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Stabilization and Regulation in Linear Multivariable Systems

P. J. ANTSAKLIS AND J. B. PEARSON

Abstract—**This paper presents a generalization of some recent results of Cheng and Pearson [1]. It is shown that the conditions obtained under certain assumptions in [1] are valid under more general assumptions. We therefore obtain the solution to a more general linear multivariable regulator problem than that previously solved by Wonham and Pearson [2].**

I. Introduction

In this paper we consider a linear time-invariant system of the form

$$\begin{aligned}\dot{x}_1(t)&=A_1x_1(t)+A_3x_2(t)+B_1u(t)\\ \dot{x}_2(t)&=A_2x_2(t)\\ y(t)&=C_1x_1(t)+C_2x_2(t)+C_3u(t)\\ z(t)&=D_1x_1(t)+D_2x_2(t)+D_3u(t)\end{aligned}\tag{1}$$

which may be interpreted as a plant with state vector $x_1(t)$ coupled to an exogenous system with state vector $x_2(t)$. The control input is $u(t)$; $y(t)$ is the measured output and $z(t)$ is the regulated output. The control problem is to find a feedback controller with input $y(t)$ and output $u(t)$ such that for every initial state of the closed-loop system we obtain $\lim_{t\to\infty} z(t)=0$.

A less general version of this problem ($C_3=0$, $D_3=0$) was formulated and solved in [2] using the geometric approach. In this paper a frequency domain formulation is used, thus permitting the general case to be treated directly without the additional complications involved in the geometric approach. A robust version of this more general problem has been treated by Davison and Goldenberg [3], but the *robust* regulator problem is quite different from the problem posed here.

Laplace transforming (1) and denoting initial conditions by a constant vector w, we may write

$$\begin{aligned}\hat{y}(s)&=-H_1(s)\hat{u}(s)+G_1(s)w\\ \hat{z}(s)&=H_2(s)\hat{u}(s)+G_2(s)w\end{aligned}\tag{2}$$

where $\hat{y}$, $\hat{z}$, and $\hat{u}$ represent the Laplace transforms of y, z, and u, respectively, and

$$\begin{aligned}-H_1(s)&=C_1(s-A_1)^{-1}B_1+C_3\\ H_2(s)&=D_1(s-A_1)^{-1}B_1+D_3.\end{aligned}\tag{3}$$

We seek a proper controller $C(s)$ of the type

$$\hat{u}(s)=C(s)\hat{y}(s)$$

such that

$$\left[H_2(s)C(s)(I+H_1(s)C(s))^{-1}G_1(s)+G_2(s)\right]$$

is analytic for $\mathrm{Re}\, s\geq 0$ and

$$\psi_1(s)\psi_C(s)|I+H_1(s)C(s)|$$

is a Hurwitz polynomial where $\psi_1(s)$ and $\psi_C(s)$ are the characteristic polynomials of $H_1(s)$ and $C(s)$, respectively, and $|X(s)|$ represents the determinant of a square matrix $X(s)$. This problem is called RPIS (regulator problem with internal stability), and when $H_1(s)=F(s)P(s)$ and $H_2(s)=P(s)$, it has been solved by Cheng and Pearson [1] under the assumptions that $F(s)$ and $P(s)$ have no hidden unstable modes and that the pair $(F(s),P(s))$ is admissible, i.e.,

$$\psi_F^+(s)\psi_P^+(s)=\psi_{FP}^+(s)$$

where $\psi^+(s)$ represents the unstable factor of the polynomial $\psi(s)$.

In this paper we show how the results obtained in [1] apply directly to the problem specified by (1) or (2). Basically, this involves replacing the admissibility of $(F(s),P(s))$ by the stabilizability and detectability of (C_1,A_1,B_1) and establishing that this is all that is needed to allow the proof of Theorem 1 [1] to go through as originally stated. In Section II we furnish a new proof, using the results of [1], of the well-known fact that a system (A,B,C,D) can be stabilized by output feedback through a proper controller if and only if (C,A,B) is stabilizable and detectable. In Section III stabilizability and detectability of (C_1,A_1,B_1) are introduced, a lemma to replace Lemma 1 of [1] is proved, and the main results of the report are stated. Our results are completely self-contained and rely only on the solution of RPIS given in [1].

II. Stabilizability of a Linear System

Consider the problem of regulating

$$\begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t)\\ y(t)&=Cx(t)+Du(t)\\ z(t)&=\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}\end{aligned}\tag{4}$$

or

$$\begin{aligned}\hat{y}(s)&=[C(s-A)^{-1}B+D]\hat{u}(s)+C(s-A)^{-1}x(0)\\ \hat{x}(s)&=(s-A)^{-1}B\hat{u}(s)+(s-A)^{-1}x(0).\end{aligned}$$

Identify

$$\begin{aligned}-F(s)P(s)&=C(s-A)^{-1}B+D\\ P(s)&=\begin{bmatrix}(s-A)^{-1}B\\ I\end{bmatrix}\\ G_1(s)&=C(s-A)^{-1}\\ G_2(s)&=\begin{bmatrix}(s-A)^{-1}\\ 0\end{bmatrix}.\end{aligned}$$

Since $-F(s)=[CD]$, the pair $(F(s),P(s))$ has no unstable hidden modes if and only if (A,B) is stabilizable and is admissible if and only if (C,A,B) is stabilizable and detectable. In order to use the results of [1], define (omit the argument s for simplicity) the minimal fraction representations

$$FP=Q_1^{-1}P_1=P_2Q_2^{-1}$$

$$\{Q_1G_1\}_+=R_1S_1^{-1}=\tilde{S}_1^{-1}\tilde{R}_1\tag{5[1]}$$

$$\{PYQ_1G_1+G_2\}_+=R_2S_2^{-1}\tag{6}$$

$$PQ_2=P_3Q_3^{-1}$$

[1] $\{\ \}_+$ denotes that part of a partial fraction expansion containing closed right half-plane poles.

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The authors are with the Department of Electrical Engineering, Rice University, Houston, TX 77001.

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and polynomial matrices (X, Y) and (X_1, Y_1) by

$$Q_1X + P_1Y = I$$
$$X_1R_1 + Y_1S_1 = I.$$

From [1] we know that if (F,P) is admissible, there exists a proper controller stabilizing the system (4) if and only if there exist polynomial matrices N, V, and W such that

$$S_2^{-1}S_1 = N \tag{7}$$

and

$$R_2NX_1 = P_3V + W\tilde{S}_1. \tag{8}$$

Since (F,P) is admissible if and only if (C,A,B) is stabilizable and detectable, we will prove the following (well-known) result.

Lemma 1: There exists a proper controller stabilizing the system (4) if and only if (C,A,B) is stabilizable and detectable.

Proof: Necessity is obvious. For sufficiency we show that there exist polynomial matrices N, V, and W satisfying (7) and (8).

First replace (C,A,B) by a minimal (i.e., controllable and observable) representation $(\bar{C},\bar{A},\bar{B})$. Then since

$$-FP = -Q_1^{-1}P_1 = \bar{C}(s-\bar{A})^{-1}\bar{B} + D,$$

it follows that

$$Q_1\bar{C}(s-\bar{A})^{-1}\bar{B} + Q_1D$$

is a polynomial, and since

$$(s-\bar{A})^{-1}\bar{B}$$

is minimal,

$$Q_1\bar{C}(s-\bar{A})^{-1}$$

is a polynomial. Since (C,A) is detectable,

$$\left\{Q_1\bar{C}(s-\bar{A})^{-1}\right\}_+ = \left\{Q_1C(s-A)^{-1}\right\}_+ = 0 \tag{9}$$

and so we can choose in (5) $R_1 = 0$, $\tilde{R}_1 = 0$, $S_1 = I$, $\tilde{S}_1 = I$.

Clearly,

$$FPYP_1 - FP$$

is a polynomial, and since $\bar{C}(s-\bar{A})^{-1}\bar{B}YQ_1$ is a polynomial, so is

$$\bar{C}\left[(s-\bar{A})^{-1}\bar{B}YQ_1\bar{C} + I\right](s-\bar{A})^{-1}\bar{B}.$$

Then since $(s-\bar{A})^{-1}\bar{B}$ is minimal and

$$\bar{C}\left[(s-\bar{A})^{-1}\bar{B}YQ_1\bar{C} + I\right]$$

is a polynomial, so then is

$$\bar{C}\left[(s-\bar{A})^{-1}\bar{B}YQ_1\bar{C} + I\right](s-\bar{A})^{-1}$$
$$= \bar{C}(s-\bar{A})^{-1}\left[\bar{B}YQ_1\bar{C}(s-\bar{A})^{-1} + I\right].$$

Again since $\bar{C}(s-\bar{A})^{-1}$ is minimal and $(\bar{B}YQ_1\bar{C}(s-\bar{A})^{-1} + I)$ is a polynomial, it follows that

$$(s-\bar{A})^{-1}\left(\bar{B}YQ_1\bar{C}(s-\bar{A})^{-1} + I\right)$$

is a polynomial. Because of stabilizability and detectability

$$\left\{(s-\bar{A})^{-1}\bar{B}YQ_1\bar{C}(s-\bar{A})^{-1} + (s-\bar{A})^{-1}\right\}_+$$
$$= \left\{(s-A)^{-1}BYQ_1C(s-A)^{-1} + (s-A)^{-1}\right\}_+ = 0$$

and this together with (9) yields

$$\{PYQ_1G_1 + G_2\}_+ = 0.$$

Therefore, from (6) we may choose $R_2 = 0$, $S_2 = I$.

Since $S_2^{-1}S_1 = I$, (7) is true. Since $R_2 = 0$ and $\tilde{S}_1 = I$, let V be any polynomial matrix and define

$$W = -P_3V.$$

Then (8) is true and we have completed the proof. From this result it is clear that stabilizability and detectability of (C_1,A_1,B_1) in (1) is the necessary replacement for admissibility of (F,P) in [1].

III. Main Results

We first establish the necessary generalization of Lemma 1 [1]. Let

$$(s-A_1)^{-1}B_1 = SP^{-1}$$

be a minimal fraction representation. From (3)

$$H_1 = -C_1SP^{-1} - C_3 = R_1P_1^{-1}$$
$$H_2 = D_1SP^{-1} + D_3 = R_2P_2^{-1} \tag{10}$$

where (R_1,P_1) and (R_2,P_2) are minimal.

Lemma 2: Assume (C_1,A_1,B_1) is stabilizable and detectable. Then $\{H_2P_1\}_+ = 0$.

Proof: Since (R_1,P_1) and (R_2,P_2) are minimal, it follows from (10) that

$$P = P_1G_1 = P_2G_2$$

and $|G_1|$ is stable by assumption. Thus,

$$H_2P_1 = R_2P_2^{-1}P_1 = R_2G_2G_1^{-1}$$

and so

$$\{H_2P_1\}_+ = 0.$$

Our main result is stated as follows.

Theorem: Given the system (2) with $H_1(s)$ and $H_2(s)$ defined by (3). Assume (C_1,A_1,B_1) is stabilizable and detectable. There exists a proper controller which solves RPIS if and only if there exist polynomial matrices N, V, and W satisfying

$$S_2^{-1}S_1 = N$$
$$R_2NX_1 = P_3V + W\tilde{S}_1$$

where

$$H_1 = Q_1^{-1}P_1 = P_2Q_2^{-1}$$
$$\{Q_1G_1\}_+ = R_1S_1^{-1} = \tilde{S}_1^{-1}\tilde{R}_1$$
$$\{H_2YQ_1G_1 + G_2\}_+ = R_2S_2^{-1}$$
$$H_2Q_2 = P_3Q_3^{-1}$$

are minimal representations and X_1 and Y are polynomial matrices defined by

$$Q_1X + P_1Y = I$$
$$X_1R_1 + Y_1S_1 = I.$$

Proof: The proof is identical to the proofs of Theorems 1 and 4 in [1]. The key step is to replace Lemma 1 in [1] by Lemma 2 as stated above.

IV. Conclusion

In this paper we have shown how the results of Cheng and Pearson [1] apply to a more general regulation problem than has been previously studied. The results are not completely satisfactory in that hybrid (i.e., part state space, part frequency domain) conditions have been used. This

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has been necessary because of the key relation that exists between $H_1(s)$ and $H_2(s)$ as stated by Lemma 1 [1] when $H_1 = FP$ and $H_2 = P$ and by Lemma 2 in the present case. In the general case, all we can say at the present time is that if H_1 has no unstable hidden modes and the relation given by Lemma 2 is satisfied by H_1 and H_2, the existence of N, V, and W satisfying the above equations is sufficient for RPIS to be solvable.

The particular formulation used in Section II may be useful in parameterizing all stabilizing controllers through the rational function K used in [1] to solve RPIS. This could possibly lead to a better understanding of minimal-order stabilizing controllers.

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On Alternative Methodologies for the Design of Robust Linear Multivariable Regulators

H. G. KWATNY AND K. C. KALNITSKY

***Abstract*—This paper presents two synthesis algorithms which embody the two major variants of the numerous methodologies which have been proposed for the design of multivariable linear regulators which exhibit the property of disturbance rejection with or without additional robustness qualities. It is shown that these two procedures generally lead to substantively different compensator structures.**

I. Introduction

During the past decade a number of techniques have been proposed for the design of multivariable linear regulators enjoying the properties of disturbance rejection and, in some cases, structural stability [1]–[12], [14], [15], [18], [26]–[30]. Many of these procedures have been fairly widely applied [3], [13], [16], [17], [19], [20], [30], and in view of the interest which has been evidenced by theoretician and practitioner alike, it is clear that with the development of these concepts the day of application of "modern" multivariable control theory is at hand. It appears that although many investigators have independently evolved their own specific design methodologies, these can be grouped into two distinct variations—those employing estimates of (possibly artificial) disturbance states, and those employing dynamic error augmentation. Examples of the former have been proposed by Kwatny *et al.* [15], [18], Balchen *et al.* [30], Parker [29], Sebakhy and Wonham [26], and Francis [27]. Examples of the latter have been proposed by Davison *et al.* [6]–[11], Young and Williams [12], and Calovic and Cuk [14].

In the present authors' work both types of design procedures have been applied—specifically those methods of Kwatny *et al.* and Davison *et al.* Experience has shown that closed-loop transient behavior can be significantly different even when the designs are carried out with the intent of attaining the same performance requirements. The question naturally arises as to whether these differences come about because of the inherent latitude the designer has at various points within the design processes or whether they are, in fact, due to fundamental differences in structure. This paper reports on studies intended to provide at least a partial answer to this question.

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H. G. Kwatny is with the Department of Mechanical Engineering and Mechanics, Drexel University, Philadelphia, PA 19104.

K. C. Kalnitsky is with The Analytic Sciences Corporation, Reading, MA 01867.

In Section II, the regulator problem and the robust regulator problem are defined as they will be discussed in this paper. The adopted formulation is somewhat less general than can be treated and that can be found discussed in several of the papers cited above. Nevertheless, this choice has been made in order to avoid obscuring the main ideas with a host of nonessential technical detail. Section III presents two basic design algorithms which typify the essential variants to be found in the references. Section IV identifies the fundamental difference between these methodologies and correlates this result with the classical compensator design techniques. Section V presents a brief concluding statement.

II. The Regulation Problem and Compensator Structure

This paper is concerned with a linear time-invariant system defined by the equations

$$\begin{aligned} \dot{x} &= Ax + E\omega + Bu \\ \dot{\omega} &= Z\omega \\ y &= Cx + F\omega \\ \bar{y} &= G\omega \\ e &= y - \bar{y} \end{aligned} \tag{1}$$

where x is an n-dimensional plant state vector, y is an r-dimensional output vector, $\bar{y}$ is an r-dimensional reference output, u is an m-dimensional input vector, ω is a q-dimensional vector representing a combined state for the exogenous disturbance and output reference, and e is an r-dimensional error vector. In what follows it is assumed that (A,B) is controllable and (C,A) is observable and that B and C are of full rank. With some restriction in generality it is assumed that the composite pair $\left\{ [\, C \,\vdots\, (F-G)\,], \left[\begin{array}{c:c} A & E \\ \hdashline U & Z \end{array} \right] \right\}$ is observable. Where this condition is required it can usually be relaxed to detectability. The discussion in Francis [27] on this assumption is pertinent. The *regulator problem* is the construction of a feedback controller such that the closed-loop system—excluding the disturbance states ω, is stable (internal stability) and $e(t) \to 0$ as $t \to \infty$ for all initial states (output regulation).

A *robust* (or *structually stable*) solution of the regulator problem has the desirable property that closed-loop stability and output regulation are preserved under specified classes of perturbations of plant and controller parameters.

Numerous researchers have studied the regulator problem from various viewpoints in recent years [1]–[12], [14], [15], [18], [26]–[30]. The notion of robust solutions of the regulator problem appears to have originated with Davison [9] and has been further examined by Davison and Goldenberg [11], Pearson *et al.* [12], Francis and Wonham [25], Sebakhy and Wonham [26], and Francis [27].

Necessary and sufficient conditions for the existence of a solution to the regulator problem and the robust regulator problem have been stated by several authors, notably Davison [9], Davison and Goldenberg [11], Francis and Wonham [25], and Francis [27]. These conditions are summarized for the problem as stated above in the following theorem.

Theorem 1: A necessary and sufficient condition for the existance of a solution to the regulator problem is that the following conditions hold:

1) (A,B) is stabilizable;

2) (C,A) is detectable;

3) There exists an $n \times q$ matrix X and an $m \times q$ matrix U satisfying the relations

$$\begin{aligned} AX - XZ + BU &= E \\ CX &= F - G. \end{aligned}$$

A necessary and sufficient condition for the solution to the robust regulator problem is obtained if 3) is replaced by

4) $\operatorname{rank} \begin{bmatrix} A-\lambda_i & B \\ C & 0 \end{bmatrix} = n + r$, for each λ_i in the spectrum of Z.

Proof of Theorem 1 for the regulator problem is given by Francis [27].