Frege's Reduction

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Frege's logicism, the thesis that "the laws of arithmetic are analytic"[1]is standardly taken to be an important epistemological thesis. The traditional view of Frege's work is that his "reduction" of arithmetic to logic was intended to provide the cornerstone of an argument that the truths of arithmetic are knowable *a priori* and independently of anything which Kant would have labelled "intuition". The truth of Fregean logicism would, on this view, have repudiated the explanations of arithmetical knowledge offered by Kant and by Mill. It would have provided an explanation of arithmetical knowledge which was acceptable from a generally empiricist perspective, and which preserved the intuition that arithmetical truths are necessary and knowable *a priori*.

As against this received view, Paul Benacerraf has recently argued in "Frege: The Last Logicist"[2] that Frege's project was not an epistemological one, and was in particular not an attempt to counter Kant's view of the nature of arithmetical knowledge. Though Frege explicitly claims to be engaged in demonstrating the analytic, *a priori* nature of arithmetical truth, Benacerraf claims that Frege has so re-construed the notions of analyticity and *a priori* truth that the entire project is, from the very beginning, a non-epistemological one. As Benacerraf puts it, Frege's "attempt to establish the analyticity of arithmetic [is] not to be construed as an attempt to enter an ongoing philosophical debate between Kant and the empiricists, and indeed ... his very construal of the question took it out of that arena."

Benacerraf's claim is an alarming one. First of all, Frege's project clearly looks like an epistemological one, and one intended to provide an alternative to Kant's view of the nature of arithmetical knowledge. In the conclusion of the *Grundlagen*, Frege sums up that work as follows: "I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgments and consequently *a priori*,"[3] and that in showing this result, "we achieved an improvement on the view of Kant."[4] The early *Grundlagen* discussion of the "distinctions between *a priori* and *a posteriori*, analytic and synthetic" are accompanied by the note: "I do not, of course, mean to assign a new sense to these terms, but only to state accurately what earlier writers, Kant in particular, have meant by them." [5]

Secondly, the incompatibility of Frege's project with epistemological goals is due largely, on Benacerraf's reading, to the fact that Frege allows multiple reductions of the numbers to logical objects. But if the inference here is warranted, then it is difficult to see how any reductionist project, whether logicist or not, can ever be of epistemological significance. For no standard reduction of the numbers can be claimed to be uniquely correct. Thus if Benacerraf is right, it seems that we must not look to arithmetical reductions in the attempt to understand the nature of arithmetical knowledge.

It is the purpose of this paper to defend the epistemologically-loaded reading of Frege's project. Frege's view of the purpose and method of mathematical reductions was strikingly different from those conceptions which have supplanted his. I hope to make it clear that the view of Frege's project as having been irrelevant to epistemological concerns is due to a confusion of Frege's reduction with a reduction of a more modern variety. As a corollary, I take it that Frege's own reductionist project, in general outline, is one which must be taken seriously by those who are concerned with the nature of arithmetical knowledge.

1. The Project

Analyticity, for Frege, is first and foremost a proof-theoretic concept: an analytic truth is anything provable using only definitions and laws of logic.[6] As Frege views proofs,

> The aim of proof is ... not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another.[7]

The relations of dependence revealed by proofs is the key to what Frege regards as the epistemological status of the proposition proven. In particular, to have shown upon which propositions a given truth depends is to have demonstrated that knowledge of those propositions suffices to ground knowledge of the truth in question. As Frege puts it in 1897,

> ...I was looking for the fundamental principles or axioms upon which the whole of mathematics rests. Only after this question is answered can it be hoped to trace successfully the springs of knowledge upon which this science thrives.[8]

A crucial feature of Frege’s conception of analyticity is that the concept applies never to sentences, but only to the nonlinguistic items expressed by sentences.[9] This view is owed to the close connection in Frege's system between analyticity, provability, and dependence, along with the view that dependence is a relation between nonlinguistic items. Regarding dependence, Frege holds that

> We have to distinguish between the external, audible, or visible which is supposed to express a thought, and the thought itself. It seems to me that the usage prevalent in logic, according to which only the former is called a sentence, is preferable. Accordingly, we simply cannot say that one sentence is independent of other sentences; for after all no one wants to predicate this independence of what is audible or visible.[10]

Thus:

> What we prove is not a sentence, but a thought. And it is neither here nor there which language is used in giving the proof.[11]

The claim that the truths of arithmetic are analytic is a claim that each of a particular collection of nonlinguistic propositions is provable from, and hence dependent solely upon, laws of logic and whatever is expressed by the relevant definitions.

Two things go, for Frege, by the name "definition." The first is simply a stipulation that a newly-introduced piece of notation is to abbreviate a longer, already-meaningful series of marks. Only definitions of this kind occur in Frege's formal proofs, and all of these introduce conservative extensions of the formal system and hence are eliminable.[12] Such definitions express no extra-logical content. Thus a proof using laws of logic and definitions will demonstrate that the proven proposition depends for its truth solely on principles of logic. As Frege puts it, the result will be that "every proposition of arithmetic [is] a law of logic, albeit a derivative one."[13]

The second thing which Frege calls "definition" is the much less clearly-characterized conceptual analysis which precedes stipulative definition. Here Frege starts with an ordinary arithmetical concept and "reduces" it to simpler or more general concepts. The question of the definability in this sense of the concept of cardinal number is, Frege claims, "the point which [the *Grundlagen*] is meant to settle."[14] In general, the reduction of a concept to concepts expressible in primitive *Begriffsschrift* notation is followed in the formal work by a stipulative definition, one which introduces a new term as shorthand for the purported analysans.[15] The abbreviation is then used in proofs, instead of its less rigorously-defined ordinary counterpart. We return to the central issue of the relationship between the ordinary and defined terms in §2. The point to be stressed here is that (as Frege himself stresses) the definitions which occur within the proofs are always stipulative and eliminable.

Despite their dispensability within proofs, stipulative definitions play a crucial role in the overall project. A sentence containing a defined term expresses just what its definitional transcription expresses, and hence the proposition expressed by a sentence containing defined terms is determined to a large extent by the definitions of those terms. Had Frege's definitions been different from what they are, the propositions expressed by sentences containing defined terms would have been correspondingly different.

Where a proof is a series of propositions, the last of which is the proposition proven, let us use the word "derivation" for a series of sentences which together express a proof. As long as each sentence in a derivation either (a) expresses a truth of logic, (b) stipulates a notational convention, or (c) expresses a proposition which follows by logical principles from previous lines, then a derivation will demonstrate that the proposition expressed by its last line is analytic. Hence the importance of Frege's stipulative definitions: *which* propositions will have been shown to be analytic depends to a large extent on the definitions of those defined terms occurring in the derived sentences.

2. Definitions, Propositions and Multiple Reductions

If Frege's proofs are to be, strictly speaking, proofs of the truths of arithmetic, his definitions must, together with the other semantic features of the language, ensure that the derived sentences express arithmetical claims. Assuming that we express arithmetical truths in our ordinary arithmetical discourse, Frege's proofs will succeed in this way only if his derived sentences express the same propositions as do sentences of ordinary arithmetic. Taking a cue from Frege's talk of analysis and reduction, we might fairly presume that sentences containing defined terms are intended to express claims about "fully-analysed" versions of arithmetical objects and concepts, and that the success of the analyses underwrites the identity (or sufficient similarity) between proven proposition and ordinary arithmetical truth. If this is the appropriate description of the project, then the constraints on Frege's analyses and definitions are clear: The definition of an arithmetical term in the formal language must assign a meaning sufficiently close to that of the ordinary arithmetical term that sentences incorporating the defined term express the same propositions as (or ones sufficiently similar to) those expressed by sentences incorporating the ordinary term.

The question, now, is whether Frege's definitions do in anything like this sense "preserve the meanings" of the sentences of ordinary arithmetic. A central part of Benacerraf's argument is that, unlike the clearly epistemologically-motivated logicism of the logical positivists, Frege's logicism does not involve the kind of meaning-preservation required for epistemological significance.

We should note, first, that the importance of some kind of semantic-value preservation is clear to Frege, and underlies various of his complaints against competing foundationalist projects. In particular, Frege's response to Mill includes the objection that "Mill understands the symbol + in such a way that it will serve to express the relation between the parts of a physical body or of a heap and the whole body or heap; but such is not the sense of that symbol;" and that "In order to be able to call arithmetical truths laws of nature, Mill attributes to them a sense which they do not bear."[16] In response to Hankel's attempt to define addition in terms of our intuitions of magnitude, Frege objects:

> The definition can perhaps be constructed, but it will not do as a substitute for the original propositions; for in seeking to apply it the question would always arise: Are Numbers magnitudes, and is what we ordinarily call addition of Numbers addition in the sense of this definition?[17]

The response to Newton contains a similar worry. Even if Newton's definitions can be given in a non-circular way, Frege claims, "Even so, we should still remain in doubt as to how the number defined geometrically in this way is related to the number of ordinary life..."[18]

Given this concern with the preservation of ordinary meaning, one might expect from Frege some clear criterion of sameness of meaning, or of sameness of proposition expressed. But there seems to be no such workable criterion. The closest Frege comes to a clear conception of proposition-identity is the post-1890's relation of identity of sense, or of thought expressed. But the criterion of sense-identity most readily reconstructed from Frege's writings of this period is one on which two sentences express the same sense only if they are fairly obvious synonyms. And such a criterion is clearly not what Frege had in mind: no derived sentence in Frege's system is anything like an easily-recognizable synonym of its ordinary counterpart. Nor should it be, of course; if we are to have informative reductions of any kind, we cannot require that reduced and reducing sentences be in any immediate and obvious sense the same in meaning.[19]

An alternative criterion to which one might naturally hold Frege's definitions is that the defined singular terms and predicative expressions must refer to the same things as do their ordinary counterparts.[20] The idea here would be that one first determines which objects and relations are referred to by the parts of ordinary arithmetical sentences, and then assigns these objects and relations as the referents of the corresponding terms in Frege's formulae. The goal of the definitions would be, on this picture, to "preserve the reference" of the arithmetical terms.

But such atomistic reference-preservation is, as Benacerraf points out, clearly not what Frege had in mind. Frege's definition of natural number turns on the definition of "number which belongs to the concept ..."; that definition is as follows:

> "The Number which belongs to the concept F is the extension of the concept 'equinumerous with the concept F'."[21]

This definition is used to provide the definitions of each of the natural numbers; each number is on this account the number which belongs to some first-level concept. Thus each number is the extension of a particular second-level concept.[22] Immediately having given the above definition, Frege remarks in a footnote to which Benacerraf draws attention that

> I believe that for "extension of the concept'' we could write simply "concept.''

A similar claim occurs in the concluding pages of the *Grundlagen*. Here Frege says: "I attach no decisive importance to bringing in the extensions of concepts at all.''[23]

The thrust of these passages is that though he has in fact defined the numbers as extensions, the purposes of the project could as easily have been met by defining them as concepts.[24] As Benacerraf puts it, "Frege ... allows that different definitions, providing different referents ... might have done as well;'' consequently: "The moral is inescapable. Not even reference needs to be preserved.''[25]

One might hope to avoid Benacerraf's conclusion by pointing out that the passages in question are inconsistent with much of the rest of the *Grundlagen*, and are repudiated by the time of the *Grundgesetze.* Frege argues at length in the *Grundlagen* that the numbers are objects, and that they cannot be concepts. Further, one of the demands of logicism is that the existence of the natural numbers be established by purely logical means.[26] Thus if the numbers are to be objects, they must be the kind of objects which are guaranteed to exist via the principles of Frege's logic. Since the only such objects are the extensions of concepts, Frege's claim to "attach no decisive importance" to the use of extensions shows that he was perhaps not entirely clear about the demands of the project at this point.

Frege seems quickly to realize the indispensability of extensions. For the problematic *Grundlagen* claim is never repeated after 1885,[27] and we find in the *Grundgesetze* the claim about extensions (courses of value) that "we just cannot get on without them."[28] Even this repudiation, however, does not clarify matters entirely with respect to the issue of multiple reducibility. For though it is clear that, as Frege claims, the numbers must be defined as extensions, it is not at all clear that there is a unique sequence of extensions which must serve as the referents of the formal numerals.

In fact, it appears that Frege himself proposes two distinct reductions of the numbers to extensions. In the *Grundlagen*, the numerals refer to the extensions of second-level concepts, while in the *Grundgesetze* they refer to the extensions of first-level concepts.[29] This would seem to clinch the case in favor of Frege's tolerance of multiple reductions.[30] But here again there are difficulties. For there are no clear criteria by means of which to conclude that the relevant first-level and second-level extensions are in fact distinct. First, though every well-formed identity-sentence of the *Grundgesetze* receives a truth-value, no sentence which identifies the extension of a second-level concept with that of a first-level concept occurs in the *Grundgesetze* notation. For there are no singular terms in this notation for extensions of second-level concepts.[31] Secondly, there is no natural way to extend the *Grundlagen's* or *Grundgesetze*'s comprehension principles so as to deal with such "cross-level" identities.[32] Were Frege to extend the *Grundgesetze* notation to include names for the extensions of second-level functions, he would need to stipulate which of the new identity-sentences were true and which false.[33] But in the absence of such a stipulation, there is simply no fact of the matter about whether the extensions in question are the same or distinct.[34]

Despite the fact that the indeterminacy here raises alternatives to reading Frege as admitting multiple reductions, it does nothing to restore the view of

Frege's definitions as essentially reference-preserving. For this indeterminacy illustrates a general feature of Frege's reduction: that the answers to a wide range of identity-questions concerning the referents of the numerals are simply irrelevant to the ability of these objects to play the appropriate role within the reduction. In the case of the *Grundlagen*, the only answerable identity-questions concerning the referents of the numerals are those which question the identity of these objects with extensions of second-level concepts. In the *Grundgesetze*, the only numerical identity-questions which receive answers are those concerning the identity of a numeral's referent with an extension of a first-level concept. Consider now the referents of the numerals of ordinary arithmetic. Unless it is already clear that these objects are, or that they are not, first- (second-) level extensions, then the question of their identity with the *Grundlagen's* or *Grundgesetze's* numerals simply receives no answer. Whatever Frege's definitions do, then, they do not pick out a sequence of objects independently defensible as "the numbers" and assign them to the Fregean numerals.

3. Mathematics, Epistemology and Preservation of Structure

The decidedly non-linguistic flavor of Frege's reduction underscores the central claim of Benacerraf's paper: that Frege's logicism was of a very different kind from that of his twentieth-century followers. And as Benacerraf points out, Frege's acceptance of apparently distinct reductions reinforces this point: the positivist aim of preserving meaning term-for-term is no part of Frege's goal.

But the further lesson which Benacerraf draws from this difference is one which, I shall argue, we ought to resist. On Benacerraf's view, the distinction between Frege and the logical positivists marks a distinction between a primarily mathematical concern with foundations and a primarily philosophical concern:

> [A] concern [with the foundations of arithmetic] might be interpreted in two different ways, corresponding to the interests of a philosopher and to those of a mathematician. Typically, the philosopher takes a body of knowledge as given and concerns himself with epistemological and metaphysical questions that arise in accounting for that body of knowledge, fitting it into a general account of knowledge and the world. That is Kant's stance. He studies the nature of mathematical knowledge in the context of an investigation of knowledge as a whole. And that

> was the positivist's stance, though they reached quite different conclusions.
>
> But a mathematician's interest in what might be called "foundations" is importantly different. *Qua* mathematician, he is concerned with substantive questions about the truth of the propositions in question ...[35]

As Benacerraf remarks, this distinction cannot be drawn sharply, and "the interests of the two groups are not disjoint." But Frege does have primarily, as Benacerraf sees it, the mathematician's concern; his proofs were intended to demonstrate the truth of propositions which stood in need of such demonstration, and thereby to provide a needed justification for our acceptance of the truths of arithmetic. They were not, on Benacerraf's reading, intended to contribute to the kind of philosophical project in which Kant was engaged.

Before deciding whether the failure of Frege's definitions to preserve reference should incline us toward such a reading, it is worth noting that Frege himself seems to draw substantially the distinction drawn above by Benacerraf, but places himself on the "philosophical" side of the divide:

> My purpose necessitates many departures from what is customary in mathematics. ... Generally people are satisfied if every step in the proof is evidently correct, and this is permissible if one merely wishes to be persuaded that the proposition to be proved is true. But if it is a matter of gaining an insight into the nature of this 'being evident', this procedure does not suffice; we must put down all of the intermediate steps, that the full light of consciousness may fall upon them. Mathematicians generally are indeed only concerned with the content of a proposition and with the fact that it is to be proved. What is new in this book is not the content of the proposition, but the way in which the proof is carried out and the foundations on which it rests. That this essentially different viewpoint calls for a different method of treatment should not surprise us.[36]

We should note, secondly, that Frege's acceptance of multiple reductions and the consequent distinction between his and positivist logicism does not by itself offer any clear support to a "purely mathematical" reading of the Fregean project. If the failure of Frege's definitions to preserve reference entails that the final sentences of his derivations do not express the truths of arithmetic, then his proofs will not provide an immediate demonstration of the analyticity *or* of the truth of these propositions.

But this is not to say that epistemological and mathematical significance need in general go hand-in-hand. Had Frege's reduction been an attempt to provide e.g. a model for the true sentences of arithmetic, then his project would arguably have been of the kind outlined by Benacerraf. The availability of distinct ω-sequences makes the existence of distinct models for the arithmetic of the natural numbers immediate. And the provision of a model would suffice to establish the truth of arithmetical claims while contributing little of value to the epistemological debate. For any sentence which expresses a truth about an ω-sequence (or about members thereof) and which is provable solely from the fact that the sequence *is* an ω-sequence, will continue to express a truth when interpreted as being about the natural numbers. But the analyticity of such a proposition will in no way guarantee the analyticity of the corresponding arithmetical truth. For this further conclusion demands the additional argument that the fact that the natural numbers form an ω-sequence is itself a matter of logic - and not, e.g., a matter of geometry. The mere existence of ω-sequences can tell us nothing about whether, say, Mill is right about the nature of arithmetical knowledge.

But it is clear that Frege's extensions and courses of value are intended to provide more than just a model. Consider, first, Frege's rejection of attempts to found arithmetic on geometrical or formalist constructions. Two of the central objections here are that such foundations would mis-construe the meanings of arithmetical claims, and that they would fail to explain the applicability of arithmetic to non-spatial questions.[37] These objections make no sense from the point of view of a project whose goal is simply to assign to the numerals members of a collection which exhibits the right structure. Additionally, the rejected projects provide perfectly good models: there are any number of ways of constructing ω-sequences out of geometrical or syntactic objects.[38]

Further, Frege explicitly rejects the model-theoretic approach in his objections to Hilbert's "reduction" of geometry to arithmetic. On Frege's view, the use of partially-interpreted sentences in the axiomatization of geometry entails that the axioms are not in fact about geometry at all.[39] Were Frege's reduction of arithmetic to logic simply the provision of a model for partially-interpreted arithmetical sentences, precisely the same objection would apply: the resulting interpretation in terms of extensions would bear only a superficial resemblance to arithmetic. We should note also in this connection that Frege does not see in Hilbert's work any reason to modify the view that geometry is synthetic. If Frege's reductions were answerable only to the criteria met by Hilbert's, then there would be no sense to be made of Frege's view that the foundations of arithmetic and geometry are fundamentally different.

To return to the central issue, then. The final sentences of Frege's derivations are not in any standard sense synonymous with the sentences of ordinary arithmetic. This is clear even without the phenomenon of multiple reducibility, and is sufficiently clear that presumably no such correlation as obvious synonymy was intended by Frege to hold between these sentences. On the other hand, the fact that the referents of Frege's formal numerals can be mapped in an order-preserving way onto the referents of the ordinary numerals is, while necessary, not sufficient for the success of the project as Frege conceives it. The connection between the sentences of ordinary arithmetic and the various formal counterparts which are acceptable from a Fregean point of view must lie somewhere between these two extremes.

4. Another Response to Benacerraf

An account of Frege's program which attempts to reconcile the acknowledgement of multiple reducibility with the epistemological project has recently been offered by Joan Weiner.[40] On Weiner's account, the reason that Frege's assignment of referents to the formal numerals was not required to preserve the reference of the ordinary numerals is that these numerals have, prior to Frege's work, no reference. Frege's goal, on this reading, is to provide the numerals (and other parts of the language) with reference, as part of the project of providing a replacement for ordinary arithmetic. The replacement is to be an improvement on its ordinary counterpart in terms of precision and rigor. The point of Frege's definitions would be the provision of reference for the arithmetical terms in such a way that all and only the previously-accepted arithmetical sentences (or their formal counterparts) end up expressing truths.

If Frege's goal were indeed that of providing the numerals with reference for the first time, then there would certainly be several different ways of doing this, consistent with the demands of the overall project. But note that such a project must abandon the idea that the things we know (prior to Frege's work) are both the truths of arithmetic and the things expressed by ordinary arithmetical sentences. For if the ordinary numerals lack reference, then the sentences of ordinary arithmetic lack truth-value. The claims they express are not truths of any kind, let alone truths of arithmetic. If we do (prior to Frege's work) know any truths of arithmetic, then the things we know are not expressed by the sentences we use to communicate. This conclusion cannot be Frege's, in light of his view that the things "grasped" in judgment and thought are precisely the semantic values of our utterances. On the other hand, if the things we know are expressed by the sentences we use, then on this account we know, prior to Frege's work, no truths of arithmetic. And in this case, it is difficult to see what the epistemological import of Frege's work could have been. For he will have demonstrated of some interesting collection of propositions that they are all analytic, but these will have been propositions which, prior to his work, nobody had ever known.

A large number of issues are raised by this interpretation of Frege, most of which must wait until another time. The central issue here is that if we are to view Frege's project as an attempt to explain the epistemological status of (or even to demonstrate the truth of) claims which are actually known by those who do ordinary arithmetic, then Weiner's reading is not the appropriate one to give. We cannot explain Frege's acceptance of multiple reductions on the grounds that there are no antecedent referents to be "preserved" via these reductions.[41]

5. Frege's Analyses

Despite his own emphasis on the provision of analyses, Frege himself gives no clear account of the nature of analysis, or of the conditions which must be met by an adequate analysis.[42] But we should recall that the crucial relationship is that between the proven propositions and the truths of arithmetic; if Frege's analyses and definitions are such that the analyticity of the former propositions guarantees the analyticity of the latter, then the definitions will have been successful. Note that a proposition is analytic in Frege's sense just in case all of its logical equivalents are as well. This section is an examination of Frege's analysis of the finite cardinals, from which, I shall argue, a picture of Frege's procedure emerges according to which: (a) the definitions are intended to provide provable propositions whose analyticity fairly clearly entails that of the truths of arithmetic; and (b) the existence of multiple reductions is to be expected, due to the fact that obvious logical equivalents are as good as one another for Frege's purposes.

That feature of number with which Frege is primarily concerned is that numbers are used to count things. In particular, Frege's analysis of number centers on an analysis of statements of the form "there are n F's." With respect to such statements, the claim which Frege later regards as "the most fundamental of [his] results" is that "a statement of number expresses an assertion about a concept." The *Grundgesetze*'s account of arithmetic, we are told, "rests upon this" result.[43] What Frege means here is best illustrated by his own examples:

> If I say "Venus has 0 moons", ... what happens is that a property is assigned to the *concept* "moon of Venus", namely that of including nothing under it. If I say "the King's carriage is drawn by four horses", then I assign the number four to the concept "horse that draws the King's carriage."[44]

The argument for this view comes entirely from considerations about ordinary arithmetical discourse, and occupies the bulk of sections 18 through 52 of *Grundlagen*. Frege's claim here would appear to be that, in some ordinary sense of analysis, claims of the form "there are n F's" are adequately analyzed as claims which "assign a number" to a concept.

The account of what it is to *assign a number* to a concept forms the basis of the analysis of number in general. In particular, to say what it is to "assign" zero to a concept, or to say that zero is the number which *belongs to* a concept, will be to give an account of the number zero. To say what it is to assign the *successor* of a number to a concept will be to give an account of the relation of successor. This, then, is intended to suffice for the account of the natural numbers, as these are to be defined in terms of zero and successor.

As a preliminary but ultimately unsatisfactory account of such assignments, Frege suggests that to assign zero to a concept F is to say, intuitively, that there are no F's, and that to assign n+1 to a concept F is to say that for some object a falling under F, there are n F's other than a. More precisely, Frege's suggestion is that

> (□) zero is the number which belongs to the concept F

and

> (□) n+1 is the number which belongs to the concept F

be analysed as, respectively,

> (A) The proposition that a does not fall under F is true universally, whatever a may be,

and

> (B) There is an object a falling under F and such that the number n belongs to the concept "falling under F, but not a."

These preliminary definitions, says Frege, "suggest themselves so spontaneously in the light of our previous results, that we shall have to go into the reasons why they cannot be reckoned satisfactory."[45] The spontaneousness with which the definitions suggest themselves is not difficult to feel, if the suggestion is that the sentences (A) and (B), as normally understood, make essentially the same claims as do "there are no F's" and "there are n+1 F's". There is of course room for argument here. But what is crucial is that (A) and (B) do offer, arguably, analyses of the claims expressed by their ordinary counterparts.

Despite their naturalness, Frege of course rejects (A) and (B). Their unsatisfactoriness is due to the fact that they give no account of the uses of numerical terms in contexts other than those of the form "the number of F's". In particular, they do not allow us to make sense of ordinary identity-statements involving numerals. But this is not to say that Frege rejects the view that (A) and (B) do provide analyses of the ordinary claims. Far from it. For the definitions with which Frege is eventually satisfied, both in the *Grundlagen* and the *Grundgesetze*, assign to (□) and (□) claims which are logically equivalent with (A) and (B).[46]

The final *Grundlagen* account of (□) rests on analyses of the relation "number which belongs to the concept," and of the number zero. The definitions corresponding to these analyses are as follows:

> (D1) "The number which belongs to the concept F" is shorthand for "the extension of the concept *equinumerous with the concept F.*"
>
> (D2) "0" is shorthand for "the number which belongs to the concept *not self-identical.*"

The concept of equinumerosity involved in (D1) is itself cashed out in terms of one-one mappings.

Using these definitions, (□) is shorthand for a cumbersome sentence which is provably equivalent, using Frege's laws of logic, with (A). The situation is similar with successor. Using the *Grundlagen*'s analysis of successor, (□) is logically equivalent with (B). The analysis of the rest of the natural numbers is then straightforward: for every first-level concept F and numeral *n*, Frege's "n is the number which belongs to the concept F" is logically equivalent with the analysis one would have obtained using the preliminary (A) and (B). Each such claim will in fact be logically equivalent with our own familiar first-order rendition of "there are n F's."

If Frege is right that the natural numbers are to be treated as essentially measures of cardinality,[47] then this last result is crucial: it shows that to whatever extent ordinary claims about the numbers can be analyzed as claims about the cardinality of concepts, they are logically equivalent with claims about the referents of Frege's numerals. Much more is needed in the way of analysis before it is clear that for each arithmetical claim there is such an equivalent. In particular, in order to be able to give logical equivalents of such purely arithmetical claims as the Peano axioms, Frege needs to give an account of how claims about addition and multiplication are to be cashed out in terms of the cardinality of concepts. But the groundwork has been laid; the accounts of equinumerosity, of zero, of successor, and of natural number suffice to generate logical equivalents of a variety of arithmetical claims involving reference to and quantification over the natural numbers. Consider for example Frege's version of "every natural number has a successor." Unpacking definitions, the Fregean claim is logically equivalent with the claim that for every finite first-level concept F there is a concept G under which one more object falls.[48] If one accepts the Fregean account on which natural numbers are essentially numbers *of* concepts, this latter claim provides an adequate analysis of the ordinary claim that every natural number has a successor. The derivation of the Fregean sentence would have demonstrated the analyticity of the ordinary claim.

Do Frege's definitions "preserve meaning"? Probably not, for to say that the number of F's is zero is presumably not to make the same claim as that made by (▯), where we take (▯) to be shorthand for its definitional transcription. But in this simple case at least, they preserve what is crucial: If we wanted to know, for a particular F, whether the fact that there are zero F's depends on purely logical principles, we would have to look no further than (▯).

So far, it is a straightforward matter to make sense of the multiple "reducibility" of the numbers. Replacing definition (D1) above with

> (D1') "The number which belongs to the concept F" is shorthand for "the extension of the concept under which fall all and only the *extensions* of concepts equinumerous with F"

gives us (a natural-language version of) the *Grundgesetze*'s account.[49] Whether the object referred to by "the number which belongs to the concept F" is the same in the *Grundgesetze* as it is in the *Grundlagen* is no longer of any significance. The *Grundgesetze*'s version of (□), using (D1'), is logically equivalent with the *Grundlagen*'s version, and hence with (A). Similarly, the *Grundgesetze*'s version of (□) is, like its predecessor, logically equivalent with (B). If the goal is to come up with identity-sentences which are logically equivalent with the proposed analyses, then there are infinitely many ways to do this. It is in this case no longer surprising that the change from the *Grundlagen* version to the technically nicer *Grundgesetze* version was something which Frege saw no need to defend.

It is now clear why not just any □-sequence would do for Frege's purposes. Supplied with any such sequence $<\square_0, \square_1, ...>$, one can correlate concepts with members of the sequence in such a way that all and only the concepts under which nothing falls are correlated with $\square_0$, and a concept F is correlated with $\square_{n+1}$ iff there's an object a falling under F and such that the concept "falling under F but not identical with a" is correlated with $\square_n$. It will then be the case that for each concept F and natural number n, F is correlated with $\square_n$ iff there are exactly n F's. But this is as close as such an arbitrary reduction will get to a Fregean reduction. The claim that $\square_n$ is correlated with F will in general be only materially, and not logically, equivalent with the ordinary claim that there are n F's. In particular, if the existence of the sequence $<\square_0, \square_1, ...>$ is grounded in principles of e.g. geometry or set theory, then no appeal to laws of logic will enable one to derive the "reduced" from the ordinary claim.[50]

Herein lies a crucial distinction between Frege's reduction and modern reductions of arithmetic to set theory. A standard reduction to set theory clarifies the relationships between properties shared by all isomorphic structures. By showing that certain truths about a collection of objects and relations follow solely from structural features of that collection, the reduction to set theory will help explain why arithmetic, if it is about such a structured collection, must contain these truths. So too will a reduction of arithmetic to geometry. But neither of these reductions will help to clarify what Frege calls the "ultimate grounds" of the truths of arithmetic. For there is no reason to suppose that the claims of ordinary arithmetic are true for the same reasons as are their set-theoretic or geometrical surrogates. In particular, no reduction of arithmetic to set theory can help to say whether the structural features of the arithmetical subject-matter are due to empirical facts (as Mill would have it), facts about the structure of intuition (as Kant would have it), or something else altogether. For none of these alternatives has the least bearing on the "reducibility" of arithmetic to set theory. If Frege had been right, however, and if his "logic" had not succumbed to Russell's paradox, the situation would have been quite different with his own reduction. For Frege was in a position to argue that the claims of ordinary arithmetic were logically equivalent with their Fregean surrogates, and hence that the grounds of the latter sufficed to ground the former.

6. Frege and the Tradition

Both Frege and Kant view proofs as establishing the *grounds* of the proven proposition.[51] And both take these grounds to be epistemologically significant. Assuming the correctness of Frege's view of the natural numbers as essentially measures of cardinality, and of equi-cardinality as a matter of the existence of 1-1 mappings, Frege's proofs would have shown arithmetical truths to be grounded in principles which he counted as purely logical. Setting aside for a moment the implications of his error in this last regard, we can ask whether the kind of analyticity which would have been conferred upon the truths of arithmetic via these proofs would have sufficed to show them analytic in Kant's eyes.

There are some clear differences between the two accounts of analyticity, perhaps the most striking of which concern Frege's rejection of the importance of the subject-predicate nature of propositions, and his conception of analytic truths as capable of carrying existential import. Further, analytic judgments are never for Kant ampliative, while they are clearly so for Frege.[52]

But there is a striking parallel, one which arguably justifies Frege's claim to be using a modernized version of Kant's own notion.[53] For Kant, the notion of contradiction is central; an analytic truth is often characterized by him as one whose negation is self-contradictory. That the principle of contradiction suffices to ground analytic judgments is tied to the fact that, for Kant, the truth of an analytic proposition follows just from an analysis of the concepts involved in that proposition. Synthetic truths on the other hand, for Kant, "cannot possibly spring from the principle of analysis, namely, the law of contradiction, alone. They require a quite different principle from which they may be deduced..."[54] Here Frege is in full agreement. The laws of logic are, for Frege, laws which can only be denied on pain of self-contradiction.[55] An analytic truth is one for which an analysis of constituent concepts, according to the rules of logic, suffices to demonstrate its truth. Synthetic truths, on the other hand, are those which cannot be proven "without making use of truths which are not of a general logical nature." If we assume a shared understanding of "contradiction" here, Frege and Kant are directly at odds over arithmetic. For Kant, no amount of analysis of arithmetical concepts will ever show that an arithmetical truth is true on pain of self-contradiction. For Frege, this is precisely what his analyses and proofs demonstrate.

Thus no Kantian can agree with Frege's analyses and with the view that Frege's laws of logic become self-contradictory when denied. Similarly, no Millian can accept the analyses and the weaker view that Frege's laws are knowable a priori. The fact that Frege's logicism poses no threat to these programs is owing simply to the fact that his laws turned out to satisfy neither description; the lack of a clear confrontation is due to Russell's paradox, not to the irrelevance of Frege's program to the epistemological debate.

7. The Failure of the Project

Russell's paradox shows that Frege's comprehension principle, which he had taken to be a "fundamental law of logic,"[56] is simply false. This principle is central to the analyses outlined above: without the law of comprehension, Frege's method of arriving at logical equivalents of the truths of arithmetic is entirely undermined. Thus Russell's paradox does not just destroy the validity of Frege's proofs; it also shows that the derived sentences fail to express truths whose analyticity would guarantee that of the truths of arithmetic. As Frege puts it, "My efforts to become clear about what is meant by number have resulted in failure."[57]

The failure of the comprehension principle spells the failure of the logicist reduction as Frege conceives it. For there is little reason to believe, in the absence of such a principle, that the existence of an appropriate sequence of objects can be regarded as a matter of logic. Thus there is little hope that, for instance, claims about the equinumerosity of concepts can be regarded as logically equivalent with claims about the identity of objects. If the appropriate analysis of arithmetical claims is one which treats the numbers as objects, then arithmetical truth is not grounded solely in logical truth. This is the conclusion Frege himself comes to by about 1925, if we are to take his late diary entries and unpublished manuscripts seriously. The conclusion is again an epistemological one for Frege: Because arithmetic is about objects, and because the "logical source of knowledge" is insufficient to "yield us any objects," arithmetical knowledge turns out not to be simply a species of logical knowledge.[58]

If we accept Frege's realism about the numbers, then we must agree with his conclusion that there is more to arithmetic than pure logic. Whether we should agree in this way is, I think, still an open question. But one thing which should be clear is that the Fregean conception of reduction is one which has an epistemological payoff. If, as Frege speculated late in life, arithmetic is reducible in the Fregean manner to geometry, then the source of arithmetical knowledge is to be found in our knowledge of the principles of geometry. For such a reduction would be a demonstration that, properly analyzed, the truths of arithmetic are themselves truths about geometrical objects and relations. If on the other hand there is, contra Frege, an analysis of arithmetic which makes no reference to numerical objects, then the door is open once again to a Fregean reduction of arithmetic to logic.[59] And such a reduction would be evidence that the truths of arithmetic are, in something very like Kant's sense, analytic.[60]

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[1]*Grundlagen* 99. Abbreviations used in this paper for Frege's works are:
Bs (Begriffsschrift): "*Begriffsschrift,* a formula language, ..." in *From Frege to Godel*, van Heijenoort (ed), Harvard 1967, pp 5-82.
CP: *Collected Papers*, McGuinness (ed); Blackwell 1984.
FG: *On the Foundations of Geometry and Formal Theories of Arithmetic*, Kluge (ed), Yale 1981.
Gg (Grundgesetze): *The Basic Laws of Arithmetic*, Furth (ed), California 1964.
Gl (Grundlagen): The Foundations of Arithmetic, Austin (ed), Blackwell 1980.
PMC: *Philosophical and Mathematical Correspondence*, McGuinness (ed) Chicago 1980.
PW: *Posthumous Writings*, Hermes et al (eds), Chicago 1979.

[2]*Midwest Studies in Philosophy VI*, French, Uehling, & Wettstein (eds); Minnesota, 1981, pp 17-36. Hereafter, "FLL".

[3]*Gl* 99.

[4]*Gl* 118-9.

[5]*Gl* 3.

[6]*Gl* 4

[7]*Gl* 2

[8]"On Mr. Peano's Conceptual Notation and My Own", CP 235. See also *Gg* 3: "Because there are no gaps in the chains of inference, every 'axiom', every 'assumption', 'hypothesis,' or whatever you wish to call it, upon which a proof

is based is brought to light, and in this way we gain a basis upon which to judge the epistemological nature of the law that is proved."

[9]I.e., to *contents of possible judgment*, or simply *contents* in the pre-1890's writings (see e.g. *Bs* pp 11-13); to senses of sentences, or *thoughts* in the later writings. See footnote 11.

[10]FG 101. Here Kluge has translated ``*Satz*'' as ``proposition''; I take it that the context makes it clear that in this case ``sentence'' is preferable.

[11]PW 206. See also FG 101: "When one uses the phrase "prove a proposition'' in mathematics, then by the word "proposition'' we clearly mean not a sequence of words or a group of signs, but a thought; something of which one can say that it is true." The view of nonlinguistic items as the bearers of analyticity/provability dates back to the *Begriffsschrift*, in which the nonlinguistic *judgment* is derived, as opposed to its representation "in signs." (*Bs* 28). In the *Grundlagen*, Frege reports that he has already, in the *Begriffsschrift*, "given a proof of a proposition, which might at first sight be taken for synthetic, which I shall here formulate as follows: ..." (*Gl* 103). The proposition proven is neither the *Begriffsschrift* sentence (formula 133) nor the *Grundlagen* sentence which follows this remark; it is the nonlinguistic counterpart which these purportedly share. The thing which has been demonstrated to be analytic, that thing "which might at first sight have been taken for synthetic,'' is the nonlinguistic item.

[12]This is again a view which remains constant from Frege's early work through his latest writings. The first definition of the *Begriffsschrift* is accompanied with the explanation that "we can do without the notation introduced by this proposition and hence without the proposition itself as its definition; nothing follows from the proposition that could not also be inferred without it. Our sole purpose in introducing such definitions is to bring about an extrinsic simplification by stipulating an abbreviation" (*Bs* §24). In the *Grundlagen*, a definition "only lays down the meaning of a symbol" (*Gl* 78), and in the *Grundgesetze:* "The definitions ... merely introduce abbreviated notations (names), which could be dispensed with were it not that lengthiness would then make for insuperable external difficulties" (*Gg* 2). See also Frege's 1899 letter to HIlbert (PMC 36), and FG 23, FG 24. Note that the newly-introduced sign will have the same meaning (same content and, later, same sense and reference) as does the definiens (see e.g. *Gg* §33, PW 211). This is to be distinguished from the issue of meaning-preservation raised in the next section.

Benacerraf (FLL 28) points out the apparent tension between this view and the *Grundlagen* view that definitions must be "fruitful" (*Gl* 81). I take it that the tension is no more than apparent, as the "definitions" which Frege takes to be fruitful are the conceptual analyses (see next paragraph) which precede stipulative definitions and do not occur within proofs, while Frege consistently observes in his formal work the stipulative and eliminable character of definitions which do appear in proofs. For further discussion of fruitfulness and stipulative definitions, see Joan Weiner: "The Philosopher Behind the Last Logicist" in *Frege: Tradition and Influence*, Wright (ed), Blackwell 1984, and *Frege in Perspective*, Cornell 1990, pp 89-92, and M. Ruffino: "Context Principle, Fruitfulness of Logic and the Cognitive Value of Arithmetic", *History and Philosophy of Logic* 12 (1991) 185-194. My own understanding of the fruitfulness requirement has benefitted from an unpublished paper by Jamie Tappenden (though he would not agree with the above).

[13]*Gl* 99. Frege does not seem to share his successors' concern that logical truths involve only "logical" concepts. It would appear that anything provable using only the self-evident fundamental laws of logic will count as a (derivative) truth of logic. The purpose of the conceptual analysis (of, e.g. the concept of number) is not to eliminate "non-logical" concepts, but to present the logical structure to which one must appeal in giving the proof. See e.g. *Gl* 4. In any case, concepts which we would count as non-logical will occur only inessentially in any truths which Frege counts as logical.
[14]*Gl* 5
[15]It is of course irrelevant whether the "new" term is typographically distinguishable from its ordinary counterpart.
[16]*Gl* 13.
[17]*Gl* 18
[18]*Gl* 26
[19]The importance for the epistemological project of a semantic connection between the ordinary and formal discourse is noted by Gregory Currie ("Frege, Sense and Mathematical Knowledge", *Australasian Journal of Philosophy* 60 (1982) 5-19). On Currie's view, the definitions are intended to preserve sense, as a means to preserving reference. But as Currie notes, this requires that we attribute to Frege at least two distinct notions of sense; we are left with no clear criteria of identity for the relevant one. Regarding reference-preservation, see below. A second criterion of proposition-identity which occurs occasionally in Frege (see e.g. letter to Husserl of 9 December 1906) is that two sentences express the same proposition iff they are logically equivalent. But note that (a) this criterion would imply that if logicism is true, there is only one truth of arithmetic; and (b) the criterion cannot be applied, as the relation of logical equivalence applies in the first instance, for Frege, to propositions.
[20]Frege seems to suggest such a criterion in his 1894 Review of Husserl's *Philosophy of Arithmetic* in response to Husserl's objection that his definitions do not preserve (something like) sense. On the strangeness of this suggestion, see Dummett, *Frege Philosophy of Mathematics*, Harvard Press 1991 (Hereafter, "FPM") p 142.
[21]*Gl* 79-80.
[22]That is, each *numeral* is, via these definitions, shorthand for a singular term which refers to the extension of a concept.
[23]*Gl* 117.
[24]Frege often holds that a concept-word occurring in subject-position refers not to a concept, but to an associated object. (See e.g. "On Concept and Object", CP 184) If this position is applied to these passages, then one can read Frege's proposed substitution of "concept" for "extension of the concept" as a substitution of one term referring to an *object* for another term referring to an object. As Michael Resnik notes ("Frege's Theory of Incomplete Entities"; *Philosophy of Science* 1965, pp 329-341; esp. p. 333n), Frege later suggests such a reading (*op cit* p 48). Resnik takes the suggestion to imply, further, that the associated object is the extension of the concept. This stronger claim is also argued for by Burge ("Frege on Extensions of Concepts", *Philosophical Review* XCIII, 1 (Jan 1984)) and Cocchiarella (*Logical Studies in Early Analytic Philosophy*, Ohio 1987, pp 76 ff). On this reading, Frege's proposed change in the *Grundlagen* footnote would be merely terminological.

This reading seems doubtful to me, for two reasons. (1) The identification is sufficiently simple, and important, that had Frege made it, it is reasonable to suppose that he would have been explicit about it. But he nowhere claims that the "associated objects" are extensions. (2) Following the proposal, Frege notes that it would be open to the objections that (a) it "contradicts my earlier statement that the individual numbers are objects", and (b) "concepts can have identical extensions without themselves coinciding." The first objection would be badly misguided had Frege taken the change to be merely terminological, and it is strange that he does not immediately explain this. The second objection would be beside the point; again, Frege's silence on such an easily-remedied matter is at least odd.

Note also that the proposed reading of the *Grundlagen* footnote will not help with Frege's claim that he attaches no "decisive importance" to bringing in extensions, since on this construal, the alternative definition would have "brought in" extensions. In any case, there is nothing in Frege's texts which necessitates the proposed reading; this is what is important for Benacerraf's point.

[25]FLL 30.

[26]Note e.g. that the claim that every number has a successor must be provable via the laws of logic.

[27]An endorsement of this claim seems to appear in Frege's 1885 reply to Cantor's review of the *Grundlagen*. See CP 122.

[28]*Gg* 6.

[29]Michael Resnik suggests (*op cit*) that Frege takes the second occurrence of "the concept F" in the *Grundlagen's* definition to refer to the extension of the concept F, so that this definition in fact assigns the extension of a *first*-level concept to each numeral. In this case, as Resnik suggests, the *Grundlagen* and *Grundgesetze* definitions would be "essentially the same." But note that on this reading we must take each phrase of the form "the concept ..." in the *Grundlagen's* definition of equinumeracy as referring to an extension, so that equinumerosity is properly a relation between extensions. But this conflicts with the precise rendering of this definition in the *Grundgesetze*, §§38-40, in which "the concept F" and "the concept G" are replaced by "-□□ □" and "-□□ □", which are clearly concept-terms. The *Grundgesetze's* version of the number of F's is the extension of that first-level concept under which fall all and only the *extensions* of concepts equinumerous with F, so that even though the number is the extension of a first-level concept, equinumerosity is still a relation between concepts. See footnote 49.

[30]Or to show that he changed his mind. But as Frege nowhere notes the change in definition, and nowhere claims that the new definition does a better job (or even an equally good job) at securing for his numerals the referents of the ordinary numerals, this seems unlikely.

[31]This way of putting it is not quite accurate, since of course if the extensions of second-level concepts *are* extensions of first-level concepts, then they have names in the *Grundgesetze*. More precisely: the *Grundgesetze* has no term-forming operator which applies to the name of a second-level concept and gives its course of values.

[32]For the comprehension principles tell us that the extensions of two concepts are the same iff the same things fall under those concepts. But if the concepts in question are of different levels, then the question of the identity of the "things" falling under the concepts is, on Frege's own principles, non-sensical. Specifically: Where F is the relevant first-level concept and □ the relevant second-level concept, the purported comprehension principle would

imply: ext(F) = ext(□) iff (x)(Fx iff □(x)). If the range of x includes objects, then "□(x)" is ill-formed; if it includes concepts, then "F(x)" is ill-formed.
[33]See *Gg* §10.
[34]Further, there is as far as I can see no reason that the stipulations in question ought to be made one way rather than another. (The relevant question would seem to be which such stipulations would preserve consistency - but of course this question makes little sense, in light of the inconsistency of the system as a whole.)
[35]FLL 23
[36] *Gg* 4-5.
[37]See *Gl* 18-21, 101-102.
[38]The "modelling" interpretation of Frege is not helped by including the sentences of applied arithmetic in the collection to be satisfied by the model. For the assignment of reference to numerals in such a way as to model applied sentences involving finite cardinalities requires only that there be a function *f* from first-level concepts to the referents such that for each concept F, *f*(F) = the referent of the numeral *n* iff there are n things falling under F. And the existence of such a function requires only that the referents do in fact form an □-sequence.
[39]See "On the Foundations of Geometry" (1906), in FG pp 49 - 112.
[40]*op. cit.*
[41]Weiner's positive argument for the view that ordinary numerals lack *bedeutung* is, I think, problematic in its insistence that Frege's requirement of sharp delimitation and total definition of concepts is applicable to ordinary language. These requirements are better seen as constraints on the formal language; they are in particular required for rigor in that the failure of these constraints will result in syntactically well-formed names which lack reference. This will make it impossible to give purely syntactic rules of inference which are literally truth-preserving. Another reason to doubt that the ordinary numerals lack *bedeutung* is that if they do, then it is difficult to understand those arguments of Frege's which turn on the fact that the numerals are singular terms. The term "Odysseus" is a singular term in the sense in which Weiner would construe the term "two", but Frege would not take the occurrence of the former as evidence of any kind of realism. In short: Frege's remarks about the functioning of number-words in ordinary language only carry the weight they need if the sentences which embed them have truth-values.
[42]This is pointed out by Dummett (at e.g. FPM 30-31, 143). On Dummett's account, Frege's definitions are (or ought to be) intended to "come as close as possible to capturing the existing sense." Dummett cashes out this criterion in terms of a number of conditions (see esp. FPM 152-3) which if met ensure that the system of definitions "comprises everything that must be implicitly known by anyone who understands all those expressions" (FPM 154). But there is more to be said. What advantage do definitions in terms of extensions have over definitions in terms e.g. of geometrical constructions by way of "comprising everything that must be implicitly known" about the numbers? The following is in large part an attempt to answer this further question. If the account in this section is correct, then it seems that one need not attribute to Frege a conception quite as "holist" as does Dummett (FPM 32, 154). But further pursuit of this issue must be postponed.
[43]The claim occurs in *Gl* 58-61, 67. The quoted remark occurs at *Gg* 5.
[44]*Gl* 59

[45]Gl 67
[46]Logically equivalent, that is, on Frege's problematic view of logic. The inconsistent comprehension principle is essential here.
[47]It is interesting to note that in his last writings, Frege seems to hold that his earlier view of finite cardinals as fundamental was mistaken. See "Numbers and Arithmetic" (1924/25); PW 275-7.
[48]That is: for every finite concept F, there is a concept G and an object a falling under G such that the number which belongs to the concept "falling under G but not identical with a" is the same as the number which belongs to the concept F.
[49]The *Grundgesetze*'s version of *the number which belongs to the concept F*, i.e., □F(□) is, by definition of and using modern notation for the quantifier and conjunction:
(1) □(□q)(□F(□) □ (□ □ > q) & □ □ (□F(□) □ > q)) (§ 40).
I.e., it is the extension of a concept under which an object a falls iff:
(2) (□q)(□F(□) □ (a □ > q) & a □ (□F(□) □ > q)).
Recall that for a function F (or, as Frege would write it, F(□)), □F(□) is that function's course of values (cov). Where F is a concept, this cov is the extension of F. The symbol "□" is defined (§ 34) in such a way that a□u = G(a), where G is "the u-function", i.e., the function whose cov is u. (Note that, in particular, if u is the extension of a concept G, then a□u = T iff a falls under G.)
Where □ is the q-relation (i.e., q=□□(□□)), □F(□) □ (a □ > q) above expands (using modern terminology for the material conditional) to:
(e)(d)(□(e,d) (c)(□(e,c) d=c)) & (d)(F(d) (□c)(□(d,c) & c□a))
(§§ 34, 36-39).
Similarly, a □ (□F(□) □ > q) expands to:
(e)(d)(□(d,e) (c)(□(c,e) d=c)) & (d)(d□a (□c)(□(c,d) & F(c))).
Here, "c□a" will refer to □(c), where □ is the function such that a=□□(□).
Thus (2) says that:
(3) (□q)(the q-relation maps the a-concept one-one onto F),
where by "the q-relation" we mean the relation whose cov is q, and by "the a-concept", that concept whose extension is a.
So (1) names the extension of the concept under which fall all and only the extensions of those concepts which can be mapped 1-1 onto F.
[50]Unless, contra the pre-1920's Frege, the ordinary claim is at bottom geometrical.
[51]That is, they share this view with respect to direct proofs. The view that a demonstration of grounds is the primary purpose of proofs explains Kant's and Frege's shared mistrust of indirect proofs, and Frege's view that premises and axioms must always be true. For a discussion of the legacy of the rejection of proofs by contradiction, see Paolo Mancosu's forthcoming *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century,* Oxford University Press.
[52]I suspect that this *prima facie* difference is not as striking as it at first appears, as Frege does not seem to share Kant's notion of "ampliative". But I leave this question for another time.
[53]That Frege's analytic/synthetic distinction is essentially Kant's has been argued by Philip Kitcher in "Frege's Epistemology" (*Philosophical Review* 88 (1979) 235-262). The opposite has been argued by Newton Garver in "Analyticity and Grammar" (*Kant Studies Today*, L.W. Beck (ed), Open Court: La Salle 1969).

[54]*Prolegomena to Any Future Metaphysics*, Beck translation (Liberal Arts Press 1950), p. 15

[55] See e.g. *Gl* 20-21, in which the fact that the truths of geometry can be denied without self-contradiction is a mark of their synthetic nature. See also *Gg* 15. I have recently seen a similar emphasis on contradiction in Kant and Frege in an unpublished paper by Jamie Tappenden.

[56]"Function and Concept", CP 42.

[57]Diary entry of March 28, 1924; PW p. 263.

[58]PW 278-279; manuscript of 1924/25.

[59]For such an analysis, see Harold Hodes: “Logicism and the Ontological Commitments of Arithmetic”, *Philosophical Studies* 41 (1982) 161-178.

[60]Many of the ideas in this paper have been presented in colloquia to a number of departments of philosophy; thanks to many members of these audiences for helpful discussions and criticism. I would particularly like to thank David Schmidtz, Michael Resnik, John Etchemendy, Paolo Mancosu, William Demopoulos, Andrew Irvine and Gottfried Gabriel for helpful discussion and/or comments on earlier versions.