than two, while $n(s)$ can be

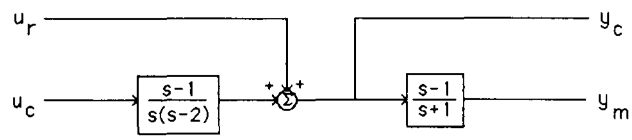

Fig. 2.

obtained from the equation

$$n(s)=(s-1)^2k(s)+d(s)$$

where $k(s)$ is chosen to make $n(s)$ close to the desired zero polynomial and relative degree of $(n(s) - d(s))/d(s)$ greater than zero. Suppose $d(s)$ is chosen to be $d(s) = (s + 1)(s + 2)(s + 3)$, and $n(s)$ is only required to have its zeros in the left-half plane, then $k(s)$ can be chosen as $k(s) = 1$, and therefore $n(s) = s^3 + 7s^2 + 9s + 7$, which has all its roots in the LHP. The solution is

$$X=G^{-1}(H_d-H)N^{-1}=\frac{-s(s-2)}{(s+2)(s+3)}$$

and the compensator F is

$$F=[I+XM]^{-1}X=\frac{-s(s+1)(s-2)}{n(s)}$$

which is proper and stable.

Comparing this example to the one in [8], we can see the method proposed here has several advantages. First it is more intuitive since it gives a direct relation between poles and zeros of G, N, H_d, and H and the existence of proper and stable solution X. Second, the constraints on H_d are fairly easy to apply when we try to pick an H_d that not only guarantees the existence of a proper and stable compensator F, but also meets the closed-loop performance specifications.

V. Conclusions

In this note, we presented a new algorithm to solve the stable model matching problems (SMMP). It is guaranteed that the solution will be stable if the condition in the existence theorem is fulfilled. This algorithm is made possible by formulating the SMMP as an inverse problem. It can be directly applied to solve the two-sided matching problem (TSMP) and the stability is also guaranteed. This algorithm is believed to be a practical systematic approach to find a solution of the SMMP. Note that a version of these results has appeared in [17].

Appendix

A. Solving Inverse Problem of Strictly Proper Plant in the State Space [14]

Assume that the plant P we are working on has the state-space form $\{A, B, C, E\}$. The algorithm in [13] only works on those where E has full row rank. In order to use this algorithm for any plant $\{A, B, C, E\}$, we use the following algorithm [14] to find P in the form:

$$P=\Phi^{-1}(s)[C_q(sI-A)^{-1}B+E_q] \tag{A-1}$$

where $\Phi(s)$ is a $p \times p$ nonsingular polynomial matrix, E_q has full row rank.

Algorithm:

Initialization: Set $i = 0$, $y_i = y$, $C_i = C$, and $E_i = E$.

Step i

I) Construct a $p \times p$ matrix U_i to compress the rows of E_i

$$U_iE_i=\begin{bmatrix} E_i' \\ 0 \end{bmatrix} \tag{A-2}$$

where E_i' has full row rank and $l_i = \text{rank}\,[E_i']$ is defined by the compression.

II) If $l_i = p$ then go to V).

If $l_i < p$, partition U_iy_i and U_iC_i conformably with the compression of E_i

$$U_iy_i=\begin{bmatrix} y_i' \\ y_i'' \end{bmatrix} \text{ and } U_iC_i=\begin{bmatrix} C_i' \\ C_i'' \end{bmatrix} \tag{A-3}$$

where y_i' and C_i' have l_i rows and y_i'' and C_i'' have $(p - l_i)$ rows.

III) Replace the last $p - l_i$ equations of (A-3) by the following:

$$\dot{y}_i''=C_i'\dot{x}=C_i'(Ax+Bu) \tag{A-4}$$

IV) Set

$$i=i+1,\ y_i=\begin{bmatrix} y_{i-1}' \\ \dot{y}_{i-1}'' \end{bmatrix},\ C_i=\begin{bmatrix} C_{i-1}' \\ C_{i-1}''A \end{bmatrix},\ \text{and } E_i=\begin{bmatrix} E_{i-1}' \\ C_{i-1}''A \end{bmatrix}. \tag{A-5}$$

Note that the new output equation is

$$y_i=C_ix+E_iu \tag{A-6}$$

Go to step i.

V) Set $q = i$. C_q and E_q are obtained with rank $[E_q] = p$, and the relation between y_q and $y_0(=y)$ can be expressed as

$$y_q=S_{q-1}(p)U_{q-1}S_{q-2}(p)U_{q-2}\cdots S_0(p)U_0y=\Phi(p)y \tag{A-7}$$

where

$$S_i(p)=\begin{bmatrix} I_{l_i} & 0 \\ 0 & I_{(p-l_i)}p \end{bmatrix},\ p=\frac{d}{dt},$$

and

$$\Phi(p)=S_{q-1}(p)U_{q-1}S_{q-2}(p)U_{q-2}\cdots S_0(p)U_0.$$

The transfer function matrix of (A, B, C_q, E_q) is given by

$$P_q=C_q(sI-A)^{-1}+E_q=\Phi(s)[C(sI_n-A)^{-1}+E]=\Phi(s)P. \tag{A-8}$$

The inverse of P_q can be derived using the algorithm below.

B. Stable Inverse Algorithm [13]

Let $E = \lim_{s\to\infty} P$ satisfying rank $[E] = p$. Assume that P is of order n and it has k zeros; then a proper nth-order right inverse can always be found with k zeros of P as its poles and remaining $(n - k)$ poles are arbitrarily assignable. Suppose that the state-space realization of P is $\{A, B, C, E\}$; the following procedure is designed to determine the inverse of P with $(n - k)$ nonfixed poles stable.

i) Find an $m \times m$ nonsingular matrix M such that $EM = [I_p{:}0]$.

ii) Calculate $[\underline{B}_1, \underline{B}_2] = BM$ and $A - \underline{B}_1C$.

iii) Find an *lsvf* matrix $\underline{F}_2$ which assigns the $n - k$ controllable poles of $\{A - \underline{B}_1C, \underline{B}_2\}$ in the LHP.

iv) The desired proper right inverse is

$$\left\{A+BM\begin{bmatrix} -C \\ \underline{F}_2 \end{bmatrix}, BM\begin{bmatrix} I_p \\ \underline{G}_2 \end{bmatrix}, M\begin{bmatrix} -C \\ \underline{F}_2 \end{bmatrix}, \begin{bmatrix} I_p \\ \underline{G}_2 \end{bmatrix}\right\}$$

where $\underline{F}_2$ was determined above and $\underline{G}_2$ is any $(m - p) \times p$ real matrix (which can be taken as 0 for convenience).

C. Selecting T in Control Design [10]

In control, T in $T = PM$ is chosen so that the system response $y = Tr$ to test input satisfies the control design specifications. The relation $N_T = N_b\underline{N}_T$ which characterizes the unstable finite zeros and the zeros that infinity T must have for a proper and stable solution M to exist, does not provide a convenient way to choose an appropriate T. Note that the transfer function entries in T are individually chosen to satisfy specifications, and their zeros do not necessarily appear as zeros of T. Therefore, there is a need for simple and direct conditions which will help the designer to choose T containing the unavoidable unstable or infinite zeros together with their directions. The following theorem on unstable zeros is given below without proof; the results may apply to infinite zeros with slight modifications.

Assume that z_i, $i = 1, \cdots, 1$, are distinct or if z_j is a multiple zero the rank reduction in $N(z_j)$ equals the multiplicity of z_j, where $P = ND^{-1}$ is in a coprime polynomial fraction form.

Theorem A: The unstable zeros of P together with their structure will appear in T if and only if

$$a_i T(z_i) = 0 \tag{A-9}$$

where a_i is determined from

$$a_i P(z_i) = 0. \tag{A-10}$$

For a simple example see Example 1, where P has only one zero at $z_1 = 1$. Here a_1 could be an arbitrary constant since $P(z_1) = [0\ 0]$, therefore, T has to satisfy $T(z_1) = [0\ 0]$.

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The Stability of a Family of Polynomials Can Be Deduced from a Finite Number $0(k^3)$ of Frequency Checks

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Abstract—**Let $\phi(s, a) = \phi_0(s) + a_1\phi_1(s) + a_2\phi_2(s) + \cdots + a_k\phi_k(s) = \phi_0(s) - q(s, a)$ be a family of real polynomials in s, with coefficients that depend linearly on parameters a_i which are confined in a k-dimensional hypercube Ω_a. Let $\phi_0(s)$ be stable of degree n and the $\phi_i(s)$ polynomials ($i \geq 1$) of degree less than n. A Nyquist argument shows that the family $\phi(s)$ is stable if and only if the complex number $\phi_0(j\omega)$ lies outside the set of complex points $-q(j\omega, \Omega_a)$ for every real ω. In a previous paper we have shown that $-q(j\omega, \Omega_a)$, the so-called "$-q$ locus," is a $2k$ convex parpolygon. The regularity of this figure simplifies the stability test. In this note we again exploit this shape and show that to test for stability only a finite number of frequency checks need to be done; this number is polynomial in k, $0(k^3)$, and these critical frequencies correspond to the real nonnegative roots of some polynomials.**

Manuscript received May 31, 1988.

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IEEE Log Number 8929488.

I. Introduction

Let $\phi(s, a)$ be a family of real polynomials denoted by

$$\begin{aligned}\phi(s, a) &= \phi_0(s) + a_1\phi_1(s) + a_2\phi_2(s) + \cdots + a_k\phi_k(s) \\ &= \phi_0(s) + q(s, a)\end{aligned} \tag{1}$$

where $\phi_0(s)$ is monic, stable, and of degree n and where the other $\phi_i(s)$ are nonzero polynomials of degree less than n. The vector of parameters $a = (a_1, \cdots, a_k)$ are confined to a hypercube Ω_a, i.e., $\Omega_a = \{a \mid a_i^- \leq a_i \leq a_i^+,\ 1 \leq i \leq k,\ a_i^- < 0,\ a_i^+ > 0\}$. It follows from the Nyquist stability criterion that the family will be stable if at each frequency ω, the set $-q(j\omega, \Omega_a)$ does not include the point $\phi_0 = (\mathrm{Re}(\phi_0(j\omega)), \mathrm{Im}(\phi_0(j\omega)))$. Since $-q(j\omega, \cdot)$ is an affine map over the hypercube Ω_a, then the $-q$ locus is a polytope. In fact, such polytopes have even more structure. In [1] it is shown that the $-q$ locus is a $2k$-convex parpolygon—a convex polygon of an even number of sides ($2k$) in which opposite sides are equal and parallel. For these shapes it is easy to determine whether ϕ_0 is contained in the $-q$ locus; easier than the so-called $H(\delta)$-theory, (see [2]) which was developed to handle general, planar, and polytopic shapes.

In this note we again exploit the regularity of these $2k$-convex parpolygons and now turn our attention to the "frequency sweeping" task; recall that one must check if ϕ_0 is outside the $-q$ locus for *all* $\omega \in [0, \infty)$. Due to the "finite bandwidth" of the polynomial family $\phi(j\omega, a)$, the frequency sweep may be conducted over a bounded interval of frequencies Δ, say $[0, M]$. This fact has been previously noted in [2], [5]. We will improve upon this result and show that only a finite number of frequencies in $[0, M]$, call them the "critical frequencies," must be checked. Most importantly, we will prove that this set of critical frequencies can be determined *a priori* and that its cardinality is $0(k^3)$ where k is the number of parameters. As we will see, these critical frequencies correspond to the real, nonnegative roots of some special, but easily constructed, polynomials. Consequently, in concert with [1], these results lead to the most computationally efficient algorithm, presently available, for deducing whether the polynomial family in (1) is stable.

II. The Two-Parameter Case

To demonstrate the nature of our methods we first consider the simple case of only two parameters a_1 and a_2. In the next section we address the general case. To this end suppose that the family of polynomials is described by

$$\phi(s, a) = \phi_0(s) + a_1\phi_1(s) + a_2\phi_2(s); \qquad a \in \Omega_a \tag{2}$$

where ϕ_1 and ϕ_2 are not identically zero. In the sequel we let Re $(\cdot)$ and Im $(\cdot)$ denote the real and imaginary parts of a complex expression and define E_ϕ and 0_ϕ to be the so-called *even* and *odd parts* of a polynomial $\phi(s)$. That is,

$$\phi(s) = E_\phi(s^2) + s0_\phi(s^2)$$

or

$$\phi(j\omega) = E_\phi(-\omega^2) + j\omega 0_\phi(-\omega^2).$$

For notational simplicity we write E_ϕ and 0_ϕ for $E_\phi(-\omega^2)$ and $0_\phi(-\omega^2)$.

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