Dynamic System Response

AME20213
Laboratory Exercise 3
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1 Summary

The overall objective of this experiment was to investigate the dynamic response characteristics of first and second order measurement systems. The first objective of this experiment was to determine the time constant of an RC circuit (first order system) using a digital oscilloscope. By using a function generator to generate a step input and by applying Kirchhoff’s voltage law, the time constant was determined to be $\tau = 660 \pm 20 \mu s$. From this value and the given value of the capacitance, the resistance was determined to be $R = 976 \pm 59 \Omega$, which was 7.3 % larger than the upper limit given for the resistor. The second objective was to determine the magnitude ratio and phase lag as functions of input frequency for an RLC circuit (second order system). For a sinusoidal input, the magnitude ratio and the phase lag were both shown to decrease with increasing input frequency. The value of the theoretical damping ratio was shown to be $0.001$, which differed by about 8 % with the experimental values. The third objective was to determine and compare the natural frequencies and damping ratios of single wall and double wall aluminum bats by striking them with a softball. The values of the natural frequency and the damping ratio of the single wall bat were determined to be 7.9 % larger and 78.6 % larger, respectively, than those of the double wall bat. This indicated that the single wall bat was stiffer than the double wall, and that the double wall vibrated longer than the single wall. It was also shown that these values could be altered by the hand position on the handgrip of the bat. The major sources of uncertainty in the determination of these quantities involved the orientation of the bat at impact, graphical methods to find the damping ratio, and the fast Fourier transform program used to find the natural frequency.

2 Findings

The first objective of this experiment was to determine the time constant, $\tau$, of an RC circuit, which is a first order system. A function generator was used to generate a square wave, which acted as a step input forcing function for the system. Using Kirchhoff’s voltage law, the output voltage $V$ of the capacitor can be expressed as

$$V(t) = E(1 - e^{\frac{-t}{RC}}),$$

where $E$ is the input voltage, $R$ is the resistance, and $C$ is the capacitance. Equation 1 can be used to show that $V = 0.632E$ when $t = \tau = RC$. To determine the value of $\tau$, a digital oscilloscope was used to plot $E(t)$ and $V(t)$. By determining the change in time from the base of the square wave to 63.2 % of $E$, it was determined that $\tau = 660 \pm 20 \mu s$. The calculated value of $\tau$ and the given value of $C = 0.68 \mu F$ were used in the relation $\tau = RC$ to determine that $R = 976 \pm 59 \Omega$. This calculated value of $R$ was 7.3 % larger than the upper limit given for the variable resistor. Since the upper limit does not fall within the calculated uncertainty of $R$, the uncertainty in the value of $C$ and the deficiencies of the step input waveform should also be considered.

The second objective of this experiment was to determine the magnitude ratio and phase lag as functions of input frequency for an RLC circuit, which is a second order system. By

\footnote{All uncertainties are reported with 95 % confidence. See Appendix for all uncertainty calculations.}
applying Kirchhoff’s voltage law to the circuit and solving the obtained differential equation, the magnitude ratio $M$ can be shown to be

$$M(\omega) = \frac{1}{\sqrt{[1 - (\frac{\omega}{\omega_n})^2] + [2\zeta(\frac{\omega}{\omega_n})]^2}},$$

(2)

where $\omega$ is the angular frequency of the input, $\omega_n$ is the natural angular frequency of the RLC circuit, and $\zeta$ is the damping ratio. The phase lag, $\Phi$, is given by

$$\Phi(\omega) = \tan^{-1}\left[\frac{2\zeta(\frac{\omega}{\omega_n})}{1 - (\frac{\omega}{\omega_n})^2}\right].$$

(3)

During the experiment, an input sinusoidal wave of known amplitude and frequency was generated using the function generator, and the output amplitude and the phase lag were measured using the oscilloscope. This was done for frequencies ranging from 100 Hz to 15 000 Hz. To determine the value of $\zeta$ for the RLC circuit, equations 2 and 3 were solved for in terms of $\zeta_M$ and $\zeta_\Phi$, respectively. The measured experimental values of $M$, $\Phi$, and $\omega_n = \frac{1}{\sqrt{LC}}$ were input into these new equations for each tested value of $\omega$ to determine the value of $\zeta_M$ and $\zeta_\Phi$ for each run. The determined values for each $\omega$ were averaged, which yielded that $\zeta_M = 5.59 \pm 0.15$ and $\zeta_\Phi = 5.57 \pm 1.48$.

The small percent difference between these two values (less than 1 %) was agreeable with theory, since there really is only one value of $\zeta$ for the RLC circuit. The values of $\zeta_M$ and $\zeta_\Phi$ were then input into equations 2 and 3 to yield $M(\frac{\omega}{\omega_n})$ and $\Phi(\frac{\omega}{\omega_n})$. The theoretical and experimental values for $M(\frac{\omega}{\omega_n})$ and $\Phi(\frac{\omega}{\omega_n})$ are shown in Figures 1 and 2.

Figure 1 shows that the experimentally calculated values of $M$ closely correlated with the theoretical values. Furthermore, all experimental values fell within the confidence intervals. This plot also shows that the output amplitude is approximately equal to the input amplitude for values of $\omega \ll \omega_n$ and that the output goes to zero as $\omega$ increases. Figure 2 shows that the experimentally calculated values of $\Phi$ also correlated with the theoretical values. Only several experimental values fell outside of the confidence intervals. This plot also indicates that there is little phase lag for $\omega \ll \omega_n$ and that the lag increases as $\omega$ increases.

The theoretical value of the damping ratio is given by $\zeta_{thy} = \frac{R}{R_c}$, with $L$ being the inductance of the circuit. A multimeter was used to determine that $R = 886.08 \pm 0.01 \Omega$. Using this value of $R$ and the given values of $C = 0.68 \mu F$ and $L = 5 \text{ mH}$, it was determined that $\zeta_{thy} = 5.1669 \pm 0.0001$. This differed by about 8 % with the average $\zeta_M$ and $\zeta_\Phi$. The determined experimental values of $M$, $\Phi$, and $\zeta$ supports the conclusion that the RLC circuit behaves as a second order system, since these values closely corresponded to the theoretical values of a second order system.

The third objective of the experiment was to determine and compare the natural frequencies and the damping ratios of single wall and double wall aluminum baseball bats. This

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2See Appendix for program used to calculate these.

3Horizontal error bars were omitted by ignoring any uncertainty in the frequency set by the function generator. The zero-order uncertainty was only considered for $M$ and $\Phi$, but these error bars were omitted since they were too small to be visible. See Appendix for chart of the combined standard uncertainties.
Figure 1: Plot of the theoretical and experimental values of $M(\frac{\omega}{\omega_n})$, with the confidence intervals for the theoretical curve.

was done by examining the responses in the bats caused by hitting a softball. The bat striking the ball can be characterized as an impulse response. For systems that are underdamped, this response is given by

$$y_s(t) = \frac{\omega_n e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t). \quad (4)$$

In the experiment, each bat was equipped with strain gauges that were used to measure the vibrations in the bat. When the hanging softball was struck by a bat, the resulting vibrations measured by the strain gauges were recorded and saved using the oscilloscope. Data was taken for two types of grips for each bat. First, the ball was struck with the hands positioned at the lower end of the hand grip (near the knob). Second, the hands were positioned at the upper end of the hand grip (near the barrel of the bat).

A MATLAB program that performs a fast Fourier transform (FFT) was used to determine the frequencies present in the response for each bat and grip type. The natural frequency and the damping ratio for each bat and grip type were determined in the following way: Each prominent frequency acquired (about two or three) from the FFT was set as the
Figure 2: Plot of the theoretical and experimental values of $\Phi(\frac{\omega}{\omega_n})$, with the confidence intervals for the theoretical curve.

natural frequency. This was then input into the equation

$$\frac{y_2}{y_1} = e^{\zeta \omega_n (t_1 - t_2)} \quad (5)$$

to determine the corresponding value of $\zeta$. The values of $y_1$ and $y_2$ are the amplitudes of the first and second peaks of the acquired signal, and $t_1$ and $t_2$ are the corresponding times. These four values were determined graphically from a plot of the acquired signal. The values $\omega_n$ and $\zeta$ were then input into equation 4, and this was plotted with the experimental signal. The theoretical plot was scaled by a factor of 0.0001 and was shifted horizontally and vertically to coincide with the experimental response. The values of $\omega_n$ and $\zeta$ that produced the best correspondence between the experimental and theoretical responses were used as the values for that bat and grip type. The obtained results are shown in Table 1. 4

The plots of the theoretical and experimental responses for each of the three cases in Table 1 are shown in Figures 1 to 3.

4A response that could be adequately analyzed using the described procedure could not be produced for the case of the double wall bat with the hands positioned near the upper end of the hand grip.
Table 1: The calculated values of $\omega_n$ and $\zeta$ for single wall and double wall aluminum bats with different hand positions.

<table>
<thead>
<tr>
<th>Bat Type</th>
<th>Hand Position</th>
<th>$\omega_n$ (rad/s)</th>
<th>$\zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single wall</td>
<td>Bottom of Handgrip</td>
<td>943</td>
<td>0.25</td>
</tr>
<tr>
<td>Single wall</td>
<td>Top of Handgrip</td>
<td>987</td>
<td>0.10</td>
</tr>
<tr>
<td>Double wall</td>
<td>Bottom of Handgrip</td>
<td>874</td>
<td>0.14</td>
</tr>
</tbody>
</table>

The results indicate that the values of $\omega_n$ and $\zeta$ are dependent on both bat type and hand position. Table 1 indicates that for the hand position at the bottom of the handgrip, the values of $\omega_n$ and $\zeta$ for the single wall bat were 7.9 % larger and 78.6 % larger, respectively, than those for the double wall bat. The difference in the values of $\omega_n$ indicates that the single wall bat is stiffer than the double wall, since the natural frequency is proportional to the square root of the stiffness [1]. The large difference in the value $\zeta$ was surprising for the following reason: A small value of $\zeta$ indicates the bat will vibrate for longer and more intensely, which can produce a sting in the hands of the batter. By increasing the value of $\zeta$, this characteristic sting is reduced. Since the double wall bat was introduced after the single wall bat [2], it could be expected that the value of $\zeta$ would be increased in this newer design to reduce the sting.

Table 1 indicates that for the single wall bat, the value of $\omega_n$ for the hand position at the top of the grip tape was 4.7 % larger than that for the other hand position, and the value $\zeta$ for the hand position at the bottom of the grip tape was 150 % larger than that for the other hand position. This indicates that the apparent stiffness of the bat and the characteristic sting can be affected by the positioning of the hands.

Figures 1 to 3 show that the theoretical and experimental responses had both similarities and differences. In some instances, the theoretical and experimental responses corresponded almost exactly, such as from about 0.0075 s to 0.015 s on Figure 2. During these cases, the theoretical equation serves as a good model of the experimental behavior. In other cases, the theoretical and experimental responses had noticeable differences, such as the end behavior of Figure 1 and the amplitude of the second trough on Figure 3. These differences can be explained by the following: The experimental responses were typically not centered around a fixed horizontal line ($y = 0$) like the theoretical responses, which made it difficult to match a theoretical response to an experimental one. The amplitude attenuation of the experimental responses over time also tended to be nonuniform, as shown in the significant amplitude reduction after the third peak of Figure 1. This produced differences in the amplitudes of the theoretical and experimental responses at certain times, as described earlier.

A problem frequently encountered in the determination of $\omega_n$ and $\zeta$ for the theoretical equation involved the FFT. The MATLAB program used plotted the relative amplitude versus the frequency of the experimental signal, and the frequency at which the relative amplitude was a maximum was supposed to be the natural frequency. However, the natural frequency that best matched the experimental response was almost always the frequency of the second highest relative amplitude. The frequency for the highest relative amplitude was typical a very low value ($\omega_n \approx 100$ rad/s) which did not correspond with the experimental
Figure 3: Plot of the experimental and theoretical outputs $y$ versus the time $t$ for the single wall bat with the hands positioned near the lower end of the handgrip.

response. This low frequency was probably determined from data acquired by the oscilloscope from the time leading up to the impact to the time after the bat had stopped oscillating. This low frequency was therefore a frequency of the signal acquired by the oscilloscope but not a frequency of the actual bat response. To resolve the problem, the FFT program could be modified to ignore frequencies below a certain value, or only the experimental data corresponding to the bat vibration could be entered into the program.

It should be noted that the values of $\omega_n$ determined for each bat type and grip type do not represent the true natural frequencies of the bats themselves. Because the response measured using the oscilloscope was that of the bat vibrating within the hands of the individual holding it, the determined values of $\omega_n$ are really the natural frequencies of the different bat-in-hand combinations. However, it is reasonable to assume that these determined values are very close to the values of $\omega_n$ for the bats themselves since a hand-held baseball bat vibrates as if it were a free body [1].

The major sources of uncertainty in the determination of the values of $\omega_n$ and $\zeta$ for each bat and grip type are described in the following: Due to the orientation of the strain gages on the bats, the orientation of the bat at impact would affect the measured response. In order to get more reliable results, a machine could be used to hit the ball with the bats for precise bat orientations. The calculation of $\zeta$ involved the determination of $y_1$, $y_2$, $t_1$, and $t_2$, which were obtained graphically. The determination of $y_1$ and $y_2$ was particularly subject to uncertainty since the equation that uses these values (equation 5) assumes the response
Figure 4: Plot of the experimental and theoretical outputs $y$ versus the time $t$ for the single wall bat with the hands positioned near the upper end of the handgrip.

is centered around $y = 0$. However, as described previously, this was not the case for the experimental data. These values were therefore calculated by first determining a horizontal line that best approximated the center of the response, and the determining the distance from the first and second peak to this line.

In conclusion, it was shown in this experiment that physical systems, including electrical and mechanical systems, can often be modeled as either first order or second order systems. Both experimental data and theory can be used to determine certain properties of the systems, such as $\tau$ for first order systems and $\omega_n$ and $\zeta$ for second order systems. Experimental data and theory can also be used to determine how the system will respond to different forcing functions.
Figure 5: Plot of the experimental and theoretical outputs $y$ versus the time $t$ for the double wall bat with the hands positioned near the lower end of the handgrip

3 References


well done