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Axioms in Frege

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As is generally well appreciated, Frege's conception of axioms is an old-fashioned one, a conception according to which each axiom is a determinate non-linguistic proposition, one with a fixed subject-matter, and with respect to which the notion of a 'model' or an 'interpretation' makes no sense. As contrasted with the fruitful modern conception of mathematical axioms as collectively providing implicit definitions of structure-types, a conception on which the range of models of a set of axioms is of the essence of those axioms' significance, Frege's view is a dinosaur.

It is the purpose of this essay to investigate some of the philosophically-important aspects of that dinosaur, in order to shed light on Frege's understanding of the foundational role of axioms, and on some of the ways in which our current conception of such axiomatic virtues as independence and categoricity have (and in some cases have not) been informed by a move away from Frege's understanding of the foundational role of axioms.

Part I: Consistency and Independence of Axioms

I.1. *Conceptual Analysis and Logical Entailment*

We begin with a brief excursion into Frege's views about the connection between conceptual analysis and proof in arithmetic. Early in *Grundlagen*, Frege explains the project of that book as follows:

> … the fundamental propositions of arithmetic should be proved, if in any way possible, with the utmost rigor; for only if every gap in the chain of deductions is eliminated with the greatest care can we say with certainty upon what primitive truths the proof depends …
>
> If we now try to meet this demand, we very soon come to propositions *which cannot be proved so long as we do not succeed in analyzing concepts which occur in them into simpler concepts or in reducing them to something of greater generality*. Now here it is above all Number which has to be either defined or recognized as indefinable. This is the point which the present work is meant to settle. On the outcome of this task will depend the decision as to the nature of the laws of arithmetic.[1]

[1] *Grundlagen* §4. Emphasis added

The basic idea here is a familiar one: truths expressed using terms such as "prime number" or "continuous function" can often only be proved once the relevant notions have been broken down into complexes of simpler ones, so that the truths themselves are expressed by sentences whose terms stand for the relative simples, and whose syntactic complexity is greater than that of the original sentences. Frege's idea is that his analysis and clarification of such fundamental concepts as that of cardinal number, *en route* to the proof of claims about cardinal numbers, is of a piece with standard instances of conceptual clarification in the history of mathematics:

> [I]n mathematics a mere moral conviction, supported by a mass of successful applications, is not good enough. Proof is now demanded of many things that formerly passed as self-evident. Again and again the limits to the validity of a proposition have been in this way established for the first time. The concepts of function, of continuity, of limit and of infinity have been shown to stand in need of sharper definition. Negative and irrational numbers, which had long since been admitted into science, have had to submit to a closer scrutiny of their credentials.
>
> In all directions these same ideals can be seen at work – rigor of proof, precise delimitation of extent of validity, and as a means to this, sharp definition of concepts.
>
> §2. Proceeding along these lines, we are bound eventually to come to the concept of Number and to the simplest propositions holding of positive whole numbers, which form the foundation of the whole of mathematics.[2]

The idea of conceptual analysis as an essential preliminary to proof is maintained throughout Frege's work. In the 1914 *Logic in Mathematics* manuscript, we find a similar sentiment:

> In the development of science it can indeed happen that one has used a word, a sign, an expression, over a long period under the impression that its sense is simple until one succeeds in analysing it into simpler logical constituents. By means of such an analysis, we may hope to reduce the number of axioms; for it may not be possible to prove a truth containing a complex constituent so long as that constituent remains unanalysed; but it may be possible, given an analysis, to prove it from truths in which the elements of the analysis occur.[3]

[2] *Grundlagen* §§1-2.
[3] Frege [1914]. *[1983]* 226 / *[1979]* 209.

The general picture painted here, and the pattern Frege follows in *Grundlagen* and *Grundgesetze*, is as follows:

(a) We begin with a thought expressed in a relatively-unanalyzed way, e.g. via a sentence of ordinary arithmetic.

(b) We analyze that thought, typically via a decomposition of some of its central concepts (e.g. *cardinal number*, *successor*), allowing an expression of the resulting, highly-analyzed thought via a more syntactically-complex sentence, sometimes a formula of Frege's formal language.

(c) We derive that latter sentence (or a definitional abbreviation thereof) from a (possibly-empty) collection of premise-sentences, showing thereby that the original thought is logically entailed by the thoughts (if any) expressed by the premise-sentences.

In the case of his own logicist project, the set of premise-sentences mentioned in (c) is empty, and the derivation establishes the purely-logical grounding of the original thought mentioned in (a), i.e. the purely-logical grounding of a truth of arithmetic.

The syntactic complexity gained by the analytic procedure is typically essential to the derivations: in paradigmatic cases, the original sentence, noted in (a), has simple terms standing for logically-complex objects or relations, and hence is insufficiently complex, syntactically, to be derived from the relevant premise-sentences via purely logical principles. Hence the utility of the conceptual analyses with respect to derivations: on the basis of an analysis, we achieve a new sentence whose syntax better represents the logical complexity of the content it expresses than does the original, syntactically simpler sentence.[4]

The first proto-arithmetical examples of this pattern in Frege's work appear in *Begriffsschrift* Part III. Here for example Frege proves the transitivity of the ancestral of a binary relation in a way that appeals essentially to his analysis of the

[4] The importance of the increased syntactic complexity of the analysans-sentence can be obscured when that sentence is abbreviated via the use of defined terms, i.e. of terms whose role is explicitly to stand as abbreviations. The virtue of the increased syntactic complexity of the un-abbreviated sentence is in such a case achieved via the use of definitions, whose effect is that of dis-abbreviation on demand.

ancestral. Where f is a binary relation, Frege refers to the strong ancestral of f as the relation of "following in the f-series." His proposal is that we analyze

(1) β follows α in the f-series

as

(1.1) $\forall F[(\forall x(f(\alpha,x) \rightarrow Fx) \ \& \ Her_f(F)) \rightarrow F\beta]$,

where "$Her_f(F)$" is short for the statement that F is f-hereditary, i.e. for the statement that

$\forall x \forall y[(Fx \ \& \ f(x,y)) \rightarrow Fy]$.

The full, unabbreviated version of (1.1) is:

(1.2) $\forall F[(\forall x(f(\alpha,x) \rightarrow Fx) \ \& \ \forall x \forall y[(Fx \ \& \ f(x,y)) \rightarrow Fy]) \rightarrow F\beta]$

As Frege demonstrates, given this analytic toolkit, the fully-analyzed version of (1), namely (1.2), is derivable via purely-logical inferences from the fully-analyzed versions of

(2) β follows γ in the f-series

and

(3) γ follows α in the f-series.

For our purposes, the important point is that this analysis and derivation together show, as Frege sees it, that the thought expressed by (1) follows logically from the thoughts expressed by (2) and (3), despite the fact that the sentence (1) itself is not derivable via pure logic from the sentences (2) and (3). By moving to the more highly-articulated sentences (1.2) and the corresponding analysantia for (2) and (3), we reveal previously-hidden logical structure, and make it possible to demonstrate, via rigorous derivations, the relations of logical dependence that obtain amongst the original thoughts.

Similarly for the thoughts and sentences of arithmetic. While the sentence

(4) $\forall x \forall y \forall z((succ(x,z) \ \& \ succ(y,z)) \rightarrow x=y)$

is not derivable from purely-logical axioms via purely-logical rules of inference, Frege's view is that (i) a conceptual analysis of the thought expressed by (4) yields as analysans a thought expressed via a more-complex sentence, and (ii) that this more-complex sentence is derivable from purely-logical axioms via rules of pure logic. As Frege sees it, this is sufficient to show that the original thought expressed by (4) is itself logically grounded in, because logically entailed by, truths of pure logic. That is to say, the analysis and subsequent demonstration show that the original arithmetical truth is itself a truth of logic.

On this picture, there is a one-way connection between formal derivability (a relation between sentences) and logical entailment (a relation between thoughts). If a sentence σ is derivable in a system like Frege's *Begriffsschrift* from a set Γ of sentences, then the thought $\tau(\sigma)$ is logically entailed by the set $\tau(\Gamma)$ of thoughts expressed by the members of Γ. (This assumes, of course, that the axioms and rules of the logical system have been well chosen.) But not vice-versa: the fact that a sentence σ is *not* derivable in such a purely-logical way from a set Γ of sentences is no guarantee that $\tau(\sigma)$ fails to follow logically from the set $\tau(\Gamma)$. For, as we've seen, such a case of non-derivability is compatible with the existence of a sentence σ' and set Γ' of sentences expressing analyses respectively of the thoughts expressed by σ and Γ, and such that σ' is derivable via pure logic from Γ'. And in this case, from Frege's point of view, we have a demonstration that the thought $\tau(\sigma)$ is logically entailed by the set $\tau(\Gamma)$.

I.2. Independence and Independence-Demonstrations

The connection between derivability and logical entailment, and especially the ways in which they fail to converge, loom large in Frege's rejection of a standard kind of independence-proof in geometry. In this section, we present the main lines of that rejection, in order to clarify some of the fundamental differences between Frege's understanding of independence and a more modern one, one that turns on the existence of models.[5]

The issue is raised most starkly in Frege's reaction to the independence-proofs given in David Hilbert's *Foundations of Geometry*. Hilbert's demonstrations employ the now-familiar strategy of reinterpreting non-logical constants, in this

[5] This issue is treated in more detail in my [1996], [2007] and [2012].

case such terms as “point,” “line,” and “between,” in terms of a background theory (here a theory of real numbers). Given a true sentence σ of geometry and a set Γ of true sentences of geometry, the fact that the terms of σ and Γ can be re-interpreted in such a way that Γ’s members express theorems of the background theory (i.e. truths about constructions on the real numbers) while σ expresses the negation of such a theorem, shows that σ is independent of Γ, relative to the consistency of that background theory. To put the point in more modern terms: a model of Γ∪~σ demonstrates the independence of σ from Γ, assuming the consistency of the theory employed in the construction of that model.

To have a concrete example in mind, let’s examine Hilbert’s first three axioms of order:

II.1 If A, B, and C are points lying on a straight line, and B lies between A and C, then B lies between C and A.

II.2 If A and C are two points lying on a straight line, there is at least one point B that lies between A and C, and at least one point D such that C lies between A and D.

II.3 Of any three points lying on a straight line, one and only one lies between the other two.[6]

The first of these is straightforwardly independent of the other two, as can be shown by providing an interpretation of “point,” “line,” “lies on,” and “lies between” under which II.1 expresses a falsehood while II.2 and II.3 express truths. Adapting one of Hilbert’s own interpretations, let “point” stand for the set of pairs <x, y> of real numbers, “line” the set of ratios [u:v:w] of real numbers, and “lies on” the set of pairs <<x, y>, [u:v:w]> such that ux + vy + w = 0.[7] Finally, let “lies between” be such that: point <x2, y2> *lies between* points <x1, y1> and <x3, y3> iff all three points lie on a line, and either x1 < x2 < x3, or (x1=x2=x3 and y1< y2< y3), where “<” is the usual

[6] Hilbert [1899] in Hallett and Majer pp 436-525; my translation.

[7] These are roughly the assignments Hilbert makes for these terms in his first consistency-proof in FG; see §9. The ratios assigned to “line” must also meet the requirement, here and in Hilbert’s presentation, that not both of u and v are zero; this will be suppressed for ease of presentation in what follows. Hilbert uses a countable subset, not the set of all reals.

ordering on the reals.[8] Then, appealing to straightforward facts about the real numbers, such as their density under the usual ordering and the asymmetry of that ordering, it is straightforward to see that II.2 and II.3 express theorems of real arithmetic, while II.1 expresses the negation of such a theorem.

This procedure straightforwardly shows the independence of the first axiom-sentence from the other two, which is to say that it shows that we can't deduce the first from the others via purely-logical steps. Here, by "logical steps" are meant those of the kind that make no appeal to the meanings of the geometric terms, but appeal only to the structure of the axioms themselves together with the meanings of such terms as "and" and "not." These inferential principles of are of a kind that can straightforwardly be rigorized as syntactic transformation-rules in a formal system such as that of Frege's *Begriffsschrift*. As shorthand, we'll call the sense of independence here, i.e. that of non-derivability via such syntactically-specifiable steps of inference, *syntactic independence*. To say that II.1 is syntactically independent of II.2 and II.3 is to say that the first sentence is not derivable from the second and third via the usual kinds of topic-neutral steps of deductive inference.

That the existence of a Hilbert-style reinterpretation does show syntactic independence is straightforward. Given an interpretation under which each member of {~II.1, II.2, II.3} expresses a theorem of R (real arithmetic), any derivation of II.1 from {II.2, II.3} would demonstrate the inconsistency of R itself. Hence, assuming the consistency of R, the interpretation shows that II.1 is not derivable from {II.2, II.3}.[9]

I.3 Frege's Complaints

Hilbert's claim to have demonstrated the consistency and independence of (sets of) axioms of geometry strikes Frege as clearly false. For Frege, because logical independence is just the converse of logical entailment, independence is a relation between thoughts, rather than between sentences. As he puts it in 1906:

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[8] This is adapted from Hilbert's assignment, in which the requirement (under which II.1 is true) on the triples is that either $x1<x2<x3$, or $y1<y2<y3$, or similarly for >.
[9] This reasoning requires that the principles of derivation are sound, in the sense of being truth-preserving over all interpretations.

> When one uses the phrase 'prove a proposition' in mathematics, then by the word 'proposition' one clearly means not a sequence of words or a group of signs, but a thought; something of which one can say that it is true. And similarly, when one is talking about the independence of propositions or axioms, this, too, will be understood as being about the independence of thoughts.[10]

A thought τ is independent of set A of thoughts iff τ isn't logically entailed by A. Due to the gap, as we've seen above, between non-derivability and independence, the fact that a particular sentence, say II.1, is not derivable from a given set (e.g. {II.2, II.3}) is no guarantee that the geometric thought ordinarily expressed by that sentence fails to be logically entailed by the set of geometric thoughts ordinarily expressed by that set.

From Frege's point of view, there are two sets of thoughts involved when one engages in an independence-demonstration of Hilbert's kind. The first is the set of thoughts expressed by the target sentences when the non-logical terms are given their ordinary, in this case geometric, interpretations – when "point" means *point*, "between" *between*, and so on. As Frege sees it, these are thoughts about Euclidean space. The second set of thoughts is that set of thoughts expressed by the same sentences when the terms in question are re-interpreted in the way provided by a Hilbert-style independence-proof, e.g. via constructions on the real numbers. In our example, the two sets of thoughts are as follows:

<u>G - the Geometric thoughts:</u>

G1 *If A, B, and C are points lying on a straight line, and B lies between A and C, then B lies between C and A*.

G2 *If A and C are two points lying on a straight line, there is at least one point B that lies between A and C, and at least one point D such that C lies between A and D.*

G3 *Of any three points lying on a straight line, one and only one lies between the other two*.

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[10] Frege [1906] 401; *[1990]* 316 / *[1984]* 332.

R – the Real-number thoughts:

R1 *If pairs* $\langle x_a, y_a\rangle$, $\langle x_b, y_b\rangle$, $\langle x_c, y_c\rangle$ *of real numbers are such that for some triple* $\langle u, v, w\rangle$ *of real numbers:*

$$ux_a + vy_a + w = ux_b + vy_b + w = ux_c + vy_c + w = 0,$$

and either $x_a<x_b<x_c$, *or* ($x_a = x_b = x_c$ *and* $y_a<y_b<y_c$),

then either $x_c<x_b<x_a$, *or* ($x_a = x_b = x_c$, *and* $y_c<y_b<y_a$)

R2 *If pairs* $\langle x_a, y_a\rangle$ *and* $\langle x_c, y_c\rangle$ *of real numbers are such that for some triple* $\langle u, v, w\rangle$ *of real numbers* $ux_a + vy_a + w = ux_c + vy_c + w = 0$, *then:*

(1) *there is at least one pair* $\langle x_b, y_b\rangle$ *such that* $ux_b + vy_b + w = 0$, *and:*

either $x_a<x_b<x_c$, *or* ($x_a = x_b = x_c$ *and* $y_a<y_b<y_c$); *and*

(2) *there is at least one pair* $\langle x_d, y_d\rangle$ *such that* $ux_d + vy_d + w = 0$, *and:*

either $x_a<x_c<x_d$, *or* ($x_a = x_c = x_d$ *and* $y_a<y_c<y_d$);

R3 *For any three distinct pairs* $\langle x_a, y_a\rangle$, $\langle x_b, y_b\rangle$, $\langle x_c, y_c\rangle$ *of real numbers such that for some triple* $\langle u, v, w\rangle$ *of real numbers:*

$$ux_a + vy_a + w = ux_b + vy_b + w = ux_c + vy_c + w = 0,$$

exactly one of the following conditions (i)-(iii) holds:

(i) *either* $x_a<x_b<x_c$, *or* ($x_a = x_b = x_c$ *and* $y_a<y_b<y_c$), *or*

either $x_c<x_b<x_a$, *or* ($x_c = x_b = x_a$ *and* $y_c<y_b<y_a$);

(ii) *either* $x_b<x_a<x_c$, *or* ($x_b = x_a = x_c$ *and* $y_b<y_a<y_c$), *or*

either $x_c<x_a<x_b$, *or* ($x_c = x_a = x_b$ *and* $y_c<y_a<y_b$);

(iii) *either* $x_a<x_c<x_b$, *or* ($x_a = x_c = x_b$ *and* $y_a<y_c<y_b$), *or*

either $x_b<x_c<x_a$, *or* ($x_b = x_c = x_a$ *and* $y_b<y_c<y_a$),

To recap: each of G1-G3 is true; R1 is false, and R2 and R3 are true. The respective truth-values of the R-thoughts serve straightforwardly to demonstrate the syntactic independence of II.1 from {II.2, II.3}.

Because there are two sets of thoughts, there are two independence-questions at issue, from Frege's point of view. The first is the geometric one, that of the independence of G1 from {G2, G3}. The second is the arithmetical one, that of R1 from {R2, R3}. The second question is unproblematically answered just by noting the truth-values of the thoughts: since proof is truth-preserving, the false R1 is clearly independent of the set {R2, R3} of true thoughts. But Frege doesn't think that this answers the first question, the question about the geometric thoughts. As he puts it to Liebmann in 1900:

> As far as the lack of contradiction and mutual independence of the axioms is concerned, Hilbert's investigation of these questions is vitiated by the fact that the sense of the axioms is by no means securely fixed. … Hilbert was apparently deceived by the wording. If an axiom is worded in the same way, it is very easy to believe that it is the same axiom. But it depends on the sense; and this is different, depending on whether the words 'point,' 'line,' etc. are understood in the sense of Euclidean geometry or in a wider sense.[11]

We can see why the inference from the independence of R1 from {R2, R3} to the independence of G1 from {G2, G3} would be fallacious from Frege's point of view. Since the question of whether a given thought is *provable* from others can turn, as Frege sees it, on peculiarities of the contents of the non-logical terms used to express those thoughts, so too the question of the *independence* of a given thought from others can turn on such peculiarities of content. That we cannot prove R1 from {R2, R3} is no guarantee that we cannot prove a fully-analyzed version of G1 from the fully-analyzed version of {G2, G3}. Just as a detailed analysis of *finite cardinal number* and of *successor* reveals that the thought

Every finite cardinal number has a successor

is logically entailed by purely-logical premises, so too it is in principle possible – or, at least, has certainly not been ruled out by Hilbert's demonstration – that a further conceptual analysis of elements of G1 (and/or of G2 or G3) would reveal that G1 is logically entailed by {G2, G3}. As Frege puts the potential for new logical connections arising as a result of re-interpretation:

> I have reasons for believing that the mutual independence of *Euclidean* geometry cannot be proved. Hilbert tries to do it by widening the area so that Euclidean geometry appears as a special case; and in this wider area he can now show lack of contradiction by examples; but only in this wider area; for from lack of contradiction in a more comprehensive area we cannot infer lack of contradiction in a narrower area; for contradictions might enter in just because of the restriction.[12]

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[11] Frege to Liebmann 29 July 1900; *[1980]* 91
[12] *Ibid*

Here, the “restriction” in question is the provision of specific content to non-logical terms.

In short, Frege’s view of Hilbert’s independence- and consistency-proofs for geometric axioms as “failures” is immediately entailed by his views (a) that the axioms of geometry are thoughts, and (b) that syntactic independence and consistency are no guarantee of the independence and consistency of real axioms.

I.4. Consistency and Implicit Definitions

In contrast to Frege’s view of the geometric axiom-sentences as expressing determinate thoughts, Hilbert takes these sentences to form parts of implicit definitions. Each set of axiom-sentences, as Hilbert sees it, defines a multiply-instantiable complex condition, in his words a “scaffolding.”

> [I]t is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points I think of some system of things, e.g. the system: love, law, chimney-sweep … and then assume all my axioms as relations between these things, then my propositions, e.g. Pythagoras’ theorem, are also valid for these things. In other words: any theory can always be applied to infinitely many systems of basic elements.[13]

Each axiom-sentence stipulates a condition that forms part of that whole complex condition defined by the set taken together. Thus understood, independence-questions about (what Hilbert calls) “axioms” are questions regarding the independence of particular such conditions from one another.

In the case of Axioms II.1 – II.3, the items satisfying the defined condition are those 4-tuples whose members, when taken to interpret respectively the terms “point,” “line,” “lies on,” and “lies between,” make those axiom-sentences true. The conditions defined by each of the three axiom-sentences are as follows:

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[13] Hilbert to Frege 29 December 1899; *[1980]* 40-41

[C1] *P* and *L* are nonempty sets; *LO* is a binary relation; *B* is a ternary relation. For all a, b, c ε *P*: if ∃l ε *L*(<a, l> ε *LO* & <b, l> ε *LO* & <c, l> ε *LO*), and <b,a,c> ε *B*, then <b, c, a> ε *B.*

[C2] *P* and *L* are nonempty sets; *LO* is a binary relation; *B* is a ternary relation. For all a, c ε *P*, for all l ε *L:* if (<a, l> ε *LO* & <c, l> ε *LO*), then ∃b,d ε P (<b, l> ε *LO* & <d, l> ε *LO* & <b,a,c> ε *B* & <c,a,d> ε *B*).

[C3] *P* and *L* are nonempty sets, *LO* is a binary relation; *B* is a ternary relation. For all a, b, c ε P: if ∃l ε *L*(<a, l> ε *LO* & <b, l> ε *LO* & <c, l> ε *LO*), then exactly one of conditions (i) – (iii) holds:

(i) <b,a,c> ε *B* or <b,c,a> ε *B;*

(ii) <a,b,c> ε *B* or <a,c,b> ε *B*;

(iii) <c,a,b> ε *B* or <c,b,a> ε *B*

In the cases in which Hilbert is interested, more axiom-sentences will be involved, and so more restrictions on the structure will be spelled out. But the general picture can be illustrated nicely by this minimalist example. Independence questions as applied to such defined conditions are questions of satisfiability: our example will be that of whether the satisfaction of [C2] and [C3] by an arbitrary 4-tuple implies the satisfaction by that 4-tuple of [C1]. If not, then [C1] is in the relevant sense "independent" of {[C2], [C3]}.

A Hilbert-style interpretation of the kind already discussed answers this independence-question immediately: by providing an interpretation of "point," etc under which II.1 expresses a falsehood and II.2 and II.3 express truths, we demonstrate the existence of sets *P, L, LO,* and *B* that satisfy [C2] and [C3] while failing to satisfy [C1]. Similarly, an interpretation satisfying each of the sentences II.1, II.2, and II.3 demonstrates the consistency, in the sense of the satisfiability, of the scaffolding defined by them all, i.e. of the set {[C1], [C2], [C3]}.

The existence of a model of a set of axiom-sentences demonstrates, in short, both the syntactic consistency of that set of sentences, and the satisfiability of the condition defined by those sentences. When the axiom-sentences are first-order, syntactic consistency implies the existence of a model and hence implies the satisfiability of the defined condition; for higher-order languages, syntactic consistency is a weaker condition than satisfiability.

Frege recognizes the existence of the second kind of independence-question just noted, that concerning the independence of conditions defined by partially-interpreted sentences. He also recognizes, in passing, the success of Hilbert's method of demonstration as applied to this (in his view) less-central question. His description of Hilbert's strategy, now understood as the demonstration of the independence of defined conditions, is as follows:

> … you want to show the lack of contradiction between certain determinations. 'D is not a consequence of A, B, and C' says the same thing as 'The satisfaction of A, B, and C does not contradict the non-satisfaction of D.' … After reducing everything to the same schema in this way, we must ask, What means have we of demonstrating that certain properties, requirements (or whatever else one wants to call them) do not contradict one another? The only means I know is this: to point to an object that has all those properties, to give a case where all those requirements are satisfied. … If you are merely concerned to demonstrate the mutual independence of axioms, you will have to show that the non-satisfaction of one of these axioms does not contradict the satisfaction of the others. (I am here adopting your way of using the word 'axiom'.) But it will be impossible to give such an example in the domain of elementary Euclidean geometry because all the axioms are true in this domain. By placing yourself in a higher position from which Euclidean geometry appears as a special case of a more comprehensive theoretical structure, you widen your view so as to include examples which make the mutual independence of those axioms evident.[14]

Similarly,

> How… are we to understand Mr. Hilbert's formulation of the question? We may assume that it does not concern the whole axioms* [*footnote: As one can see here, … I accommodate myself to Mr. Hilbert's usage.] but only those of their parts that express characeristics of the concepts to be defined. … If these did contradict one another, no object having these … properties could be found. … [I]f one can produce [such an object], then this means that these characteristics do not contradict one another; and in fact this is just about the way in which Mr. Hilbert proves the consistency of his axioms. In reality, however, this is merely a matter of the consistency of the characteristics. Similarly concerning independence. If from the fact that an object has a first

[14] Frege to Hilbert 6 January 1900; *[1980]* 43

> property it may generally be inferred that it also has a second, then one may call the second dependent upon the first.[15]

As applied to our mini-example, Frege's point is that in order to demonstrate that the joint satisfaction of conditions C2 and C3 does not contradict the non-satisfaction of C1, the Hilbert-style procedure must leave the realm of Euclidean geometry in order to provide the "example" in question, the 4-tuple of constructions out of the reals which satisfies C2 and C3, and fails to satisfy C1. This construction does indeed show immediately that C1 is independent of C2 and C3; and it does so in just the way Frege sketches, namely, by producing an "object" (here, a 4-tuple) that satisfies C2 and C3 without satisfying C1.

Frege's complaint about this strategy is just that the result it demonstrates, that of the independence of multiply-instantiable conditions from each other, does not imply the independence of axioms proper – i.e. of thoughts. Referring to partially-interpreted sentences, i.e. to those parts of language that refer to the multiply-instantiable conditions in question as "pseudo-propositions," and to those conditions as "pseudo-axioms," Frege notes that

> Mr. Hilbert's independence-proofs simply are not about real axioms, the axioms in the Euclidean sense; for these, surely, are thoughts. Now nowhere in Mr. Hilbert's writings do we find a differentiation that might correspond to our own between real and pseudo-propositions, between real and pseudo-axioms. Instead, Mr. Hilbert appears to transfer the independence putatively proved of his pseudo-axioms to the axioms proper, and that without more ado … . This would seem to constitute a considerable fallacy. And all mathematicians who think that Mr. Hilbert has proved the independence of the real axioms from one another have surely fallen into the same error.[16]

To see exactly how the two kinds of independence-claims differ, let's return to our minimalist example in order to see why the independence of [C1] from {[C2], [C3]} does not imply, from Frege's point of view, the independence of the thought G1 from the pair of thoughts {G2, G3}. The first independence-claim is equivalent to the consistency of the existentially-quantified thought

[Gen] $(\exists P, L, LO, B)(\sim C1 \;\&\; C2 \;\&\; C3)$,

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[15] Frege [1903] 324; *[1990]* 266 / *[1984]* 277-8.
[16] Frege [1906] 402; *[1990]* 317 / *[1984]* 333.

while the second is equivalent to the consistency of the geometric thought

[Geo]   (~G1 & G2 & G3).

Frege refers to a thought of the first kind as a "general proposition" and to a thought of the second kind as a "particular proposition that is contained in" the general one. (Notice that [Geo] is obtained from [Gen] by stripping off the existential quantifiers, and instantiating the variables "P", "L", "LO", and "B" by the Euclidean *point, line, lies-on,* and *lies-between*, respectively.) The other "particular proposition" relevant here is the structurally-similar thought about the reals:

[Real] (~R1 & R2 & R3).

([Real] is obtained as above from [Gen], instantiating those variables now by their Hilbertian constructions.) Frege notes the connection between the different independence-claims as follows:

> If a general proposition contains a contradiction, then so does any particular proposition that is contained in it. Thus if the latter is free from contradiction, we can infer that the general proposition is free from contradiction, but not conversely.
> …
> It … seems to me that there is a logical danger in your speaking of, e.g., 'the parallel axiom' as if it were the same thing in every special geometry. Only the wording is the same; the thought content is different in every different geometry … Now given that the axioms in special geometries are all special cases of general axioms, one can conclude from lack of contradiction in a special geometry to lack of contradiction in the general case, but not to lack of contradiction in another special case.[17]

The truth of [Real] immediately shows its consistency, and hence of course the consistency (and indeed the truth) of [Gen]. Frege's claim above is that neither of these consistency results suffices to establish the consistency of the false [Geo]. For the change in content when moving from [Real] to [Geo], or – equivalently – the injection of additional content when moving from [Gen] to [Geo] – can bring inconsistency in its wake. In response to Korselt's defense of Hilbert's method as a means of demonstrating the independence of axioms, Frege says:

> In saying that modern mathematics no longer designates certain facts of experience with its axioms but at best indicates them, Mr. Korselt brings the

[17] Frege to Hilbert 6 Jan 1900; *[1980]* 47-8.

> axioms of modern mathematics into contrast with those of Euclid; and doubtless we may assume that he counts himself among the modern mathematicians. Clearly, he also counts Mr. Hilbert among them and believes that with this proposition he has hit upon the latter's usage of the word 'axiom.' If this is correct, then it is a gross error to assume that Mr. Hilbert has shown anything at all about the dependence or independence of the Euclidean axioms; or that when he talks about the axiom of parallels it is the Euclidean axiom.[18]

For, again, the relation of independence between multily-instantiable conditions is no guarantee of the relation of independence between the similarly-expressed thoughts.

In sum: Frege recognizes two senses in which a collection of axiom-sentences can be used to characterize independence-questions. The first is that in which those sentences are taken to express determinate thoughts. The independence-questions at issue in this case are those of the independence of specific thoughts from collections thereof. Here for example are the questions of the independence of G1 from {G2, G3}, and of R1 from {R2, R3}. The second use of sentences is in the characterization of multiply-instantiable conditions, for which purpose some of the terms of the sentences are taken to appear schematically, with no fixed interpretation. In this case, the relevant independence-questions are those of the independence of various of the conditions so characterized from one another. In our example, these concern the relationships between conditions C1, C2, and C3; our central question was that of the independence of C1 from the pair (C2, C3). Because of the importance of specific content to questions of independence and consistency, Frege's view is that these various related consistency-questions can have different answers. Most importantly for our purposes, the consistency of a particular instance (e.g. [Real]) of a general thought (e.g. [Gen]) suffices for the consistency of that general thought, and for the truth of the equivalent concept-consistency claim (the consistency of the condition (~C1, C2, C3)), but does not suffice for the consistency of any other instance (e.g. [Geo]) of that general thought. In short: the independence of R1 from {R2, R3} demonstrates the independence of C1 from (C2, C3), but does not suffice for the independence of G1 from {G2, G3}.

[18] Frege [1906] 297 *[1990]* 284 / *[1984]* 296

One final point about the general issue of consistency and independence before moving on: Inconsistency, as Frege understands it, can be extremely difficult to discover, since it can turn on features of the thoughts in question that are only brought out after the hard work of conceptual analysis. As he puts it in *Grundgesetze* II §§143-4:

> How is it to be recognized that properties do not contradict each other? There seems to be no other criterion than to find the properties in question in one and the same object.
>
> …
>
> Or is there perhaps a different way to prove freedom from contradiction? If there were, this would be of the highest significance for all mathematicians who ascribe the power of creation to themselves. And yet hardly anyone seems concerned to find such a method of proof. Why not? Probably because of the view that it is superfluous to prove freedom from contradiction since any contradiction would surely be noticeable immediately. How nice if things were like that! The proof of the Pythagorean Theorem would then go as follows:
>
>> "Assume the square of the hypotenuse is not of equal area with the squares of the two other sides taken together; then there would be a contradiction between this assumption and the familiar axioms of geometry. Therefore, our assumption is false, and the square of the hypotenuse is of an area exactly equal to the squares of the two other sides taken together."
>
> …
>
> Absolutely any proof could be conducted following this pattern. Unfortunately, the method is too easy to be acceptable. Surely, we see that not every contradiction lies openly on the surface.[19]

That contradictions can be hard to spot is just an instance of Frege's view that logical entailments in general can be hard to spot. And one of the central reasons that entailments are hard to spot is, as Frege sees it, that the discovery of a logical entailment can take a lot of hard analytic work.

> Often it is only after immense intellectual effort, which may have continued over centuries, that humanity at last succeeds in achieving knowledge of a concept in its pure form… [20]

[19] *Grundgesetze II* §§143-4. Translation from Ebert & Rossberg.
[20] *Grundlagen* p. vii

When the conceptual clarification in question is crucial to the logical entailment of a given thought by another, or equivalently to the inconsistency of one thought with another, the hard work required to provide the clarification is a measure of the non-triviality of the demonstration of that entailment or inconsistency. Not only is it true, from Frege's perspective, that not every contradiction "lies openly on the surface;" indeed some contradictions – and entailments - will take centuries to establish.

Part II. The Axioms of Arithmetic

II.1 The Content of the Axioms

Frege's view of the logical significance of the meanings of non-logical terms plays a crucial part in his understanding of the role of axioms in arithmetic. In order to lay out this issue clearly, it will be helpful to look at the broad outlines of a comparison between Frege's conception of the arithmetical axioms and the conception of his fellow logicist Dedekind.

To begin with, Frege's means of establishing logicism was to have consisted in the demonstration that each of a handful of axioms for arithmetic was a truth of logic. This demonstration was to have been, as above, proof-theoretic: the idea was to give a good analysis of the important components of those axiom-thoughts, and then to provide rigorous proofs, from principles of pure logic, of the highly-analyzed versions of those thoughts. The role of axiom-sentences in this project is simply that of expressing the all-important axiom-thoughts.

For Dedekind on the other hand, the axiom-sentences of arithmetic are taken as only partly interpreted; their role is to characterize a multiply-instantiable structure called, in the case of the arithmetic of the natural numbers, a "simply infinite system." A simply infinite system is any system of objects ordered by a relation in the way given by those axioms; and this is just the way that the natural numbers are ordered by less-than.

Once we have characterized the structure-type *simply infinite system*, we can, as Dedekind sees it, construct a particular instance of the type, an instance whose objects are *minimal* in the sense that their nature is entirely given by their joint satisfaction of the structure-defining conditions. These minimal objects are the natural numbers. The thesis of logicism, as Dedekind understands it, is established by noting that this construction process, and the arithmetical reasoning that follows it, are free of all appeal to intuition.

> In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought. … [N]umbers are free creations of the human mind.[21]

Consider the axiom-sentence

[E] "Every finite cardinal has a successor."

From Dedekind's point of view, there are two ways of thinking of the sentence. In what one might call its "original" role, as an axiom for arithmetic, it is a part of a structure-defining collection of sentences, serving the same role as do Hilbert's partially-interpreted axiom-sentences. Its arithmetical terms have no determinate reference. In what we'll call its "secondary" role, the sentence is interpreted via the objects and relations of the Dedekindian construction, those objects and relations whose nature is given by their joint satisfaction of the original axioms. These sentences will, like the sentences expressing Frege's axioms, express truths. But unlike Frege's axiom-sentences, there is no sense in which the arithmetical terms in Dedekind's interpreted sentences have a content that can be interrogated to reveal conceptual connections of the kind that are all-important to Frege's work. The contents of Dedekind's arithmetical terms have no nature, no properties, other than those explicitly given in the axioms themselves.

Thus the axioms of arithmetic as Dedekind understands them are not the kinds of things that can be shown to be, in Frege's sense, truths of logic. Axiom-sentence [E] taken in its original role simply helps to define the structure-type in question, by establishing some of the conditions to be met by instantiations of "finite cardinal number" and "successor." In this role, it expresses no determinate thought, and certainly doesn't express anything that could be proven from truths of logic. In its secondary role, in which it expresses a truth about Dedekind's "free creations of

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[21] Dedekind [1888] Preface to 1st edn.

the human mind," it expresses a truth, but again not one that can be proven from logical principles. There is no sense in which the truth in question can be "mined" for as-yet-unforeseen logical complexity, of the kind Frege finds buried beneath the surface of the thought that every finite cardinal has a successor. And of course Dedekind has no interest in proving the axioms of arithmetic: they stand as the fundamental basis of arithmetic, defining the structure-type that serves as the subject matter of that theory.

Regarding the content of the arithmetical terms, let's focus on the nature of the numbers themselves. It is crucial, for Dedekind, that the numbers have no features other than those given by their instantiation of the structure-type in question. Any appeal during the course of arithmetical reasoning to any other particular features of numbers would constitute an intrusion of something non-arithmetical into arithmetic. Hence the fact that arithmetic is the science of those numbers means that the numbers must be, in the sense noted above, "minimal."

In contrast to this conception, the very rich nature of the individual numbers is, for Frege, crucial to the truth of logicism. It's because the numbers are the extensions of particular concepts that truths about the numbers turn out to be (as we see on careful analysis) truths about one-one mappings and so on. And it's only once we see that the arithmetical truths are in fact such finely-structured thoughts about e.g. extensions and one-one mappings that we can prove them from purely-logical principles.

II.2 The Independence of the Axioms

Are the axioms of arithmetic, e.g. the Peano-Dedekind axioms, independent of one another?

The immediate and obvious answer, from "our" point of view in the 21st century, is that they are. Two lines of thought support this claim. The first is the intuitive idea that the axioms are not redundant: the axiom of induction for example does not, in any intuitive sense, "follow from" the other Peano axioms. The second is the fact that there is a model of the other axioms that isn't a model of the axiom of induction. The existence of a model gives us two different ways of thinking of the independence: (i) Proof-theoretically: we know that there's no way to derive the induction axiom-sentence from the others; this is demonstrated by the existence of the model, given just the assumption of the consistency of whatever background theory we're using to construct that model. (ii) In terms of satisfiability: the model shows that it's possible to satisfy the condition defined jointly by the remaining axiom-sentences without satisfying the condition defined by the induction axiom-sentence.

Similarly from Dedekind's point of view. The axioms of arithmetic as Dedekind understands them are independent of one another, and demonstrably so. Taken in their original, structure-defining role, the demonstration of independence is straightforward, and proceeds in essentially the same way as does a Hilbert-style independence-proof. One shows via construction of a model that the condition defined by the induction axiom is one whose satisfaction is not entailed by the satisfaction of the conditions defined by the remaining axioms. Similarly for the axioms taken in their secondary role, that of expressing determinate truths about Dedekind's minimal objects. Because the natural numbers are defined just as those canonical objects satisfying the conditions given by the axioms, the independence of one such condition from the others entails immediately that the *claim* that the natural numbers satisfy that given condition is itself independent of the claim that the natural numbers satisfy the remaining conditions.

What about Frege?

Here is Dummett, remarking on Frege's view that in order to be able to prove independence, we would need (amongst other things) a catalogue of the "logical" notions:[22]

> While the problem of characterizing the logical constants is no doubt of some importance, Frege is surely mistaken here. <u>Even if he is correct in saying that the principle of induction in number theory is to be reduced to purely logical</u>

[22] Frege's discussion of the need for the catalogue of logical notions appears in the context of his discussion of a potential means for demonstrating independence, in part iii of [1906]. For discussion of this issue, see: Dummett [1976], Ricketts [1997], Blanchette [forthcoming].

> inferences, it is surely intelligible, and correct, to say that the principle of induction is independent of the other Peano axioms: in saying this, we are simply prescinding from the possibility of defining 'natural number' in a second-order language, in terms of 0 and successor, and our ground for doing so is just the fact that, in the Peano axioms, all three notions are presented as primitive.[23]

This, I think, is not right. As we've sketched above, Frege's conception of independence is tightly bound up with his view of conceptual analysis as capable of unearthing logical complexity in, and hence logical connections between, thoughts. Though it might take hard conceptual work to unearth the logical complexity of what's expressed via the use of syntactically-simple terms, that complexity is nevertheless, from Frege's point of view, very real, and always relevant to questions of logical entailment and independence. This is exactly why, from Frege's point of view, syntactic independence does not suffice for independence between thoughts. Similarly, the independence e.g. of the thought R1 from {R2, R3} does not suffice for the independence of the apparently structurally-similar G1 from {G2, G3}, despite the fact that the G-thoughts are expressible via the same sentences as those used to express the corresponding R-thoughts. The specific contents of the non-logical terms is, as ever, a potential source of logically-relevant conceptual connections.

The (relevant) principle of induction, for Frege, is not a sentence, and not a condition defined by a sentence. It is a particular thought about a particular collection of objects. That thought, the thought that

[In] $\forall F((F(0) \,\&\, \forall x \forall y((Fx \,\&\, s(y,x)) \rightarrow Fy)) \rightarrow \forall x(Nx \rightarrow Fx))$

is the kind of thing that, once subjected to careful analysis, can be seen (Frege thinks) to be provable from purely-logical premises. There is no sense in which that very thought can "fail to be satisfied" by a collection of objects. Indeed, there's no sense in which it can be *satisfied* by a collection of objects, either; it's the wrong kind of thing to be satisfied by anything. Similarly for the rest of the axioms of arithmetic.

The thought [In] is also not proof-theoretically independent of the other axioms of arithmetic. [In] is, once sufficiently-carefully analyzed, provable from the

[23] Dummett: [1976] 13-14.

empty set. And similarly for the other axioms of arithmetic. This fact, i.e. that the axioms of arithmetic are so provable, is precisely Frege's logicist thesis.

In short, I think we have to say that, given Frege's understanding of what *axioms* are and what *independence* is, the axioms of arithmetic are not independent of one another. They are each truths of logic, provable from any set of premises whatsoever, and not independent of anything.

Does this mean that Frege can make no sense of the strong intuitive pull of the idea that e.g. the induction axiom doesn't follow from the rest of the Dedekind-Peano axioms? Here, the answer is a mixed bag. Strictly speaking, again, for Frege the induction axiom is *not* independent of the other axioms of arithmetic. In this it is just like any other truth of logic. But recall that Frege does make clear sense of the kind of independence that obtains between those conditions defined by collections of partially-interpreted sentences. Frege can easily acknowledge that if by "axioms of arithmetic" one means just such sentences or the multiply-instantiable conditions defined by them, then (suitably formulated) each such "pseudo-axiom" *is* independent of the others. This independence is, as above with e.g. the independence of C1 from {C2, C3}, straightforwardly demonstrable via the construction of a model. That is, to put it in Frege's terms, this independence amongst conditions is demonstrable via the presentation of an "example" that satisfies some conditions while failing to satisfy others. Frege's departure from the Dedekindian, Hilbertian, and modern conceptions of this independence lies simply in the fact that once we move from talk about pseudo-propositions and their implicitly-defined conditions (pseudo-axioms) to talk about axioms proper, i.e. to the relevant thoughts about 0 and successor, this independence vanishes. And here again, the result is, as Frege sees it, demonstrable. The demonstrations of this lack of independence are given by *Grundgesetze*'s proofs.

II.3 Frege on Defining Conditions

While Frege's interest is primarily in the status of thoughts, and not in that of partially-interpreted sentences or the conditions they define, nevertheless he does recognize the significance of the latter to arithmetical interests. Here we briefly note two such instances.

At *Grundgesetze* Vol I §144-157, Frege gives a careful treatment of that complex condition referred to by the axiom-sentences for arithmetic when the terms for zero and successor are replaced by free variables. This is, essentially, Dedekind's concept of simply infinite system. Frege's proof of Theorem 263 is a demonstration that this condition is satisfied by only those collections that are isomorphic to the collection of finite cardinals under their standard ordering.[24]

The significance of this categoricity result for Frege is interestingly different from its significance for Dedekind. From Dedekind's point of view, since the purpose of the axiom-sentences is the definition of a structure-type, the categoricity result is a demonstration of success: the axiom-sentences, if categorical, have done just what they were intended to do. The kind of success demonstrated for Frege's project by the categoricity result is not so complete. The parts of the real axioms that have been ignored for the purposes of this result – i.e. the contents of the terms for zero and successor – are essential for Frege's brand of logicism, and must be gotten right if the logicist demonstration is to succeed. Much more is required than mere categoricity. Nevertheless, given that the ordering of the finite cardinals is essential to the purely arithmetical truths regarding those objects, the successful characterization of that ordering is a crucial mark of success for Frege's project. Had the condition in question not been categorical, some necessary arithmetical feature of the finite cardinals would have been missing from Frege's treatment.

A more explicit focus on defining conditions appears at Vol II §175, in which Frege is engaged in defining the concept of *positival class*.[25] A positival class is a class of relations of a kind that plays a central role in Frege's account of the real numbers. The definition is given by stipulating four conditions, the joint satisfaction

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[24] For an enlightening discussion of this part of *Grundgesetze*, see Heck [1993] and Heck [1995]. For a discussion of the extent to which Frege was influenced here by Dedekind, see Sundholm [2001], and Heck's reply in the postscript to Heck [1995] as reprinted in Heck [2011] pp 66-68.

[25] For a helpful discussion of this issue, see Schirn, "Frege on Quantities and Real Numbers…"

of which suffices for falling under that concept. Frege comments as follows on the definition:

> In laying down this definition, I have tried to make use only of specifications that are both necessary and independent of each other. That this has been successful does not, to be sure, allow of proof; but it becomes probable when multiple attempts to reduce these specifications to others fail. In particular, it does not seem possible to dispense with the line,
>
> *[here Frege provides one of the four conditions]*.
>
> Should such an attempt nevertheless later succeed, then even if no logical error has been demonstrated in our definition, still a blemish would have been discovered.[26]

Frege's claim that the independence in question does not "allow of proof" might be thought to be of a piece with his better-known rejections of independence proofs, of the kind sketched above in the debate with Hilbert. But that Frege is not making the same complaint is clear from the note appended at the completion of *Grundgesetze* to this passage.

> That the presented specifications cannot be proven to be independent of each other should not be put forward unconditionally. It is of course thinkable that classes of relations could be found for each of which all but one of the specifications applies, so that each of them does not apply to one of the examples. Whether it is possible, however, to give such examples at this stage of the enquiry, without presupposing geometry or the fractions, negative and irrational numbers, or empirical facts, is doubtful.[27]

The independence of each of the four conditions from the other three is in principle demonstrable in just the way Frege sketches in his correspondence with Hilbert, i.e. by the provision of examples (in this case, classes of relations) satisfying all but one of the target conditions. Once again, since the entities in question are multiply-instantiable conditions rather than thoughts, there is no in-principle barrier to the demonstration of independence.

Incidentally, Frege's intuition that the fourth condition is in fact independent of the others is verified by the 1987 paper of Adeleke, Dummett, and Neumann, and

[26] Gg II §175

[27] **citation**

is settled in just the way Frege would appreciate, by the provision of an example satisfying conditions 1-3 and the negation of 4.[28]

Here we can see the importance of the fact that Frege is concerned in this part of *Grundgesetze* with the definition of a concept, and with a handful of multiply-instantiable conditions whose joint satisfaction amounts to satisfaction of that concept. The central point for our purposes is that while just such a situation is broadly speaking what Dedekind takes to be the setting when he provides axioms of arithmetic, it is radically different from Frege's understanding of the situation in which he, Frege, gives what appear superficially to be roughly the same axioms. Taking the axioms as Dedekind does, one obtains specifications of conditions amongst which claims of independence are straightforwardly, from Frege's (and Dedekind's) point of view, demonstrable. But such conditions are not, from Frege's point of view, the axioms of arithmetic. Because they're not thoughts, they can't stand as the foundation of anything. And because they are not essentially about the finite cardinal numbers, they don't have a content that can be mined in such a way as to make evident their purely-logical status.

Conclusion

As Frege says in 1906:

> Axioms are simply not characteristics of concepts. Therefore from the very first the consistency of the axioms must be distinguished from the consistency of the concepts introduced.[29]

There are two reasons, from Frege's point of view, to insist on the distinction between concepts and axioms. The first is that concepts are the wrong things to stand as the foundation of a science. The second is that the concepts indicated by partially-interpreted axiom-sentences generally have significantly different logical properties than do the thoughts expressed by those sentences as fully interpreted.

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[28] Adeleke, Dummett, and Neumann [1987]

[29] Frege [1906] 399; *[1990]* 314 / *[1984]* 330

As to the first: it's essential for Frege that the axioms of a theory be determinate, true thoughts. Because a theory itself is a collection of (hopefully) true claims about the world, its foundation, i.e. its collection of axioms, must be as well. Axioms must also be the kinds of things that can logically entail the remainder of the theory; that is their entire foundational role. And for Frege, the only items that bear the fundamental logical relation of entailment to one another are thoughts.

More importantly from the point of view of the present discussion, the thoughts expressed by a set of sentences can, for Frege, be considerably richer in logically-relevant content than are any of the multiply-instantiable conditions indicated by those sentences. For as Frege sees it, the contents of non-logical terms are not inert; they are in principle a source of important logical complexity. It is in part because of the complexity discovered in the contents of such terms as "continuous function" and "finite" that large areas of mathematics are reducible to impressively simple bases. And it is in part because of the further complexity discoverable in the contents of such terms as "0" and "successor" that the truths of arithmetic, and indeed the axioms of arithmetic themselves, turn out to be reducible, as Frege sees it, to logic. The fact that thoughts can in principle be mined so as to yield more logical complexity than is evident from the surface structure of the sentences that express them means that the consistency and independence of those concepts expressed when non-logical terms appear only schematically is no guarantee of the consistency and independence of the real axioms expressed by fully-interpreted sentences. And it's because Frege views just such rich content as essential to a theory like arithmetic that, from his point of view, axioms in the modern, schematic sense are pale imitations of real axioms.[30]

[30] Some of the material in this essay was presented to the "Classical Model of Science II" conference in Amsterdam, August 2011. Thanks to many members of the audience for helpful comments, and to Arianna Betti for the opportunity to participate.

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